Zapiski predavanj iz Digitalnega procesiranja signalov

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18. januar 2000

Kazalo

0 Uvod

Literatura:

- John G. Proakis, Dimitris G. Manolakis: *Digital Signal Processing: Principles, Algorithms and Applications*, 3rd edition, Prentice Hall, 1995
- L.R. Rabiner, B. Gold: *Theory and Applications of Digital Signal Processing*, Prentice Hall, Englewood Cliffs, 1975

Osnovni pojmi:

- Signal (lat. *signum*) je nekaj, kar služi za prenos informacij; optični, električni, ...
- Načini obdelave signalov:
  - analogni: občutljivi na toploto, staranje, vlago
  - digitalni: matematična natančnost

Učni načrt:

1. Teorija diskretnih, linearnih, časovno-invariantnih sistemov
2. Digitalni filtri; dve vrsti:
   (a) KEO (*FIR – finite impulse response*)
   (b) NEO (*IIR – infinite impulse response*)
3. DFT in FFT algoritem
4. Spektralna analiza; spekter $\equiv$ Fourierova analiza signalov

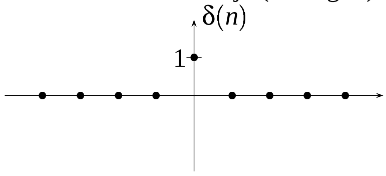
1 Teorija diskretnih, linearnih, časovno-invariantnih sistemov

$$x \longrightarrow \boxed{\text{LDČIS}} \longrightarrow y$$

Signale predstavimo z zaporedji:

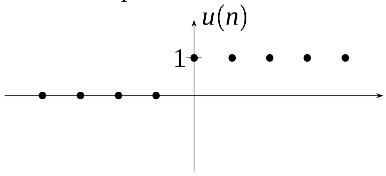
$$\begin{array}{ll} x(n), n = 0, \pm 1, \pm 2, \ldots & \text{– diskretni signal} \\ x(t) & \text{– analogni signal} \end{array}$$

- Diracova delta funkcija (analogno) – enotin impulz (diskretno)



$$\delta(n) \equiv \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

- enotina stopnica



$$u(n) \equiv \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$x(n) \longrightarrow \boxed{\Phi(x(n))} \longrightarrow y(n)$$

$$y(n) = \Phi[x(n)]$$

- Linearen sistem:

  $y_1(n)$ odziv na $x_1(n)$
  $y_2(n)$ odziv na $x_2(n)$
  $ax_1(n) + bx_2(n) \longrightarrow ay_1(n) + by_2(n) \iff$ sistem je linearen

- Časovno invarianten sistem:

  $y(n)$ odziv na $x(n)$
  $x(n - n_0) \longrightarrow y(n - n_0),\ n_0 \in \mathcal{Z} \iff$ sistem je časovno invarianten

  Bolj splošen termin (ko n ni čas): invariantnost na pomik.

1.1 Osnovni izrek za linearne, časovno-invariantne sisteme

$$\delta(n) \longrightarrow \boxed{} \longrightarrow h(n)$$ – odziv na enotin impulz

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$\boxed{y(n) = \sum_{m=-\infty}^{+\infty} x(m)h(n-m) = \sum_{m=-\infty}^{m=+\infty} h(m)x(n-m)}$$

Dokaz:

$$\begin{array}{ll} x(m) = x(n)\delta(n-m), & n = 0, \pm 1, \pm 2, \ldots \\ x(m+1) = x(n)\delta(n-(m+1)), & n = 0, \pm 1, \pm 2, \ldots \\ x(m+2) = x(n)\delta(n-(m+2)), & n = 0, \pm 1, \pm 2, \ldots \\ \hline x(m) = \sum_{n=-\infty}^{+\infty} x(n)\delta(n-m), & m = 0, \pm 1, \pm 2, \ldots \end{array}$$

odziv:

$$\begin{array}{l} \left.\begin{array}{l} x(m)h(n-m) \\ x(m+1)h(n-(m+1)) \\ x(m+2)h(n-(m+2)) \end{array}\right\} \text{zaradi linearnosti in časovne inv.} \\ \hline \sum_{m=-\infty}^{+\infty} x(m)h(n-m) \end{array}$$

Q.E.D.

Lastna funkcija:

$f(t) \longrightarrow \boxed{} \longrightarrow a \cdot f(t-\tau)$ – zakaj je sinusna napetost

Lastnosti sistemov:

- Stabilnost sistema: na končni vhod dobimo končni izhod
  Potreben in zadostni pogoj za stabilnost:

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$$

  Dokaz:

  - potrebnost:

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \infty, \quad x(n) = \begin{cases} +1, & h(-n) \geq 0 \\ -1, & h(-n) < 0 \end{cases}$$

$$y(0) = \sum_{m=-\infty}^{+\infty} x(m)h(-m) = \sum_{m=-\infty}^{+\infty} |h(-m)| = \sum_{m=-\infty}^{+\infty} |h(m)| = \infty$$

  - zadostnost:

$$|x(n)| \leq M$$

$$\begin{aligned} |y(n)| = |\sum_{m=-\infty}^{+\infty} x(m)h(n-m)| \leq & \sum_{m=-\infty}^{+\infty} |x(m)||h(n-m)| \leq \\ \leq & \; M \sum_{m=-\infty}^{+\infty} |h(n-m)| < \infty \end{aligned}$$

Q.E.D.

- Kavzalnost sistema: posledica ni pred vzrokom

Če je $h(n) = 0$ za $n < 0 \Rightarrow$ sistem je kavzalen

ali konstanta???

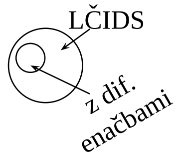
1.2 Sistemi, ki so podani z diferenčnimi enačbami

Diferenčne enačbe:

$$\sum_{i=0}^{N} b_i y(n-i) = \sum_{i=0}^{M} a_i x(n-i)$$

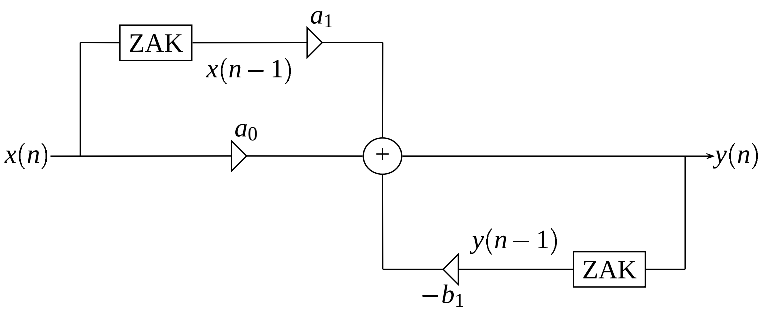
$$y(n) = \sum_{i=0}^{M} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i), n \geq 0$$

$x(n) \longrightarrow \boxed{\begin{matrix} a_i \\ b_i \end{matrix}} \longrightarrow y(n)$



1. nerekurzivne diferenčne enačbe: $b_1, b_2, \dots, b_N = 0$
2. rekurzivne diferenčne enačbe: vsaj en $b_i \neq 0$, $i = 1..N$

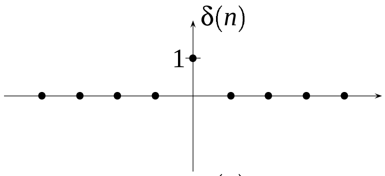
$$y(n) = a_0 x(n) + a_1 x(n-1) - b_1 y(n-1)$$



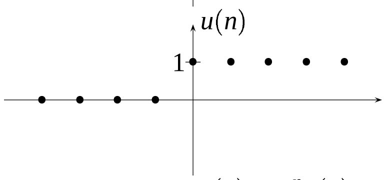
1.3 $\mathcal{Z}$-transformacija

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n},\ z \in \mathcal{C}$$

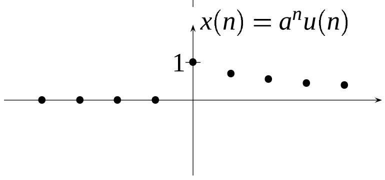
$$x(n) = \frac{1}{2\pi j} \oint_{C} X(z) z^{n-1}\, dz,\ j = \sqrt{-1} = i$$



$$\mathcal{Z}(\delta(n)) = 1$$



$$\mathcal{Z}(u(n)) = \frac{1}{1 - z^{-1}}$$



$$\mathcal{Z}(a^n u(n)) = \frac{1}{1 - az^{-1}}$$

1.4 Lastnosti $\mathcal{Z}$-transformacije

1. linearnost:

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n}$$

$$X_2(z) = \sum_{n=-\infty}^{+\infty} x_2(n) z^{-n}$$

$$\mathcal{Z}(ax_1 + bx_2) = aX_1(z) + bX_2(z)$$

2. premik:
$$\mathcal{Z}(x_1(n-k)) = z^{-k}X_1(z)$$

3. konvolucija:
$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)h(n-m)$$
$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} x(m)h(n-m)z^{-n} = \\ &= \overbrace{\sum_{m=-\infty}^{+\infty} x(m)z^{-m}}^{X(z)} \overbrace{\sum_{n=-\infty}^{+\infty} h(n-m)z^{-(n-m)}}^{H(z)} = \\ &= X(z)\cdot H(z) \end{aligned}$$
$$H(z) = \frac{Y(z)}{X(z)} \quad \text{– prevajalna funkcija}$$

4. produkt:
$$y(n) = x(n)h(n)$$
$$\frac{1}{2\pi j}\oint X(u)H\left(\frac{z}{u}\right)u^{-1}\,du$$

1.5 Frekvenčni odziv diskretnega sistema

$$x(n) = A\cos(\omega_0 n + \Phi) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = A'\cos(\omega_0 n + \Phi')$$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$$
$$x(n) = A\cos(\omega_0 n + \Phi) = \frac{A}{2}e^{j\Phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\Phi}e^{-j\omega_0 n}$$
$$y(n) = \frac{A}{2}e^{j\omega_0 n}e^{j\Phi}\sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega_0 k} + \frac{A}{2}e^{-j\omega_0 n}e^{-j\Phi}\sum_{k=-\infty}^{+\infty} h(k)e^{j\omega_0 k}$$

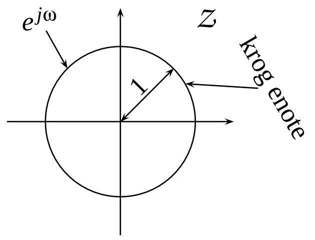
$$\boxed{H(e^{j\omega}) \equiv \sum_{k=-\infty}^{+\infty} h(k)e^{j\omega k} = |H(e^{j\omega})|e^{j\Theta(e^{j\omega})}} \text{ – frekvenčni odziv sistema}$$

Dokaz, da je sinusoidalna funkcija lastna funkcija diskretnih, linearnih, časovno-invariantnih sistemov:

$$y(n) = A|H(e^{j\omega_0})|\cos(\omega_0 n + \Phi + \Theta(e^{j\omega_0}))$$

Frekvenčni odziv: $\mathcal{Z}$-transformacija izračunana na krogu enote (na enotski krožnici):

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$$



$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$Y(z) = H(z)X(z)$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

1.6 Fourierove “transformacije”

1. zvezni signali

   - periodični: $x(t) = x(t + nT)$, $n = 0, \pm 1, \pm 2, \ldots$

$$\left.\begin{aligned} a_k &= \frac{1}{T}\int_0^T x(t)e^{-jk\frac{2\pi}{T}}\,dt \\ x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t} \end{aligned}\right\}\text{Fourierova vrsta}$$

   - neperiodični:

$$\left.\begin{aligned} X(\Omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}\,dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{j\Omega t}\,d\Omega \end{aligned}\right\}\begin{array}{l}\text{Fourierov integral ali}\\ \text{Fourierova transformacija}\end{array}$$

2. diskretni signali

   - periodični: $x(t) = x(t + nN), \quad n = 0, \pm 1, \pm 2, \ldots$

$$\left.\begin{aligned} a_n &= \frac{1}{N}\sum_{n=0}^{N-1} x(k)e^{-jk\frac{2\pi}{N}n} \\ x(k) &= \sum_{n=0}^{N-1} a_n e^{jk\frac{2\pi}{N}n} \end{aligned}\right\}\text{diskretno-časovna Fourierova transf.}$$

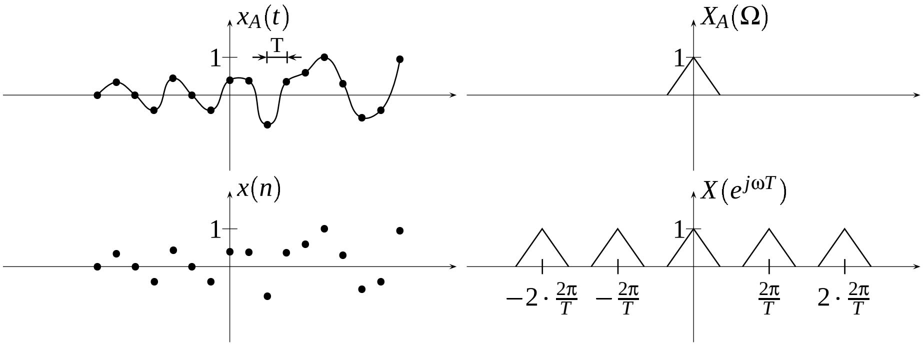
$$\left.\begin{aligned} X(n) &= \sum_{k=0}^{N-1} x(k)e^{-jk\frac{2\pi}{N}n} \\ x(k) &= \frac{1}{N}\sum_{n=0}^{N-1} X(n)e^{jk\frac{2\pi}{N}n} \end{aligned}\right\}\text{DFT – diskretna Fourierova transf.}$$

- neperiodični:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$$

$$x(k) = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega k}\,d\omega = \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} X(e^{j\omega T})e^{j\omega Tk}\,d\omega$$

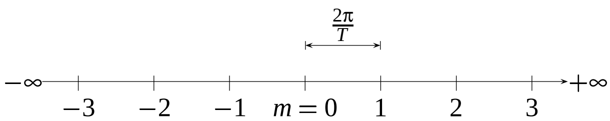
1.7 Vzorčenje



$$\left.\begin{aligned} X_A(\Omega) &= \int_{-\infty}^{+\infty} x_A(t)e^{-j\Omega t}\,dt \\ x_A(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_A(\Omega)e^{j\Omega t}\,d\Omega \end{aligned}\right\}\text{za analogne}$$

$$\left.\begin{aligned} X(e^{j\omega T}) &= \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega Tn} \\ x(n) &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} X(e^{j\omega T})e^{j\omega Tn}\,d\omega \end{aligned}\right\}\text{za diskretne}$$

$$\begin{aligned} x(n) &= x_A(nT) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_A(\Omega)e^{j\Omega nT}\,d\Omega \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{\frac{2m\pi}{T}}^{\frac{2(m+1)\pi}{T}} X_A(\Omega)e^{j\Omega nT}\,d\Omega \\ &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}}\left(\frac{1}{T}\sum_{m=-\infty}^{+\infty} X_A\left(\omega+\frac{2\pi}{T}\cdot m\right)\right)e^{j\omega nT}\,d\omega \end{aligned}$$

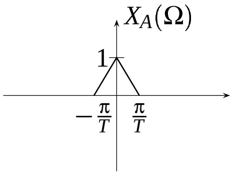


$$\Omega = \omega + \frac{2\pi}{T}\cdot m$$

Iz primerjave sledi: $\boxed{X(e^{j\omega T}) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X_A\left(\omega + \frac{2\pi}{T}\cdot m\right)}$

Torej: spekter vzorčenega signala je enak neskončni vsoti spektrov zamaknjenih signalov.

1.8 Izrek o jemanju vzorcev



$$X_A(\Omega) = 0 \text{ za } |\Omega| > \frac{\pi}{T}$$

$$X(e^{j\omega T}) = \frac{1}{T}X(\omega) \text{ velja samo, če velja } -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

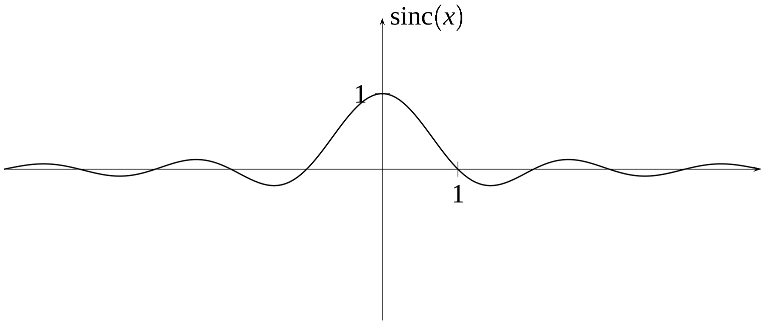
$$x_A(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_A(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} X_A(\omega)e^{j\omega t}\,d\omega = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} X(e^{j\omega T})e^{j\omega t}\,d\omega$$

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega Tn} = \sum_{n=-\infty}^{+\infty} x_A(nT)e^{-j\omega Tn}$$

$$x_A(t) = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} \left(\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega nT}\right)e^{j\omega t}\,d\omega = \sum_{n=-\infty}^{+\infty} x(n)\frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{j\omega(t-nT)}\,d\omega, \quad x(n) = x_A(nT)$$

$$\boxed{x_A(t) = \sum_{n=-\infty}^{+\infty} x_A(nT)\underbrace{\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}}_{\operatorname{sinc}\left(\frac{t}{T}-n\right)}} \text{ – interpolacijska formula}$$

$$\operatorname{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$$



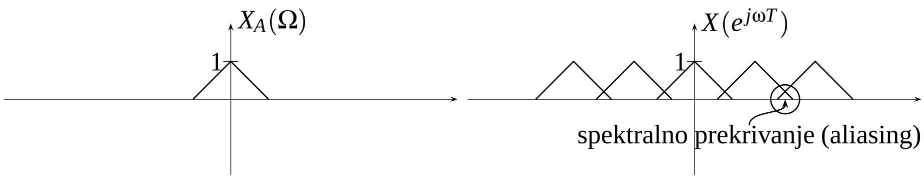
Pogoj za obratnost vzorčenja:

$$\frac{\pi}{T} \geq \text{ najvišja frekvenca } (\Omega) \text{ vzorčenega signala}$$

oziroma

$$\frac{1}{T} = f_T > 2 \times \text{najvišja frekvenca vzorčenega signala}$$

$2 \times f_{max}$ – Nyquistova frekvenca



Antialiasing filtri

Transformacijski par za predstavitev zveznega signala na diskreten način:

$$\boxed{\begin{array}{l} x_A(t) = \sum\limits_{n=-\infty}^{+\infty} c_n \Phi_n(t) \\ c_n = x_A(nT) \end{array}}$$

$$\Phi_n(t) = \mathrm{sinc}\left(\frac{t}{T} - n\right)$$

1.9 Predstavitev signalov z ortogonalnimi funkcijami

Množica funkcij $\{\Phi_k(t)\} = \{\Phi_{-\infty}(t), \dots, \Phi_{-1}(t), \Phi_0(t), \Phi_1(t), \dots, \Phi_\infty(t)\}$ je ortogonalna na intervalu t_0 do $t_0 + \tau$, če velja:

$$\int_{t_0}^{t_0+\tau} \Phi_m(t)\Phi_n(t)\, dt = \begin{cases} a, & m = n \\ 0, & m \neq n \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k(t)\Phi_k(t)$$

$$c_k = \frac{1}{a} \int_{t_0}^{t_0+\tau} x(t)\Phi_k(t)\, dt$$

Dokaz:

$$\int_{t_0}^{t_0+\tau} x(t)\Phi_m(t)\, dt = \int_{t_0}^{t_0+\tau} \left(\sum_{k=-\infty}^{+\infty} c_k \Phi_k(t) \right) \Phi_m(t)\, dt = \sum_{k=-\infty}^{+\infty} c_k \int_{t_0}^{t_0+\tau} \Phi_k(t)\Phi_m(t)\, dt = c_m \cdot a$$

Q.E.D.

Druge družine ortogonalnih funkcij:

- Kaihunen-Loewe
- Fourier
- Walsh-Maclainard
- valčki (wavelet)
- Haav
- Legendre

Parsevalov izrek

Podaja zvezo med energijo (močjo) v časovnem in frekvenčnem prostoru in velja za vse ortogonalne funkcije.

$$x^2(t) = \sum_{k=-\infty}^{+\infty} c_k^2 \Phi_k(t) + \sum_{p=-\infty}^{+\infty} \sum_{\substack{q=-\infty \\ q\neq p}}^{+\infty} c_p c_q \Phi_p(t)\Phi_q(t)$$

$$\int_{t_0}^{t_0+\tau} x^2(t)\,dt = \sum_{k=-\infty}^{+\infty} c_k^2 \underbrace{\int_{t_0}^{t_0+\tau} \Phi_k^2(t)\,dt}_{a} + \underbrace{\sum_{p=-\infty}^{+\infty} \sum_{\substack{q=-\infty \\ q\neq p}}^{+\infty} c_p c_q \int_{t_0}^{t_0+\tau} \Phi_p(t)\Phi_q(t)\,dt}_{0}$$

$$\boxed{\frac{1}{\tau}\int_{t_0}^{t_0+\tau} x^2(t)\,dt = \frac{a}{\tau}\sum_{k=-\infty}^{+\infty} c_k^2} \text{ – Parsevalov izrek}$$

Parsevalov izrek govori o tem, kako je moč signala razporejena po koeficientih diskretne predstavitve funkcije (oz. po spektru).

1.10 Strukture za realizacijo diskretnih sistemov

Za vsak LDČIS velja: $x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k)$$

$$Y(z) = H(z)\cdot X(z)$$

$$H(z) = \frac{\sum_{l=0}^{M} a_l z^{-l}}{1+\sum_{l=1}^{N} b_l z^{-l}} = A\frac{\prod_{l=1}^{M}(1-z_l z^{-1})}{\prod_{l=1}^{N}(1-p_l z^{-1})}, \quad z_l, p_l \text{ – ničle, poli}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} a_l z^{-l}}{1+\sum_{l=1}^{N} b_l z^{-l}} \quad \text{– prevajalna funkcija} \tag{1}$$

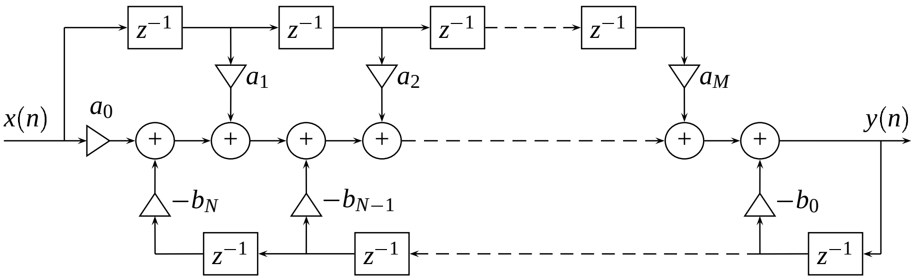
Navzkrižno pomnožimo:

$$\sum_{l=0}^{M} a_l z^{-l} X(z) = Y(z)\left(1+\sum_{l=1}^{N} b_l z^{-l}\right)$$

$$\sum_{l=0}^{M} a_l x(n-l) = y(n) + \sum_{l=1}^{N} b_l y(n-l) \quad \text{– inverzna } \mathcal{Z}\text{-transformacija gornje funkcije}$$

$$y(n) = \sum_{l=0}^{M} a_l x(n-l) - \sum_{l=1}^{N} b_l y(n-l) \quad \text{– diferenčna enačba za (1)}$$

Opisano je zelo enostaven postopek pretvorbe prevajalne funkcije v diferenčno enačbo. Prikaz z diagramom, direktna forma 1:



Če so vsi $b_i = 0$, potem je enačba nerekurzivna.

$$H(z) = \frac{Y(z)}{X(z)} = \underbrace{\left(\frac{1}{1+\sum_{l=1}^{N} b_l z^{-l}}\right)}_{H_1(z)} \underbrace{\left(\sum_{l=0}^{M} a_l z^{-l}\right)}_{H_2(z)} = H_1(z)H_2(z)$$

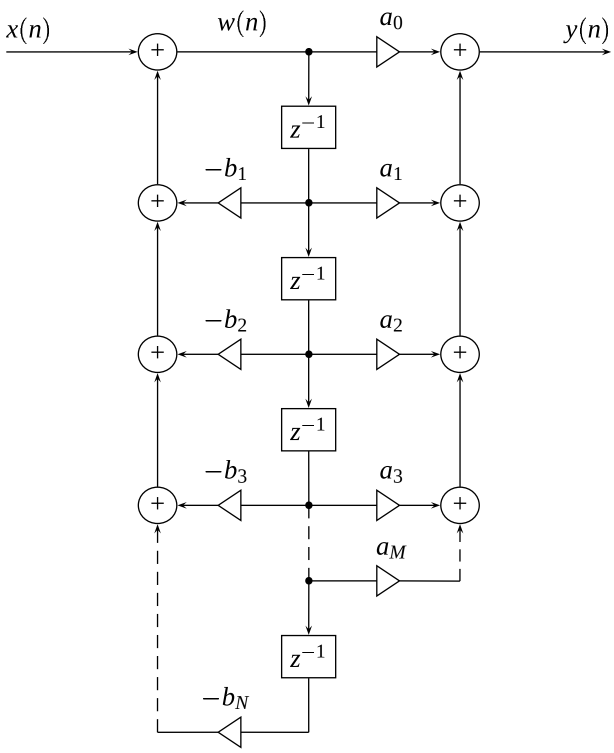
$$H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1+\sum_{l=1}^{N} b_l z^{-l}}$$

$$H_2(z) = \frac{Y(z)}{W(z)} = \sum_{l=0}^{M} a_l z^{-l}$$

$$w(n) = x(n) - \sum_{l=1}^{N} b_l w(n-l)$$

$$y(n) = \sum_{l=0}^{M} a_l w(n-l)$$

Dobimo enostavnejšo direktno formo 2 (kanonična oblika):



Težave:

- pri kvantizaciji a_l, b_l hočemo porabiti čim manj bitov za reprezentacijo funkcije,
- že majhne napake pri koeficientih direktne forme zelo spremenijo potek prevajalne funkcije – občutljivost direktne forme na napake.

Občutljivost direktne forme na kvantizacijo koeficientov a_l, b_l:

a_l, b_l	– neskončno natančni koeficienti
a_l^*, b_l^*	– kvantizirani koeficienti (končno število bitov)

$$a_l^* = a_l + \Delta a_l$$

$$b_l^* = b_l + \Delta b_l$$

$$\Delta a_l, \Delta b_l \quad \text{– kvantizacijska napaka}$$

$$P(z) = 1 + \sum_{l=1}^{N} b_l z^{-l} = \prod_{l=1}^{N} (1 - p_l z^{-l}), \quad p_l \text{ – poli}$$

$$p_l^* = p_l + \Delta p_l, \quad \Delta p_l \text{ – napaka v legi pola}$$

Če je majhna sprememba $b_l \to$ velika sprememba lege polov $\to$ občutljivost

$$p_l = p_l(b_1, b_2, \ldots, b_N)$$

$$P(z) = P(b_1, b_2, \ldots, b_N) = P(p_1, p_2, \ldots, p_N)$$

$$\Delta p_l = \sum_{k=1}^{N} \frac{\partial p_l}{\partial p_k} \Delta b_k$$

$$\frac{\partial P(z)}{\partial p_l} = \frac{\partial p_l}{\partial b_k} = \frac{\partial P(z)}{\partial b_k}$$

$$\frac{\partial p_l}{\partial b_k} = \frac{\frac{\partial P(z)}{\partial b_k}}{\frac{\partial P(z)}{\partial p_l}} = \frac{p_l^{N-k}}{\prod_{\substack{i=1 \\ i\neq l}}^{N} (p_l - p_i)}$$

$$\boxed{\Delta p_l = \sum_{k=1}^{N} \frac{p_l^{N-k}}{\prod_{\substack{i=1 \\ i\neq l}}^{N} (p_l - p_i)} \cdot \Delta b_k}$$

To je dokaz, da majhne spremembe b_i povzročijo velike spremembe p_l.
Če sta dva (ali več) pola blizu skupaj, potem je faktor v vsoti velik.

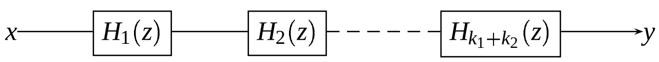
Kaskadna forma

$$H(z) = a_0 \prod_{l=1}^{k_1} H_{1l}(z) \cdot \prod_{l=1}^{k_2} H_{2l}(z)$$

$$H_{1l}(z) = \frac{1 + a_{1l} z^{-1}}{1 + b_{1l} z^{-1}}$$

$$H_{2l}(z) = \frac{1 + a_{1l} z^{-1} + a_{2l} z^{-2}}{1 + b_{1l} z^{-1} + b_{2l} z^{-2}}$$

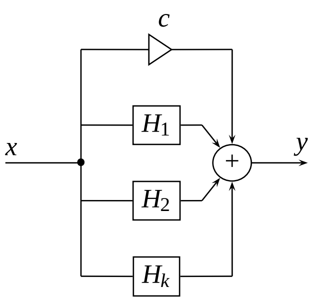
Realizacija:



Skupna občutljivost na kvantizacijske napake je bistveno manjša, ker so le-te omejene na posamezne H-je.

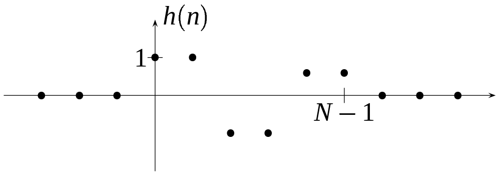
Paralelna forma

$$H(z) = c + \sum_{l=1}^{k} H_l(z)$$



2 Metode za načrtovanje digitalnih filtrov s končnim enotinim odzivom (KEO)

Če $h(n) = 0$, $n \geq N$, potem pravimo, da ima $h(n)$ končno dolžino.



– KEO sistem

Negativni indeksi so nič zaradi kavzalnosti sistema! Pri nerekurzivnih funkcijah imamo KEO sistem.
Lastnosti:

1. $H(z)$ nima polov (razen pri $z = 0$)

$$H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}$$

2. sistem je vedno stabilen:

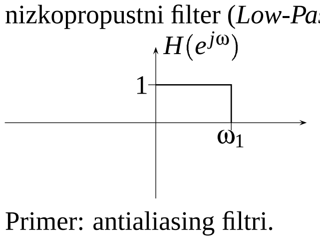
$$\sum_{k=0}^{N-1} |h(k)| = \text{končno}$$

Kaj je to načrtovanje digitalnega filtra?

1. podan imamo želeni frekvenčni potek $H_d(e^{j\omega})$
2. podano imamo dolžino N
3. iščemo tak odziv na enotin impulz $h(n)$, da bo razlika $H(e^{j\omega}) - H_d(e^{j\omega})$ čim manjša (z večanjem N manjšamo le-to)

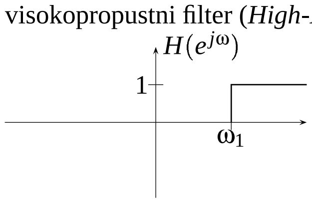
Koeficienti KEO filtra: $h(0), h(1), \ldots, h(N-1)$.
Štiri osnovne vrste filtrov (poleg teh štirih še veliko drugih):

- nizkopropustni filter (*Low-Pass*) LP

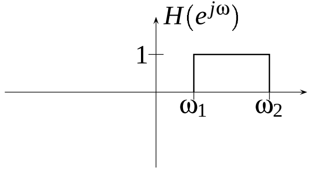
  

  Primer: antialiasing filtri.

- visokopropustni filter (*High-Pass*) HP

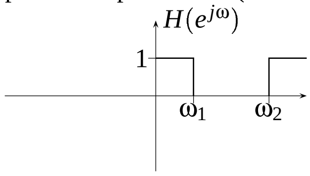
  

- pasovnopropustni filter (*Band-Pass*) BP



Primer: radio in televizija; širina pasu za radio: AM: 4.5 kHz, FM: 50 KHz (EU), 100 kHz (USA).

- pasovnozaporni filter (*Band-Stop*) BS



Obstajajo transformacije za preslikavo:

$$\begin{tikzcd} & \mathrm{HP} \\ \mathrm{LP} \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{BP} \\ & \mathrm{BS} \end{tikzcd}$$

$H_d(e^{j\omega})$ je odsekoma konstantna funkcija.
Za KEO digitalne filtre velja $h(k) = 0$ za $k \geq N$, $N =$ dolžina filtra. KEO filter je vedno stabilen, ker $\sum_{k=0}^{\infty} |h(k)| =$ končen.

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} h(k) e^{-j\omega k} = |H(e^{j\omega})| \cdot e^{j\Theta(e^{j\omega})}$$

frekvenčni odziv   amplitudni odziv   fazni odziv (zakasnitev)

$$x(k) \longrightarrow \boxed{h(k)} \longrightarrow y(k),\ x(k) = A\cos(\omega k)$$

$$\begin{aligned} y(k) &= A \cdot |H(e^{j\omega})| \cdot \cos(\omega k + \Theta(e^{j\omega})) \\ &= A \cdot |H(e^{j\omega})| \cdot \cos\left(\omega\left(k + \underbrace{\frac{\Theta(e^{j\omega})}{\omega}}_{\Delta k}\right)\right) \end{aligned}$$

$$-\frac{\Theta(e^{j\omega})}{\omega} = \text{zakasnitev}$$

Linearna faza: $\Theta(\omega) = \alpha - \beta\omega$

$$\frac{\Theta(\omega)}{\omega} = \frac{\alpha}{\omega} - \beta$$

1. Filtri z linearno fazo
2. Filtri z nelinearno fazo

2.1 KEO digitalni filtri z linearno fazo

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} h(k)e^{-j\omega k} \stackrel{\text{def}}{=} G(e^{j\omega})e^{j(\alpha-\beta\omega)}$$

realna funkcija (lahko tudi negativna)

Za realne $h(k)$: konjugirano kompleksen

$$H^*(e^{j\omega}) = H(e^{-j\omega})$$

$$|H(e j\omega)| = |H(e^{-j\omega})|$$
$$|G(e^{j\omega})| = |G(q^{-j\omega})|$$

Iz tega sledi, da je funkcija $G(e^{j\omega})$:

1. ali soda,
2. ali liha.

Primer:

1. $G(e^{j\omega}) = G(e^{-j\omega})$ soda funkcija
   $H^*(e^{j\omega}) = G(e^{j\omega}) \cdot e^{-j(\alpha-\beta\omega)}$
   $H(e^{-j\omega}) = G(e^{-j\omega}) \cdot e^{j(\alpha+\beta\omega)}$
   Zaradi $H^*(e^{j\omega}) = H(e^{-j\omega})$ sledi

$$\underbrace{G(e^{j\omega})}e^{-j(\alpha-\beta\omega)} = \underbrace{G(e^{-j\omega})}e^{j(\alpha+\beta\omega)}$$

isto

$$e^{-j(\alpha-\beta\omega)} = e^{j(\alpha+\beta\omega)}$$
$$e^{-j\alpha} \cdot e^{j\beta\omega} = e^{j\alpha} \cdot e^{j\beta\omega}$$
$$\boxed{\alpha = 0}$$

2. $G(e^{j\omega}) = -G(e^{-j\omega})$ liha funkcija

$$e^{-j\alpha} = -e^{j\alpha}$$
$$\boxed{\alpha = \frac{\pi}{2}}$$

Kakšen je β?

- Za sodo funkcijo $G(e^{j\omega})$, $\alpha = 0$

$$\begin{cases} G(e^{j\omega}) = H^*(e^{j\omega}) \cdot e^{j\beta\omega} & \text{– po definiciji} \\ G(e^{-j\omega}) = H(e^{-j\omega}) \cdot e^{-j\beta\omega} \end{cases}$$

$$H(e^{j\beta\omega})e^{j\beta\omega} = H(e^{-j\omega})e^{-j\beta\omega}$$

$$
\begin{aligned}
\left(\sum_{k=0}^{N-1} h(k)e^{-j\omega k}\right)e^{j\beta\omega} &= \left(\sum_{k=0}^{N-1} h(k)e^{j\omega k}\right)e^{-j\beta\omega} = \\
&= e^{j(N-1-2\beta)\omega}\left(\sum_{k=0}^{N-1} h(N-1-k)e^{-j\omega k}\right)e^{j\beta\omega}
\end{aligned}
$$

$$\boxed{\beta = \frac{N-1}{2}} \qquad \boxed{\alpha = 0}$$

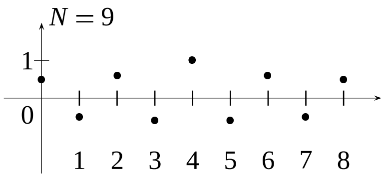
$$\boxed{h(k) = h(N-1-k)} \quad \text{– pozitivna simetrija}$$

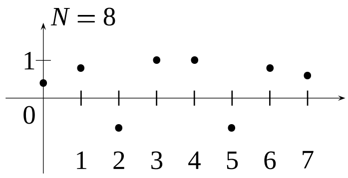
- Za liho funkcijo $G(e^{j\omega}) = -G(e^{-j\omega})$

$$\boxed{\beta = \frac{N-1}{2}} \qquad \boxed{\alpha = \frac{\pi}{2}}$$

$$\boxed{h(k) = -h(N-1-k)} \quad \text{– negativna simetrija}$$

Posledica linearne faze: pozitivna ali negativna simetrija $h(k)$





Sledi: polovica $h(k)$ je vnaprej določena, prostostna stopnja se zmanjša.
4 osnovni tipi KEO filtrov z linearno fazo:

1. pozitivna simetrija ($G(e^{j\omega}) = G(-e^{j\omega})$), $N =$ liho število

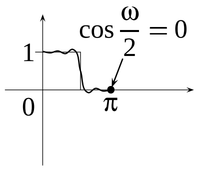
$$
\begin{aligned}
H(e^{j\omega}) &= e^{-j\omega\frac{N-1}{2}}\left(h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right)\cos(\omega k)\right) = \\
&= e^{-j\omega\frac{N-1}{2}} \underbrace{\sum_{k=0}^{\frac{N-1}{2}} a(k)\cos(\omega k)}_{\text{kosinusni polinom}} \\
a(0) &= h\left(\frac{N-1}{2}\right) \\
a(k) &= 2h\left(\frac{N-1}{2}-k\right), \quad k = 1, 2, \ldots, \frac{N-1}{2}
\end{aligned}
$$

Kosinusni polinom:

$$\cos(\omega k) = \sum_{l=1}^{k} \alpha_l \cos^l \omega = \sum_{l=1}^{k} \alpha_l x^l$$

2. pozitivna simetrija, $N =$ sodo število

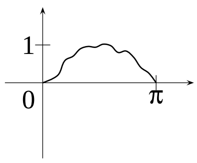
$$
\begin{aligned}
H(e^{j\omega}) &= e^{-j\omega\frac{N-1}{2}} \sum_{k=0}^{\frac{N-1}{2}-1} 2h\left(\frac{N}{2}-1-k\right)\cdot\cos\left(\omega\left(k+\frac{1}{2}\right)\right) = \\
&= e^{-j\omega\frac{N-1}{2}} \cos\frac{\omega}{2} \sum_{k=0}^{\frac{N}{2}-1} b(k)\cos(\omega k)
\end{aligned}
$$



pri $\omega = \pi$ imamo vedno $H(e^{j\omega}) = 0$

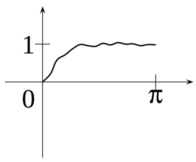
3. negativna simetrija, $N =$ liho število ($\alpha = \frac{\pi}{2}$)

$$
\begin{aligned}
H(e^{j\omega}) &= e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \left(e^{-j\frac{\pi}{2}} \underbrace{h\left(\frac{N-1}{2}\right)}_{\text{0, sicer } G(e^{j\omega}) \text{ ni liha}} + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}\right)\sin(\omega k)\right) = \\
&= e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sin\omega \sum_{k=0}^{\frac{N-3}{2}} c(k)\cos(\omega k)
\end{aligned}
$$



4. negativna simetrija, $N =$ sodo število

$$
\begin{aligned}
H(e^{j\omega}) &= e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \left(\sum_{k=0}^{\frac{N}{2}-1} 2h\left(\frac{N}{2}-1-h\right)\sin\omega\left(k+\frac{1}{2}\right)\right) = \\
&= e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sin\frac{\omega}{2} \sum_{k=0}^{\frac{N}{2}-1} d(k)\cos(\omega k)
\end{aligned}
$$

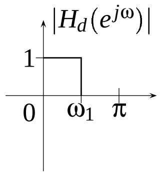


$Q(\omega)$	$P(\omega)$
1, $\cos\frac{\omega}{2}$, $\sin\omega$, $\sin\frac{\omega}{2}$	$\sum_{k=0}^{n} d(k)\cos(\omega k)$

2.2 Načrtovanje z uporabo okenskih funkcij

$H_d(e^{j\omega})$ želeni frekvenčni potek

$$H_d(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h_d(k) e^{-j\omega k}$$



$$h_d(k) = \frac{1}{2\pi} \int_0^{2\pi} H_d(e^{j\omega}) e^{j\omega k} \, d\omega$$

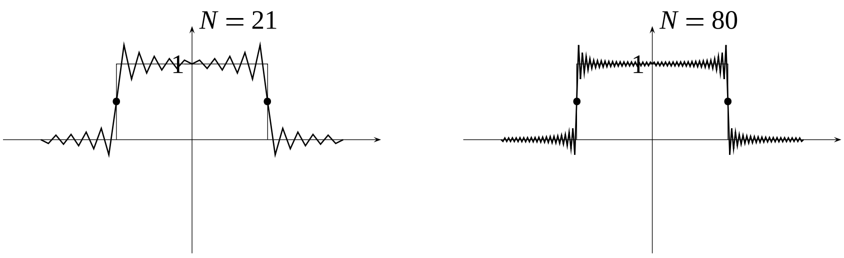
Problem: $h_d(k)$ je neskončno dolg, če vsebuje $H_d(e^{j\omega})$ nezveznosti. Rešitev:

$$h(k) = \begin{cases} h_d(k) & k = 0, 1, \ldots, N-1 \\ 0 & \text{drugod} \end{cases} \tag{2}$$

filter s končnim enotinim odzivom

2.2.1 Gibbsov pojav

1898, Albert Michelson s harmoničnim analizatorjem ugotovil posledice enačbe (2):



Gibbsov pojav: v točkah nezveznosti je vedno (za vsak končen N) 9% odstopanja (funkcija gre skozi sredino stopnice).
Kako razložimo?

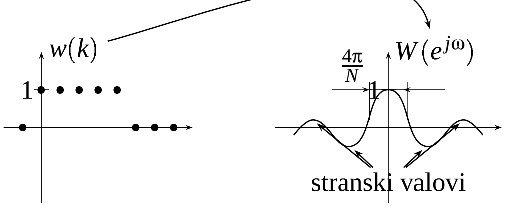
$$h(k) = h_d(k) \cdot w(k)$$

$$w(k) = \begin{cases} 1 & k = 0, 1, \ldots, N-1 \\ 0 & \text{drugod} \end{cases} \quad \text{– pravokotno okno}$$

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_0^{2\pi} H_d(e^{j\Theta}) W(e^{j(\omega-\Theta)}) \, d\Theta \quad \text{– konvolucijski integral}$$

$$W(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} w(k) e^{-j\omega k} = e^{-j\omega \frac{N-1}{2}} \underbrace{\frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}}}_{\text{digitalni sinc}}$$

Pravokotno okno:

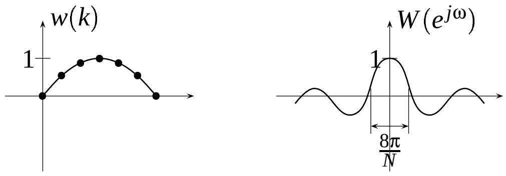


Stranski valovi povzročijo Gibbsov pojav. Funkcija brez valov je δ, vendar jo povzroči neskončno pravokotno okno. Torej iščemo funkcije s čim manjšimi valovi.

2.2.2 Okenske funkcije

1. von Hannovo okno:

$$w(k) = 0.5 - 0.5 \cdot \cos\left(\frac{2\pi k}{N-1}\right)$$



   Valovi so manjši, vendar je okno širše.

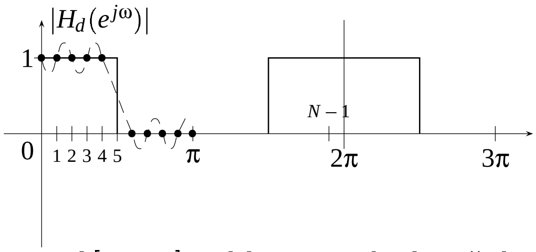
2. Hammingovo okno:

$$w(k) = 0.54 - 0.46 \cdot \cos\left(\frac{2\pi k}{N-1}\right)$$

   Podobno je von Hannovemu oknu.

3. Kaiserjeva okna (optimalna): v stranskih valovih imajo najmanj energije.
4. Dulph-Čebiševo okno
5. Bartlettovo okno
6. Trikotno okno

2.3 Načrtovanje z uporabo frekvenčnega vzorčenja



Interval $[0 \ldots 2\pi]$ razdelimo na N ekvidistančnih točk.

$$H_d(e^{j\omega_k}), \quad \omega_k = \frac{2\pi}{N} \cdot k, \quad k = 0, 1 \ldots, N-1$$

$$H_d(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

Zanimajo nas točke $\omega = \omega_k$:

$$H_d(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} h(n) e^{\frac{-2\pi jnk}{N}} \quad \longleftarrow \text{DFT: } H_d(k) = \sum_{n=0}^{N-1} h(n) e^{\frac{-2\pi jnk}{N}}$$

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H_d(k) e^{\frac{2\pi jkn}{N}} \quad \longleftarrow \text{inverzna DFT (IDFT)}$$

$$\begin{aligned}
H(z) &= \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} H_d(k) e^{\frac{2\pi jkn}{N}} \right) z^{-n} = \\
&= \sum_{k=0}^{N-1} \frac{H_d(k)}{N} \sum_{n=0}^{N-1} \left(e^{\frac{2\pi jk}{N}} \cdot z^{-1} \right)^n = \\
&= \sum_{k=0}^{N-1} \frac{H_d(k)}{N} \cdot \frac{1 - \overbrace{e^{2\pi jk}}^{=1} \cdot z^{-N}}{1 - e^{\frac{2\pi jk}{N}} \cdot z^{-1}} = \\
&= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H_d(k)}{1 - z^{-1} e^{\frac{2\pi jk}{N}}}
\end{aligned}$$

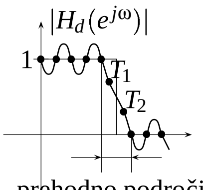
$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

$$H(e^{j\omega}) = e^{\frac{-j\omega(N-1)}{2}} \sum_{k=0}^{N-1} H_d(e^{j\omega_k}) e^{j2\pi k \frac{N-1}{2N}} \cdot \overbrace{\frac{\sin\left(N\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)\right)}{N \cdot \sin\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}}^{\text{digitalni sinc: } \frac{\sin(Nx)}{N \sin x}}$$

Pri linearni fazi:

$$H_d(e^{j\omega_k}) = |H_d(e^{j\omega_k})| e^{-j2\pi k \frac{N-1}{2N}}$$

$$\boxed{H(e^{j\omega}) = e^{\frac{-j\omega(N-1)}{2}} \sum_{k=0}^{N-1} |H_d(e^{j\omega_k})| \underbrace{\frac{\sin\left(N\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)\right)}{N \cdot \sin\left(\frac{\omega}{2} - \frac{\pi k}{N}\right)}}_{\text{povzroča “valove”} \rightarrow \text{slab filter}}}$$



prehodno področje
(*transition band*)

T_1, T_2 sta frekvenci v prehodnem področju.
Ideja metode: položaj $H(e^{j\omega})$ za $\omega = \omega_{T_1}, \omega_{T_2}$ ni pomemben. Imamo: 1,2 ali 3 točke, v katerih $|H(e^{j\omega})|$ lahko spreminjamo. Pametna izbira $|H(e^{j\omega})|_{\omega=\omega_{T_1}, \omega_{T_2}, \omega_{T_3}}$ močno zmanjša valove.
Kje dobimo pametno izbiro? Obstajajo tabele.

2.4 Načrtovanje optimalnih KEO filtrov z linearno fazo

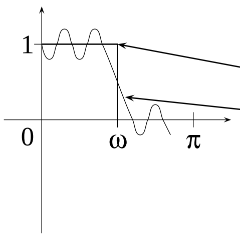
$$H(e^{j\omega}) = e^{\frac{-j\omega(N-1)}{2}} e^{jL\frac{\pi}{2}} G(e^{j\omega})$$

$L = 0$ pri pozitivni simetriji
$L = 1$ pri negativni simetriji

kosinusni polinom

$$G(e^{j\omega}) = Q(e^{j\omega}) \cdot P(e^{j\omega}) = Q(e^{j\omega}) \cdot \sum_{k=0}^{r-1} \alpha(k)\cos(k\omega)$$

1. $Q(e^{j\omega}) = 1$
2. $Q(e^{j\omega}) = \cos\frac{\omega}{2}$
3. $Q(e^{j\omega}) = \sin\omega$
4. $Q(e^{j\omega}) = sin\frac{\omega}{2}$



$H_d(e^{j\omega}) \cdot \text{faza} = D(e^{j\omega})$ $\longleftarrow$ želeni potek (brez faze)

$G(e^{j\omega})$

$$D(e^{j\omega}) = H_d(e^{j\omega}) \cdot e^{\frac{j\omega(N-1)}{2}} \cdot e^{-jL\frac{\pi}{2}}$$

funkcija napake

$$E(e^{j\omega}) = W(e^{j\omega})\left(D(e^{j\omega}) - G(e^{j\omega})\right), \qquad W(e^{j\omega}) > 0$$

utežnostna funkcija

$$E(e^{j\omega}) = \underbrace{W(e^{j\omega}) \cdot Q(e^{j\omega})}_{W'(e^{j\omega})}\left(\underbrace{\frac{D(e^{j\omega})}{Q(e^{j\omega})}}_{D'(e^{j\omega})} - P(e^{j\omega})\right)$$

$$\boxed{E(e^{j\omega}) = W'(e^{j\omega})\left(D'(e^{j\omega}) - P(e^{j\omega})\right)} \tag{3}$$

Enačba (3) je poenotenje štirih tipov filtrov na isti aproksimacijski problem.

2.4.1 Kriterij optimalnosti

aproksimacijski problem

$$\left\|E(e^{j\omega})\right\|_\infty = \min_{\{\alpha\}}\left(\max_{\omega\in\Omega}\left|W'(e^{j\omega})\left(D'(e^{j\omega}) - P(e^{j\omega})\right)\right|\right), \qquad \Omega \subseteq (0,\pi)$$

– Čebiševa norma (mat.), minimax (teh.), L_∞ norma, enakomerna aproksimacija

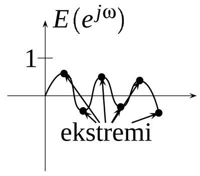
2.4.2 Alternacijski izrek

Naj bo $P(e^{j\omega})$ kosinusni polinom:

$$P(e^{j\omega}) = \sum_{k=0}^{r-1} \alpha(k)\cos(k\cdot\omega)$$

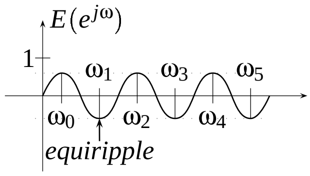
Potreben in zadosten pogoj za optimalno aproksimacijo je, da ima funkcija napake $E(e^{j\omega})$:

1. najmanj $r+1$ ekstremov:



2. če so $0 \leq \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_r \leq \pi$ točke, v katerih je ekstrem (ekstremalne frekvence), potem velja:

$$E(e^{j\omega_i}) = -E(e^{j\omega_{i+1}}), \qquad i = 0, 1, \ldots, r-1$$



$$|E_{\max}| = |E(e^{j\omega_i})|, \qquad i = 0, 1, \ldots, r-1$$

Funkcija napake:

$$E(e^{j\omega}) = W'(e^{j\omega})\left(D'(e^{j\omega}) - P(e^{j\omega})\right), \qquad P(e^{j\omega}) = \sum_{k=0}^{r-1} \alpha(k)\cos(k\omega)$$

2.4.3 Algoritem za določanje koeficientov $\alpha(k)$

Remezov algoritem; leta 1972 sta ga uporabila Parks & McClellan

1. Predpostavimo, da poznamo ekstremalne frekvence $\omega_0, \omega_1, \ldots, \omega_r$. Potem velja sistem enačb:

$$W'(e^{j\omega_k})\left(D'(e^{j\omega_k}) - P(e^{j\omega_k})\right) = (-1)^k\delta, \qquad k = 0, 1, \ldots, r$$

(drugače zapisan alternacijski izrek)

$$D'(e^{j\omega_k}) = \frac{(-1)^k}{W'(e^{j\omega_k})} \cdot \delta + \sum_{i=0}^{r-1} \alpha(i)\cos(i\omega_k), \qquad k = 0, 1, \ldots, r$$

$$\begin{bmatrix} 1 & \cos\omega_0 & \cos(2\omega_0) & \cdots & \cos((r-1)\omega_0) & \frac{1}{W'(e^{j\omega_0})} \\ 1 & \cos\omega_1 & \cos(2\omega_1) & \cdots & \cos((r-1)\omega_1) & \frac{-1}{W'(e^{j\omega_1})} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos\omega_r & \cos(2\omega_r) & \cdots & \cos((r-1)\omega_r) & \frac{(-1)^r}{W'(e^{j\omega_r})} \end{bmatrix} \begin{bmatrix} \alpha(0) \\ \alpha(1) \\ \vdots \\ \alpha(r-1) \\ \delta \end{bmatrix} = \begin{bmatrix} D'(e^{j\omega_0}) \\ D'(e^{j\omega_1}) \\ \vdots \\ D'(e^{j\omega_r}) \end{bmatrix}$$

Če poznamo $\omega_0, \omega_1, \ldots, \omega_r$, lahko izračunamo $\alpha(0), \alpha(1), \ldots, \alpha(r-1)$ in δ. Problem: ne poznamo jih. Uporabimo trik:

2. δ je mogoče izračunati, ker imamo Van der Modejevo determinanto:

$$\begin{vmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

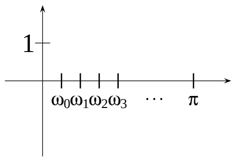
$$\delta = \frac{\gamma_0 D'(e^{j\omega_0}) + \gamma_1 D'(e^{j\omega_1}) + \cdots + \gamma_r D'(e^{j\omega_r})}{\frac{\gamma_0}{W'(e^{j\omega_0})} - \frac{\gamma_1}{W'(e^{j\omega_1})} + \cdots + \frac{(-1)^r \gamma_r}{W'(e^{j\omega_r})}}, \qquad \gamma_k = \frac{1}{\prod_{\substack{n=0\\n\neq k}}^{r} (\cos\omega_k - \cos\omega_n)}$$

3. Uporabimo interpolacijski polinom, ki gre skozi točke

$$D'(e^{j\omega_k}) - \frac{(-1)^k \delta}{W'(e^{j\omega_k})}$$

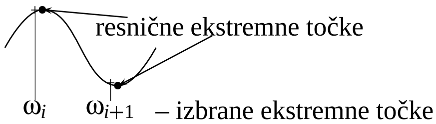
2.4.4 Remezov algoritem

1. Izberemo ekstremalne frekvence $\omega_0, \omega_1, \ldots, \omega_r$. Najboljša izbira je ekvidistančna.



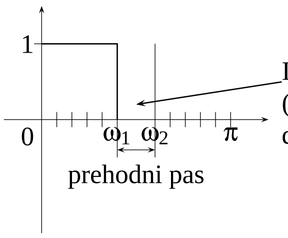
2. Izračunamo δ.

3. Potegnemo interpolacijski polinom skozi δ



4. Poiščemo resnične ekstremalne frekvence $\omega_0', \omega_1', \ldots, \omega_r'$ (iz polinoma).

5. Če so resnične ekstremalne frekvence enake prejšnjim, imamo rešitev; sicer vzamemo $\omega_k = \omega_k'$ in se vrnemo na korak 2.



Izpustimo v optimizacijskem procesu (potek nas ne zanima). Metoda zagotavlja, da bo napaka najmanjša.

$$\Omega = (0, \omega_1) \cup (\omega_2, \pi)$$

Izberemo gosto diskretno množico točk (tipično $10 \cdot r$).

2.4.5 Splošen postopek načrtovanja filtrov

1. Potrebujemo $D(\omega)$ (želeni potek), po potrebi tudi $W(\omega)$ (utežnostna funkcija).

2. Izberemo dolžino filtra N (stopnja polinoma).

3. Izračunamo koeficiente filtra (npr. optimalne).

4. Preverimo napako (razlika med želenim $D(\omega)$ in resničnim $G(\omega)$.

5. Po potrebi popravimo N in ponovimo postopek od 3. koraka naprej.

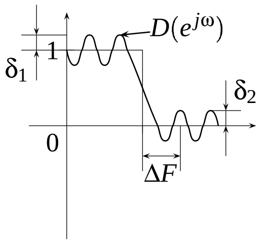
Približna formula za N:

$$D_\infty(\delta_1,\delta_2) = (N-1)\Delta F + f(\delta_1,\delta_2)\Delta F^2$$

$$f(\delta_1,\delta_2) = b_1 + b_2(\log_{10}\delta_1 - \log_{10}\delta_2)$$

$$D_\infty(\delta_1,\delta_2) = (a_1\log_{10}^2\delta_1 + a_2\log_{10}\delta_1 + a_3)\log_{10}\delta_2 + (a_4\log_{10}^2\delta_1 + a_5\log_{10}\delta_1 + a_6)$$

$$\left.\begin{array}{l} a_1,\dots,a_6 \\ b_1,b_2 \end{array}\right\} \text{v knjigi}$$



3 Metode za načrtovanje digitalnih filtrov z neskončnim enotinim odzivom (NEO filtri, *IIR*)

$$H(z) = \frac{\sum_{l=0}^{M} a_l z^{-l}}{1+\sum_{l=1}^{N} b_l z^{-l}}$$

Če niso vsi $b_l = 0 \Rightarrow$ neskončni enotin odziv $h(n)$
Pogoj za linearno fazo: $h(n) = \pm h(N-n)$

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} = \\ &= h(0) + h(1)z^{-1} + \ldots + h(2)z^{-(N-3)} + h(1)z^{(N-2)} + h(0)z^{-(N-1)} = \\ &= z^{-\frac{N-1}{2}} \underbrace{\left(h(0)(z^{\frac{N-1}{2}} + z^{-\frac{N-1}{2}}) + h(1)(z^{\frac{N-3}{2}} + z^{-\frac{N-3}{2}}) + \ldots\right)}_{\text{enako}} \end{aligned}$$

$$H(z^{-1}) = z^{\frac{M-1}{2}} \overbrace{\left(h(0)(z^{\frac{N-1}{2}} + z^{-\frac{N-1}{2}}) + h(1)(z^{\frac{N-3}{2}} + z^{-\frac{N-3}{2}}) + \ldots\right)}^{}$$

$$\begin{array}{ll} H(z^{-1}) = z^{(N-1)}H(z) & \text{pozitivna simetrija} \\ H(z^{-1}) = -z^{(N-1)}H(z) & \text{negativna simetrija} \end{array}$$

Vzemimo, da je $z_i = r_i e^{j\Theta_i}$ pol od $H(z) \Rightarrow z_i^{-1}$ je tudi pol: $\frac{1}{z_i} = \frac{1}{r_i}e^{-j\Theta_i}$.
Če je $r_i < 0 \Rightarrow \frac{1}{r_i} > 1$.
Če imamo pole (razen pri 0), je sistem pri linearni fazi vedno **nestabilen**.
Optimiziramo:

1. Močnostni odziv

$$\left|H(e^{j\omega})\right|^2 = \left|H(z)H(z^{-1})\right| \Big|_{z=e^{j\omega}}$$

2. Fazni odziv, v dveh oblikah:

   (a)

$$H(e^{j\omega}) = \left|H(e^{j\omega}\right| e^{j\beta(e^{j\omega})}$$

$$H(e^{-j\omega}) = \left|H(e^{j\omega})\right| e^{-j\beta(e^{j\omega})}$$

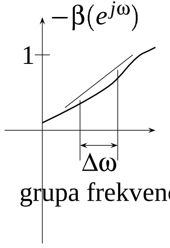
$$\beta(e^{j\omega}) = \frac{1}{2j}\ln\left(\frac{H(z)}{H(z^{-1})}\right) \qquad \text{– fazni odziv pri } z = e^{j\omega}$$

   (b) grupna zakasnitev

$$\tau_g(e^{j\omega}) = -\frac{d\beta(e^{j\omega})}{d\omega} = -jz\frac{d\beta}{dz}\bigg|_{z=e^{j\omega}}$$

   slednje pride iz

$$\frac{d\beta(e^{j\omega})}{d\omega} = \frac{d\beta(e^{j\omega})}{d(e^{j\omega})}\cdot j\cdot e^{j\omega}$$



$$\tau_g(e^{j\omega}) = -\,\mathrm{Re}\left(z\frac{\frac{dH(z)}{dz}}{H(z)}\right)_{z=e^{j\omega}}$$

1. močnostni odziv – najbolj pogost (90%)
2. močnostni odziv $+\beta$
3. močnostni odziv $+\tau_g$

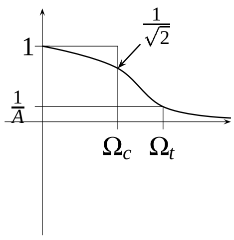
3.1 Načrtovanje z uporabo preslikave analognih filtrov

Tri najpogostejše vrste analognih filtrov:

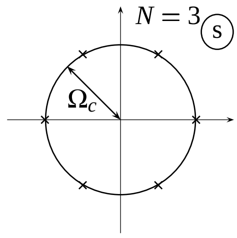
1. Butterworthovi filtri

$$|H_a(s)|^2 = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$$N = \frac{\log_{10}(A^2-1)}{2\log_{10}\frac{\Omega_t}{\Omega_c}}$$



- ima samo pole (nima ničel)
- monotono padajoča funkcija
- poli so na krožnici s polmerom Ω_c



$$H(s) = \frac{\sum_{l=0}^{M} a_l s^{-l}}{1 + \sum_{l=1}^{N} b_l s^{-l}} \qquad \text{– splošen analogen filter, } s = \text{Laplaceov operator}$$

2. Čebiševi filtri

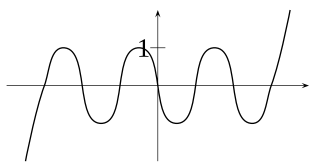
Tip 1

$$|H_a(s)|^2 = \frac{1}{1 + \varepsilon^2 T_N\left(\frac{s}{j\Omega_c}\right)}$$

$$T_N(x) = \begin{cases} \cos(N \cdot \arccos x) & \text{za } |x| \leq 1 \\ \cosh(N \cdot \mathrm{Arch}\, x) & \text{za } |x| > 1 \end{cases}$$

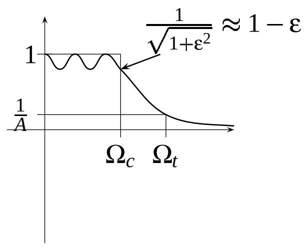
Čebiševi polinomi:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ &\dots \\ T_{N+1}(x) &= 2xT_N(x) - T_{N-1}(x) \end{aligned}$$



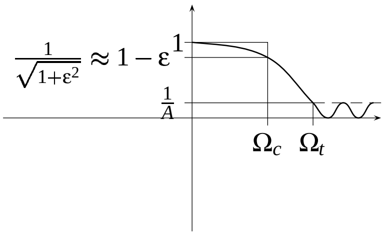
$$g = \sqrt{\frac{A^2 - 1}{\varepsilon^2}}$$

$$N = \frac{\log_{10}\left(g + \sqrt{g^2 - 1}\right)}{\log_{10}\left(\frac{\Omega_t}{\Omega_c} + \sqrt{\left(\frac{\Omega_t}{\Omega_c}\right)^2 - 1}\right)}$$



Tip 2 (inverzni Čebišev filter)

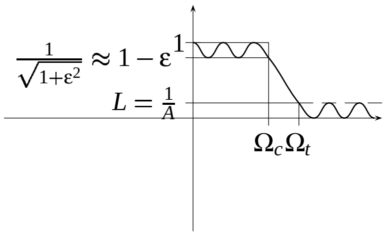
$$|H_a(s)|^2 = \frac{1}{1 + \varepsilon^2 \left(\frac{T_N\left(\frac{\Omega_T}{\Omega_c}\right)}{T_N\left(\frac{j\Omega_T}{s}\right)}\right)^2}$$



3. Cauerjevi ali eliptični filtri

$$|H_a(s)|^2 = \frac{1}{1+\varepsilon^2 R_N^2\left(\frac{s}{j\Omega_c}, L\right)}$$

$R_N =$ Jacobijeva eliptična funkcija 1. vrste

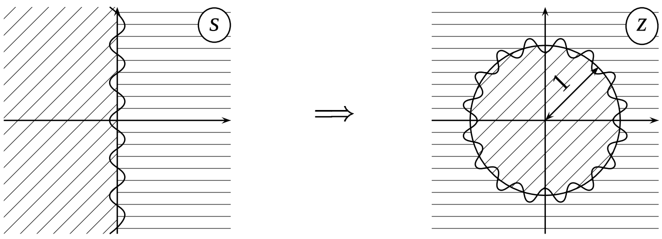


Ta filter je vedno boljši od Čebiševih filtrov in veliko boljši od Butterworthovih filtrov.

3.2 Preslikava z bilinearno transformacijo

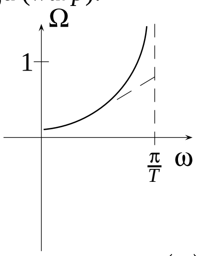
$$s = \frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}, \qquad z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$$

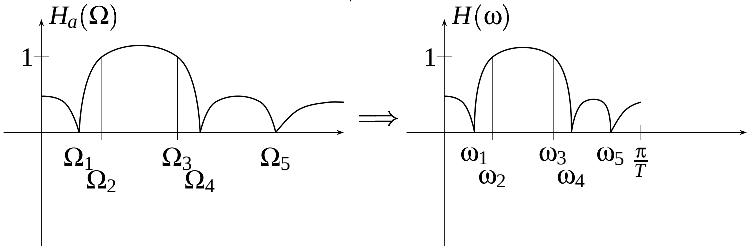
$$H(z) = H(s)\Big|_{s=\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}}$$



Transformacija zagotavlja (ohranja) stabilnost.
Problem: pride do izkrivljenja (*warp*):

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$





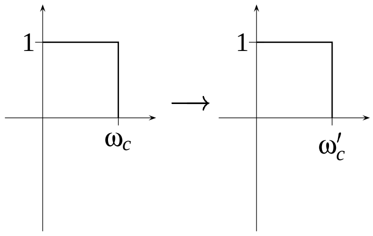
Rešitev: predizkrivljanje (*pre-warping*)
S pomočjo $\Omega = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$ preslikamo $\omega_1, \omega_2, \ldots$ v $\Omega_1, \Omega_2, \ldots$
To je uspešno samo, če je želen potek odsekoma konstantna funkcija.

3.3 Filtrske transformacije

$$H(z') = H(z)\Big|_{z=f(z')}$$

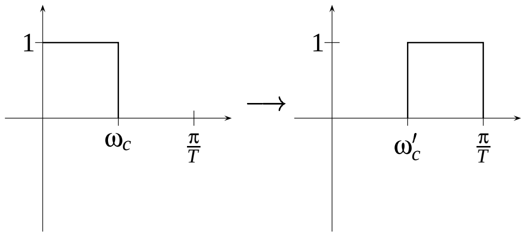
$f(z')$ je filtrska transformacija (Constantinidesove transformacije).

1. LP $\rightarrow$ LP



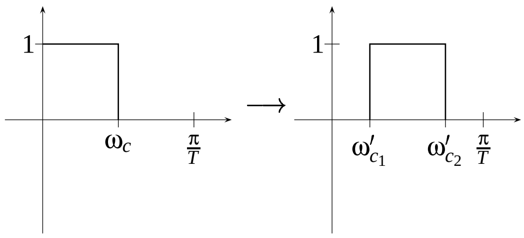
$$z = \frac{z' - \alpha}{1 - \alpha z'}, \quad \alpha = \frac{\sin\left(\frac{\omega_c - \omega'_c}{2} \cdot T\right)}{\sin\left(\frac{\omega_c + \omega'_c}{2} \cdot T\right)}$$

2. LP $\rightarrow$ HP



$$z = -\frac{z' - \alpha}{1 - \alpha z'}, \quad \alpha = \frac{\cos\left(\frac{\omega_c - \omega'_c}{2} \cdot T\right)}{\cos\left(\frac{\omega_c + \omega'_c}{2} \cdot T\right)}$$

3. LP $\rightarrow$ BP



$$z = -\frac{z'^2 - \frac{2\alpha k}{k+1} + \frac{k-1}{k+1}}{1 - \frac{2\alpha k}{k+1} z' + \frac{k-1}{k+1} z'^2}$$

$$\alpha = \frac{\cos\left(\frac{\omega'_{c_2} + \omega'_{c_1}}{2} \cdot T\right)}{\cos\left(\frac{\omega'_{c_2} - \omega'_{c_1}}{2} \cdot T\right)}$$

$$k = \tan\left(\frac{\omega_c T}{2}\right) \cdot \cot\left(\frac{\omega'_{c_2} - \omega'_{c_1}}{2} \cdot T\right)$$

3.4 Načrtovanje s Padejevo aproksimacijo

- aproksimacija racionalne funkcije s potenčno funkcijo

$$H(z) = \frac{\sum_{l=0}^{M} a_l z^{-l}}{1 + \sum_{l=1}^{N} b_l z^{-l}} = \sum_{k=0}^{\infty} h(k) z^{-k}$$

- iz želenega frekvenčnega poteka $H_d(e^{j\omega})$ lahko vedno izračunamo $h_d(k)$
-

$$\begin{aligned} P = N + M + 1, \quad & h(k) = h_d(k) \text{ za } k = 0, 1, \dots, P-1 \quad \leftarrow P \text{ enačb} \\ & h(k) \neq h_d(k) \text{ za } k \geq P \end{aligned}$$

$$y(n) = \sum_{l=0}^{M} a_l x(n-l) - \sum_{l=1}^{N} b_l y(n-l)$$

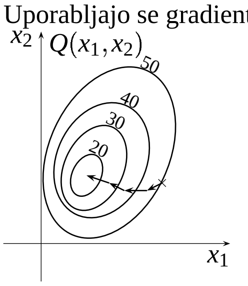
$$\begin{aligned} h(0) &= a_0 \\ h(1) &= -b_1 h(0) + a_1 \\ h(2) &= -b_1 h(1) - b_2 h(0) + a_2 \\ &\vdots \\ h(M) &= -b_1 h(M-1) - b_2 h(M-2) - \ldots - b_N h(M-N) + a_M \\ h(M+1) &= -b_1 h(M) - b_2 h(M-1) - \ldots - b_N h(M+1-N) \\ &\vdots \\ h(M+N+1) &= -b_1 h(M+N) - b_2 h(M+N-1) - \ldots - b_N h(M+1) \end{aligned}$$

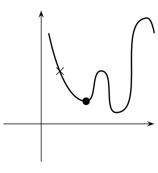
$$\left.\begin{array}{l} N \text{ neznank } b_1, b_2 \ldots, b_N \\ M \text{ enačb} \end{array}\right\} \text{izračunamo } b_l$$

nato izračunamo še a_l

3.5 Načrtovanje z uporabo nelinerane optimizacije

Uporabljajo se gradientne metode; primer: Fletcher-Powell.





$$\text{grad}\, Q = \left(\frac{\partial Q}{x_1}, \frac{\partial Q}{x_2} \right)$$

$$H(z) = A \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}}$$

$$Q(\vec{\Theta}) = \sum_{l=1}^{M} \left(\left|H_d(e^{j\omega_l})\right| - \left|H(e^{j\omega_l})\right|\right)^{2p}$$

$$p \to \infty: \quad \max_{l=1,2,\ldots,M} \left|\left|H_d(e^{j\omega_l})\right| - \left|H(e^{j\omega_l})\right|\right| \qquad \leftarrow L_\infty \text{ norma}$$

Padejeva aproksimacija da začetno točko.

3.6 Načrtovanje z uporabo lineranega programiranja (*simplex* algoritem)

$$H(z) = \frac{\sum_{l=0}^{M} a_l z^{-l}}{1 + \sum_{l=1}^{N} b_l z^{-l}} = \frac{\sum_{l=0}^{M} a_l z^{-l}}{\sum_{l=0}^{N} b_l z^{-l}}, \quad b_0 = 1$$

$$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{c_0 + 2\sum_{l=1}^{M} c_l \cos\omega l}{d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l}$$

$$c_l = \sum_{k=0}^{M-l} a_k a_{k+l}, \qquad d_l = \sum_{k=0}^{M-l} b_k b_{k+l}$$

pogoj:

$$\left|\left|H(e^{j\omega})\right|^2 - \left|H_d(e^{j\omega})\right|^2\right| < \delta \qquad \Rightarrow$$

problem: minimiziraj δ

pogoja:

$$\left|\left|H(e^{j\omega})\right|^2 - \left|H_d(e^{j\omega})\right|^2\right| - \delta \le 0$$
$$-\left|\left|H(e^{j\omega})\right|^2 + \left|H_d(e^{j\omega})\right|^2\right| - \delta \le 0$$

problem: minimiziraj δ

$$\frac{c_0 + 2\sum_{l=1}^{M} c_l \cos\omega l}{d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l} - \left|H_d(e^{j\omega})\right|^2 - \delta \le 0$$

ne gre, ker se pojavijo produkti $\delta \cdot d$

Drugače: δ spremenimo v konstanto, vpeljemo ε in prevedemo na linearni problem.
problem: minimiziraj ε
pogoj:

$$\left|\frac{c_0 + 2\sum_{l=1}^{M} c_l \cos\omega l}{d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l} - \left|H_d(e^{j\omega})\right|^2\right| \le \delta - \frac{\varepsilon}{d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l}$$

Izberemo δ; če je $\varepsilon > 0$, potem rešitev za dani δ obstaja. Ponavljamo, dokler ni $\varepsilon \approx 0$, kjer imamo optimalno rešitev.
Minimiziramo ε pri pogojih:

$$\varepsilon + c_0 + 2\sum_{l=1}^{M} c_l \cos\omega l - \left(d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l\right)\left(\left|H_d(e^{j\omega})\right|^2 + \delta\right) \le 0$$

$$\varepsilon - c_0 - 2\sum_{l=1}^{M} c_l \cos\omega l + \left(d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l\right)\left(\left|H_d(e^{j\omega})\right|^2 - \delta\right) \le 0$$

$$\left.\begin{array}{l} -c_0 - 2\sum_{l=1}^{M} c_l \cos\omega l \le 0 \\ -d_0 - 2\sum_{l=1}^{N} d_l \cos\omega l \le 0 \end{array}\right\} \text{izhaja iz } \left|H(e^{j\omega})\right|^2 = \frac{c_0 + 2\sum_{l=1}^{M} c_l \cos\omega l}{d_0 + 2\sum_{l=1}^{N} d_l \cos\omega l}$$

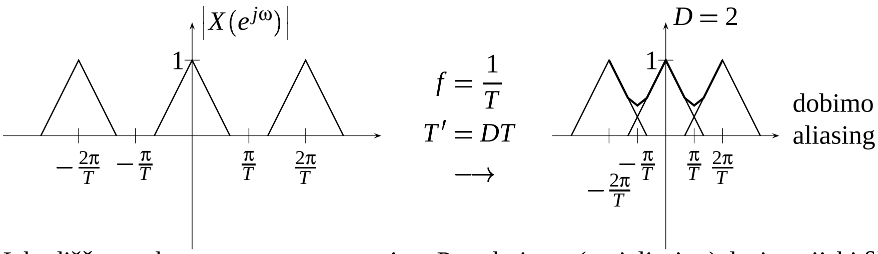
4 Decimacija in interpolacija

- decimacija (redčenje) je zmanjševanje vzorčevalne frekvence
- interpolacija je povečevanje vzorčevalne frekvence

4.1 Decimacija

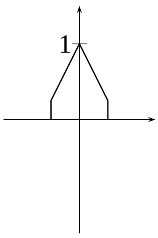
$x(n)$ – signal; frekvenco vzorčenja želimo zmanjšati za faktor D, $D \in \mathbb{N}$.

$$v(n) = x(Dn)$$

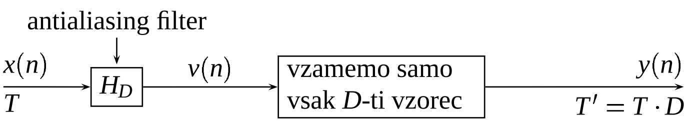


Izhodišče: spekter ostane nespremenjen. Potrebujemo (antialiasing) decimacijski filter:

$$H_D(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{DT} \\ 0, & \text{drugod} \end{cases}$$



Recept:



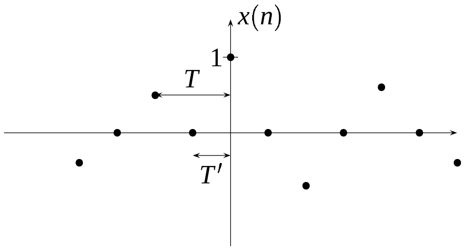
$$y(m) = v(mD) = \sum_{k=0}^{\infty} h_D(k)x(mD-k)$$

$$v(n) = \sum_{k=0}^{\infty} h_D(k)x(n-k)$$

4.2 Interpolacija

$x(n)$ – signal; frekvenco vzorčenja želimo povečati za faktor I, $I \in \mathbb{N}$.

$$f = \frac{1}{T}, \quad T' = \frac{T}{I}$$



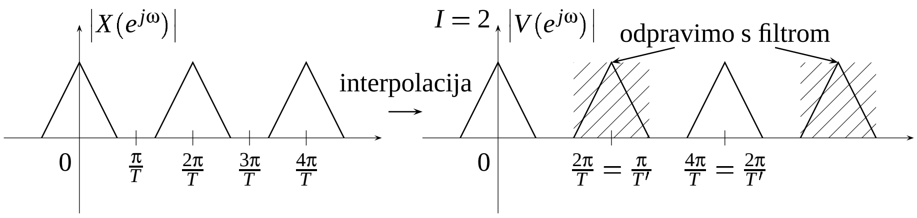
$$v(n) = \begin{cases} x\left(\frac{n}{I}\right) & \text{za } n = 0, \pm I, \pm 2I, \dots \\ 0 & \text{drugod} \end{cases}$$

Izhodišče: spekter ostane nespremenjen, signal mora biti tak, kot bi bil, če bi prvotni analogni signal vzorčili s T'.

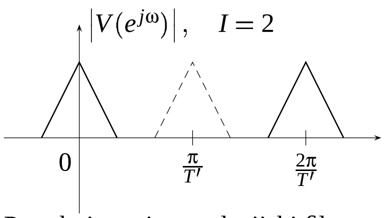
$$V(z) = \sum_{n=-\infty}^{+\infty} v(n) z^{-n} = \sum_{n=-\infty}^{+\infty} x(n) z^{-nI} = X(z^I)$$

$$T' = \frac{T}{I}$$

$$V(e^{j\omega}) = X(e^{j\omega I}) \Rightarrow V(e^{j\omega T'}) = X(e^{j\omega T' \cdot I}) = X(e^{j\omega T})$$

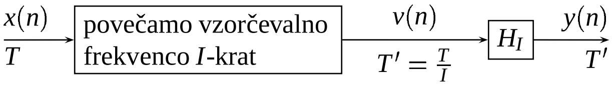


Če takoj vzorčimo analogni signal z novo frekvenco (brez interpolacije), dobimo:



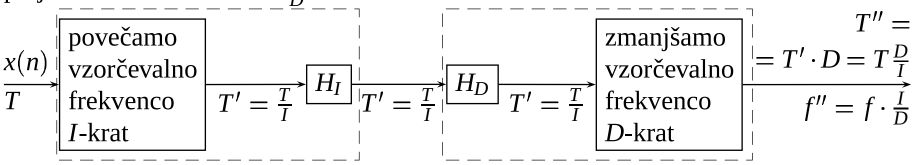
Potrebujemo interpolacijski filter:

$$H_I(e^{j\omega}) = \begin{cases} I, & |\omega| \le \frac{\pi}{IT'} \\ 0, & \text{drugod} \end{cases}$$



$$y(n) = \sum_{k=0}^{\infty} h_I(k) v(n-k) = \sum_{k=-\infty}^{+\infty} h_I(n-k) v(k) = \sum_{k=-\infty}^{+\infty} h_I(n-kI) x(k)$$

Boljše je vzeti KEO filter, ker zmanjšamo število računskih operacij.
S kombinacijo decimacije in interpolacije lahko spremenimo vzorčevalno frekvenco za poljuben racionalni faktor $\frac{I}{D}$:



H_I in H_D združimo v en filter:

$$H = \begin{cases} I, & |\omega| \leq \left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0, & \text{drugod} \end{cases}$$

5 FFT algoritem in njegova uporaba

- Leta 1965 FFT ponovno odkrijeta Cooley in Tukey.
- FFT (*Fast Fourier Transform*) je algoritem za računanje DFT (diskretna Fourierjeva transformacija).

$$x_p(k+mN) = x_p(k), \quad m = 0, \pm 1, \pm 2, \ldots$$

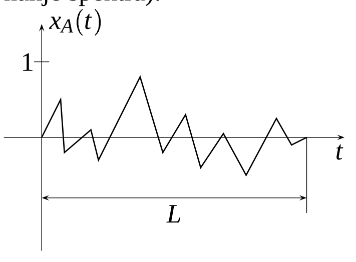
$$X_p(k) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x_p(n) e^{-2\pi jk\frac{n}{N}}, \quad k = 0, 1, \ldots, N-1$$

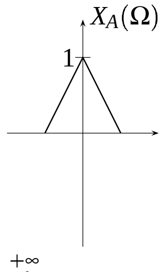
$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_p(k) e^{2\pi jk\frac{n}{N}}, \quad n = 0, 1, \ldots, N-1$$

$$\text{primitivni } N\text{-ti koren od 1: } w_N = e^{-\frac{2\pi j}{N}}, \quad w_N^r \neq 1 \text{ za } 0 < r < N$$

$$\sum_{k=0}^{N-1} w_N^{km} w_N^{-kl} = \begin{cases} N, & n - l = 0 \pmod N \\ 0, & \text{drugod} \end{cases}$$

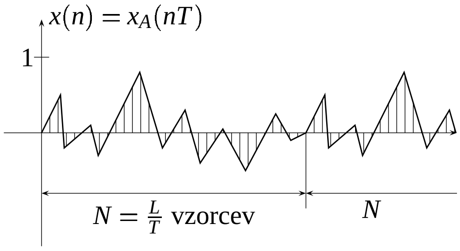
Zahtevnost DFT: N^2 operacij.
Zveza med DFT in Fourierovim integralom (oz. kdaj smemo DFT uporabiti za računanje spektra):

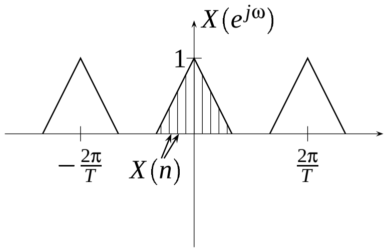




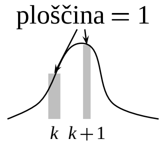
$$X_A(\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t}\, dt$$

Predpostavka: ni aliasinga.





$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$



$$X_A(\Omega) = \int_0^L x_A(t) e^{-j\Omega t}\, dt = \sum_{k=0}^{N-1} \int_{kT}^{(k+1)T} x_A(t) e^{-j\Omega t}\, dt = \sum_{k=0}^{N-1} x_A(kT) e^{-j\Omega kT}$$

$$\text{DFT: } X(n) = \sum_{k=0}^{N-1} x(k) e^{-2\pi jk\frac{n}{N}}, \quad x(k) = x_A(kT)$$

$$\boxed{X(n) = X_A(\Omega)\Big|_{\Omega = \frac{2\pi}{T}\cdot\frac{n}{N}}}$$

FFT algoritmi: $N^2 \to N \log N$, dobro se dajo paralelizirati

- decimacija v času
- decimacija v frekvenci
- radix 2, 4, 8 in praštevilski ($N = 2^n, 4^n, 8^n, p$)
- Winogradov FFT algoritem – najboljši možen

5.1 Radix 2 algoritem, decimacija v času

$N = 2^n$, razdelimo na sode ter lihe indekse, upoštevajoč $w_N^2 = w_{N/2}$:

$$\begin{aligned} X(l) &= \sum_{r=0}^{\frac{N}{2}-1} x(2r) w_N^{2rl} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) w_N^{(2r+1)l} = \\ &= \sum_{r=0}^{\frac{N}{2}-1} x(2r) w_{N/2}^{rl} + w_N^l \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) w_{N/2}^{rl} \end{aligned}$$

Dobimo par enačb, imenovan metuljček:

$$X(l) = G(l) + w_N^l F(l), \quad l = 0, 1, \ldots, \frac{N}{2} - 1$$
$$X\left(l + \frac{N}{2}\right) = G(l) - w_N^l F(l), \quad l = 0, 1, \ldots, \frac{N}{2} - 1$$

$F(L)$, $G(l)$ – DFT dolžine $\frac{N}{2}$; število operacij: $2\left(\frac{N}{2}\right)^2 + \frac{N}{2}$

$$G(l) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) w_{N/2}^{rl} = \underbrace{\sum_{r=0}^{\frac{N}{4}-1} x(4r) w_{N/4}^{rl}}_{A(l)} + w_{N/2}^l \underbrace{\sum_{r=0}^{\frac{N}{4}-1} x(4r+2) w_{N/4}^{rl}}_{B(l)} = A(l) + w_{N/2}^l B(l)$$

$$G(l) = A(l) + w_{N/2}^l B(l), \quad l = 0, 1, \ldots, \frac{N}{4} - 1$$
$$G\left(l + \frac{N}{4}\right) = A(l) - w_{N/2}^l B(l), \quad l = 0, 1, \ldots, \frac{N}{4} - 1$$

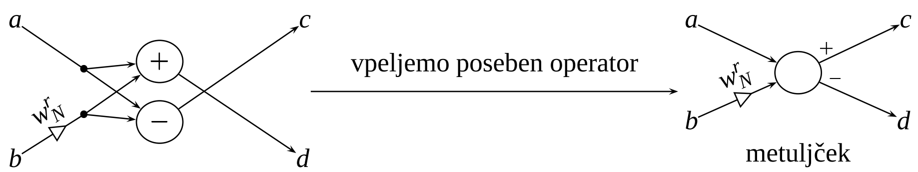
$$F(l) = \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) w_{N/2}^{rl} = \sum_{r=0}^{\frac{N}{4}-1} x(4r+1) w_{N/4}^{rl} + w_{N/2}^l \sum_{r=0}^{\frac{N}{4}-1} x(4r+3) w_{N/4}^{rl} =$$
$$= C(l) + w_{N/2}^l D(l)$$

$$F(l) = C(l) + w_{N/2}^l D(l), \quad l = 0, 1, \ldots, \frac{N}{4} - 1$$
$$F\left(l + \frac{N}{4}\right) = C(l) - w_{N/2}^l D(l), \quad l = 0, 1, \ldots, \frac{N}{4} - 1$$

Število operacij: $2\left(2 \cdot \left(\frac{N}{4}\right)^2 + \frac{N}{4}\right) + \frac{N}{2} = 4 \cdot \left(\frac{N}{4}\right)^2 + \frac{N}{2} + \frac{N}{2}$.
Splošna formula za število operacij: $\frac{N}{2} \log_2 N$.
Od kod ime metuljček:
metuljček je par enačb

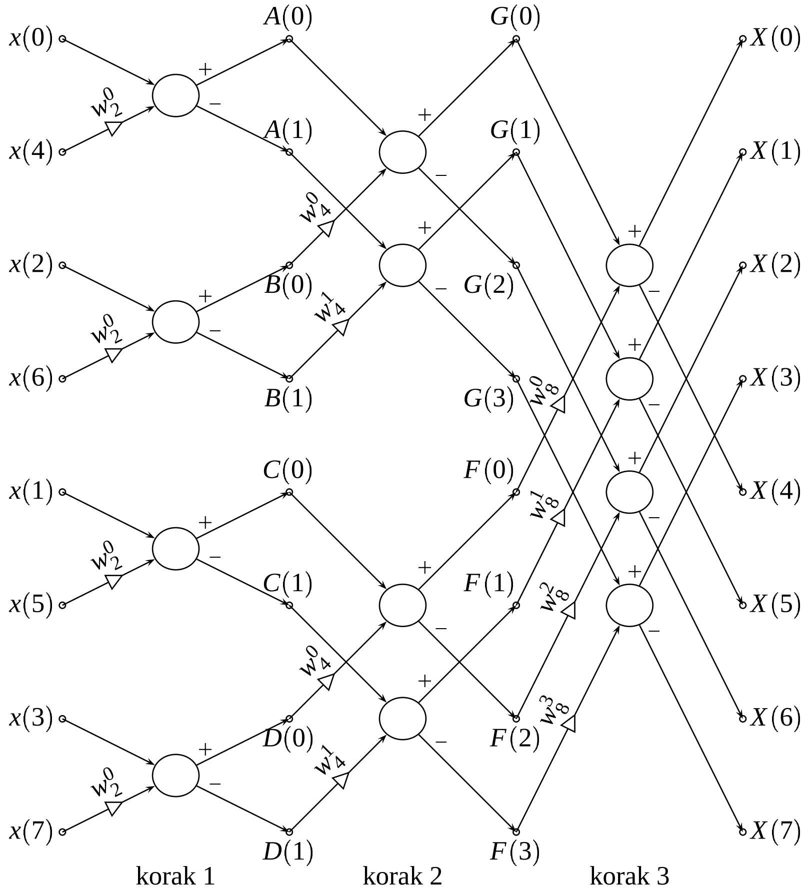
$$c = a + w_N^r b$$
$$d = a - w_N^r b$$

vhodni podatki: a, b



Uporablja se bit-reverzno mešanje indeksov (velja samo za radix-2 algoritem): index $x(i)$; $i = b_{n-1}b_{n-2}\ldots b_1 b_0 \to i = b_0 b_1 \ldots b_{n-2} b_{n-1}$

Primer za $N = 8 = 2^3$:



n korakov, v vsakem koraku $\frac{N}{2}$ metuljčkov.

5.2 Radix 4 algoritem

Približno 30% prihranek pri prehodu iz radix 2 na radix 4 algoritem.
$N = 4^n$, razdelimo člene glede na ostanek pri deljenju s 4:

$$\begin{aligned} X(l) &= \sum_{r=0}^{\frac{N}{4}-1} x(4r) w_{N/4}^{rl} + w_N^l \sum_{r=0}^{\frac{N}{4}-1} x(4r+1) w_{N/4}^{rl} + \\ &+ w_N^{2l} \sum_{r=0}^{\frac{N}{4}-1} x(4r+2) w_{N/4}^{rl} + w_N^{3l} \sum_{r=0}^{\frac{N}{4}-1} x(4r+3) w_{N/4}^{rl} = \\ &= A(l) + w_N^l B(l) + w_N^{2l} C(l) + w_N^{3l} D(l) \end{aligned}$$

Upoštevamo zveze:

$$w_N^{N/4} = -j, \quad w_N^{N/2} = -1, \quad w_N^{3N/4} = j$$

$$
\begin{aligned}
X(l) &= A(l) + w_N^l B(l) + w_N^{2l} C(l) + w_N^{3l} D(l)\\
X\left(l + \frac{N}{4}\right) &= A(l) - j w_N^l B(l) - w_N^{2l} C(l) + j w_N^{3l} D(l)\\
X\left(l + \frac{N}{2}\right) &= A(l) - w_N^l B(l) + w_N^{2l} C(l) - w_N^{3l} D(l)\\
X\left(l + \frac{3N}{4}\right) &= A(l) + j w_N^l B(l) - w_N^{2l} C(l) - j w_N^{3l} D(l)
\end{aligned}
$$

Zahtevnost algoritma: $3 \cdot \frac{N}{4} \log_4 N = \frac{3}{8} N \log_2 N$

5.3 Računanje inverzne DFT

$$x(k) = \frac{1}{N} \sum_{l=0}^{N-1} X(l) w_N^{-lk}$$

$$N x^*(k) = \sum_{l=0}^{N-1} X^*(l) w_N^{lk}$$

Trik:

$$x(k) = \frac{1}{N} \left(\sum_{l=0}^{N-1} X^*(l) w_N lk \right)^*$$

5.4 Računanje DFT za realna zaporedja

$x(k)$ – realno zaporedje dolžine $2N$; tvorimo kompleksno zaporedje dolžine N:

$$y(k) = x(2k) + jx(2k+1) = \underbrace{y_1(k)}_{\text{sodi}} + j \underbrace{y_2(k)}_{\text{lihi}}, \quad k = 0, 1, \dots, N-1$$

$$Y(l) = \sum_{k=0}^{N-1} y(k) w_N^{kl} = \sum_{k=0}^{N-1} \left(y_1(k) + j y_2(k)\right) w_N^{kl} = Y_1(l) + j Y_2(l)$$

$$Y^*(N-l) = \sum_{k=0}^{N-1} y^*(k) w_N^{-(N-l)k} = \sum_{k=0}^{N-1} \left(y_1(k) - j y_2(k)\right) w_N^{kl} = Y_1(l) + j Y_2(l)$$

$$Y_1(l) = \frac{1}{2} \left(Y^*(N-l) + Y(l)\right)$$

$$Y_2(l) = \frac{j}{2} \left(Y^*(N-l) - Y(l)\right)$$

DFT od realnega zaporedja $x(k)$ dolžine $2N$:

$$X(l) = \sum_{k=0}^{2N-1} x(k) w_{2N}^{kl} = \sum_{k=0}^{N-1} x(2k) w_{2N}^{2lk} + \sum_{k=0}^{N-1} x(2k+1) w_{2N}^{2k+1} =$$

velja: $w_{2N}^2 = w_N$

$$= \sum_{k=0}^{N-1} x(2k) w_N^{kl} + w_{2N}^l \sum_{k=0}^{N-1} x(2k+1) w_N^{kl} = Y_1(l) + w_{2N}^l Y_2(l)$$

$$X(l) = \frac{1}{2}\left(Y^*(N-l) + Y(l) + jw_{2N}^l\,(Y^*(N-l) - Y(l))\right), \quad l = 0,1,\dots,N-1$$

$$X(l+N) = \frac{1}{2}\left(Y^*(N-l) + Y(l) - jw_N^l\,(Y^*(N-L) - Y(l))\right), \quad l = 0,1,\dots,N-1$$

Prihranek: namesto DFT dolžine $2N$ potrebujemo DFT dolžine N ter N množenj. Za kompleksno zaporedje dolžine $2N$:

$$\frac{2N}{2}\log_2(2N) = N\,(1 + \log_2 N)\,,$$

z upoštevanjem realnosti:

$$\frac{N}{2}\log_2 N + N$$

5.5 Hitro računanje konvolucije

5.5.1 Periodična zaporedja

Dani sta zaporedji x_p, h_p

$$y_p(n) \stackrel{\text{def}}{=} \sum_{l=0}^{N-1} x_p(l)h_p(n-l) = \sum_{l=0}^{N-1} h_p(l)x_p(n-l)$$

Dokaz za zadnjo enačbo izpelji sam.

$$Y_p(l) = H_p(l)X_p(l), \quad l = 0,1,\dots,N-1$$

$$H_p(l) = \sum_{k=0}^{N-1} h_p(k)w_N^{kl}$$

$$X_p(l) = \sum_{k=0}^{N-1} x_p(k)w_N^{kl}$$

DFT konvolucije = produkt DFT-jev

Hitro računanje konvolucije:

1. izračunamo DFT od x_p in h_p: $2 \times \frac{N}{2}\log_2 N$ množenj
2. pomnožimo transforma $H_p \cdot X_p$: N množenj
3. izračunamo inverzno DFT od H_pX_p: $\frac{N}{2}\log_2 N$ množenj

Zahtevnost:

$$3 \cdot \frac{N}{2}\log_2 N + N$$

5.5.2 Neperiodična zaporedja

$x(n)$ je zaporedje dolžine N_1: $x(n) = 0$ za $n \begin{cases} \geq N_1 \\ < 0 \end{cases}$

$h(n)$ je zaporedje dolžine N_2: $h(n) = 0$ za $n \begin{cases} \geq N_2 \\ < 0 \end{cases}$

$$y(n) = \sum_{l=0}^{\infty} h(l)x(n-l) = \sum_{l=0}^{\infty} h(l)x(n-l)$$

$y(n)$ je zaporedje dolžine $N_1 + N_2 - 1$: $y(n) = 0$ za $n \begin{cases} n \geq N_1 + N_2 - 1 \\ < 0 \end{cases}$

Pri periodični konvoluciji imajo vsa tri zaporedja (x, h, y) enako periodo.

$$x_p(n) = \begin{cases} x(n) & \text{za } n = 0, 1, \ldots, N_1 - 1 \\ 0 & \text{za } n = N_1, N_1 + 1, \ldots, N_1 + N_2 - 2 \end{cases} \qquad \begin{array}{l} \text{periodično zaporedje} \\ \text{s periodo } N_1 + N_2 - 1 \end{array}$$

$$h_p(n) = \begin{cases} h(n) & \text{za } n = 0, 1, \ldots, N_2 - 1 \\ 0 & \text{za } n = N_2, N_2 + 1, \ldots, N_2 + N_1 - 2 \end{cases} \qquad \begin{array}{l} \text{periodično zaporedje} \\ \text{s periodo } N_1 + N_2 + 1 \end{array}$$

$$y_p(n) = \sum_{l=0}^{N_1+N_2-2} h_p(l)x_p(n-l) = \sum_{l=0}^{n} h(l)x(n-l) = y(n), \quad n = 0, 1, \ldots, N_1 + N_2 - 2$$

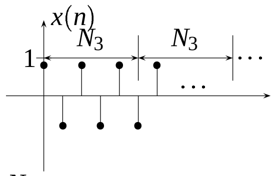
Algoritem:

- izberemo število L ($L = 2^n$) tako, da je $L \geq N_1 + N_2 - 1$
- če je $x(n)$ neskončno dolgo zaporedje, potem uporabimo trik: hitro konvolucijo po odsekih

5.5.3 Hitra konvolucija po odsekih

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \sum_{l=0}^{\infty} h(l)x(n-l) = \sum_{l=0}^{N_2-1} h(l)x(n-l)$$

$h(n)$ – dolžina N_2
$x(n)$ – dolžina ∞



trik: izberemo število N_3

$$x(n) = \sum_{k=0}^{\infty} x_k(n), \quad x_k(n) \stackrel{\text{def}}{=} \begin{cases} x(n) & kN_3 \leq n \leq (k+1)N_3 - 1 \\ 0 & \text{drugod} \end{cases}$$

$$y(n) = \sum_{l=0}^{\infty} h(l)x(n-l) = \sum_{l=0}^{n} h(l)x(n-l) = \sum_{l=0}^{n} h(l) \sum_{k=0}^{infty} x_k(n-l) =$$

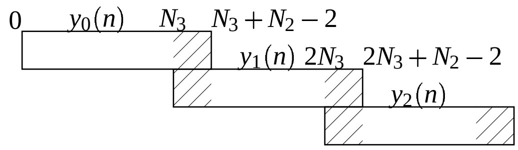
$$= \sum_{k=0}^{\infty} \underbrace{\sum_{l=0}^{n} h(l)x_k(n-l)}_{y_k(n)} = \sum_{k=0}^{\infty} y_k(n)$$

odsek konvolucije

$$y_k(n) = \sum_{l=0}^{n} \underbrace{h(l)}_{N_2} \underbrace{x_k(n-l)}_{N_3}, \quad N_2 + N_3 - 1$$

$$y_0(n) \neq 0 \text{za } n = 0, 1, \ldots, N_3 + n_2 - 2$$
$$y_1(n) \neq 0 \text{za } n = N_3, N_3 + 1, \ldots, 2N_3 + N_2 - 2$$
$$y_2(n) \neq 0 \text{za } n = 2N_3, 2N_3 + 1, \ldots, 3N_3 + N_2 - 2$$

prekrivanje pri indeksih $N_3, N_3 + 1, \ldots, N_3 + N_2 - 2$
prekrivanje pri indeksih $2N_3, 2N_3 + 1, \ldots, 2N_3 + N_2 - 2$



Algoritem `OVERLAP-ADD`:

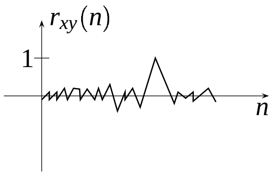
1. Izberemo N_3 ($L = N_3 + N_2 - 1 = 2^n$)
2. Tvorimo $h_p(n)$ dolžine L (dodamo $N_3 - 1$ ničel)
3. Izračunamo DFT od $h_p(n) \to H_p(l)$
4. $k = 0$
5. Tvorimo $x_{kp}(n)$ dolžine L (dodamo $N_2 - 1$ ničel k $x_k(n)$)
6. Izračunamo DFT od $x_{kp}(n) \to X_{kp}(l)$
7. $Y_{kp}(l) = H_p \cdot X_{kp}(l)$, $l = 0, 1, \ldots, L-1$
8. Izračunamo inverzno DFT – dobimo $y_k(n)$ za $n = kN_3, kN_3 + 1, \ldots, kN_3 + N_2 - 1$
9. Tvorimo $y(n)$:
   (a) $k = 0$: $y(n) = y_k(n)$ za $n = 0, 1, \ldots, N_3 + N_2 - 2$
   (b)
   $$k \geq 1: y(n) = y_k(n) + y(n) \qquad \text{za } n = kN_3, kN_3 + 1, \ldots, kN_3 + N_2 - 2$$
   $$y(n) = y_k(n) \qquad \text{za } n = kN_3 + N_2 - 1, kN_3 + N_2, \ldots, (k+1)N_3 + N_2 - 2$$
10. $k = k + 1$

Večji $L \to$ večja hitrost.

6 Križna korelacija in avtokorelacija

$$r_{xy}(n) \stackrel{\text{def}}{=} \sum_{l=-\infty}^{+\infty} x(l)y(l+n) = \sum_{l=-\infty}^{+\infty} x(l-n)y(l), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$r_{xy}(n) = r_{yx}(-n) \quad \leftarrow \text{ni komutativno!}$$



6.1 Avtokorelacija

$$r_{xx}(n) \stackrel{\text{def}}{=} \sum_{l=-\infty}^{+\infty} x(l)x(l+n)$$

$$r_{xx}(n) = r_{xx}(-n)$$

$$r_{xx}(0) \geq r_{xx}(n), \quad \text{za } n \neq 0$$

6.2 Hitro računanje korelacije

6.2.1 Periodična zaporedja

$x_p(n), y_p(n)$:

$$r_{pxy}(n) \stackrel{\text{def}}{=} \sum_{l=0}^{N-1} x_p(l)y_p(l+n), \quad n = 0, 1, \ldots, N-1$$

$$R_{pxy}(k) = \sum_{n=0}^{N-1} r_{pxy}(n)w_N^{kn} = \sum_{n=0}^{N-1}\sum_{l=0}^{N-1} x_p(l)y_p(l+n)w_N^{kn} =$$
$$= \sum_{l=0}^{N-1} x_p(l)w_N^{-kl} \sum_{n=0}^{N-1} y_p(l+n)w_N^{k(l+n)}$$

$$\sum_{n=0}^{N-1} y_p(\overbrace{l+n}^{=u})w_N^{k(l+n)} = \sum_{n=l}^{N-1+l} y_p(n)w_N^{kn} = \sum_{u=l}^{N-1} y_p(u)w_N^{kn} + \sum_{u=N}^{N-1+l} y_p(u)w_N^{kn} =$$
$$u' = u - N$$
$$= \sum_{u=l}^{N-1} y_p(u)w_N^{kn} + \sum_{u'=0}^{l-1} y_p(u')w_N^{ku'} = \sum_{u=0}^{N-1} y_p(u)w_N^{ku}$$

$$R_{pxy} = \sum_{l=0}^{N-1} x_p(l)w_N^{-kl} \sum_{u=0}^{N-1} y_p(u)w_N^{ku} = X_p(-k)Y_p(k)$$

$$R_{pxy}(k) = X_p(-k)Y_p(k) = X_p(N-k)Y_p(k)$$

Za realna zaporedja: $R_{pxy}(k) = X_p^*(k)Y_p(k)$, ker: $X_p(-k) = X_p^*(k)$ za realna zaporedja $x(n)$.
Hitro računanje:

- izračunamo DFT od x_p, y_p
- izračunamo $R_{pxy}(k) = X_p^*(k)Y_p(k)$
- izračunamo IDFT od $R_{pxy}(k) \rightarrow r_{pxy}(n), \quad n = 0, 1, \ldots, N-1$

6.2.2 Končno dolgi neperiodični zaporedji *x*, *y*

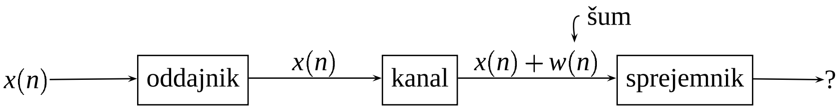
$$\begin{aligned} x(n) &\text{ dolžina } N_1 \\ y(n) &\text{ dolžina } N_2 \\ r_{xy}(n) &\text{ dolžina } N_1 + N_2 - 1 \end{aligned}$$

$$r_{xy}(n) = \sum_{l=-\infty}^{+\infty} x(l)y(l+n) = \sum_{l=0}^{N_2-1-n} x(l)y(l+n)$$

$$r_{xy}(n) = 0 \text{ za } \begin{cases} l+n < 0 & \Rightarrow n < -(N_1-1) \\ l+n > N_2-1 & \Rightarrow n > (N_2-1) \end{cases}$$

$$r_{xy}(n) \neq 0 \text{ v } N_1 - 1 + N_2 - 1 + 1 = N_1 + N_2 - 1 \text{ točkah}$$

6.3 Detekcija signala v šumu

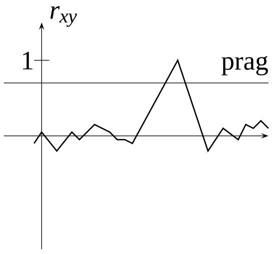


$$y(n) = x(n) + w(n)$$

$$r_{xy}(n) = \frac{1}{K}\sum_{l=0}^{K-1} x(l)y(l+n) = \frac{1}{K}\sum_{l=0}^{K-1} x(l)\left(x(l+n) + w(l+n)\right) = r_{xx}(n) + r_{xw}(n) \approx r_{xx}(n)$$

Ker: $r_{xw}(n) \approx 0$; velja, če $x(n)$ in $w(n)$ nista korelirana (aditivni šum).

$$K \gg n$$



$$r_{yy}(n) = \frac{1}{K}\sum_{l=0}^{K-1} y(l)y(l+n) = \frac{1}{K}\sum_{l=0}^{K-1} (x(l) + w(n))\left(x(l+n) + w(l+n)\right) =$$

$$= r_{xx}(n) + r_{xw}(n) + r_{wx}(n) + r_{ww}(n)$$

vrhovi pri $n = 0, n = N, n = 2N, \ldots$

7 Spektralna analiza

- Določanje močnostnega spektra

$$X_a(\omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{+\infty} |x_a(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_a(\omega)|^2\, d\omega \quad \text{– Parsevalov izrek}$$

- Spekter je Fourierov (harmonični) spekter,
  obstajajo tudi drugi spektri

- Nedeterministični signali: poznamo samo avtokorelacijo r_{xx}, x_a pa ne:

$$r_{xx}(\tau) = \int_{-\infty}^{+\infty} x_a^*(t) x_a(t+\tau)\, dt$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} r_{xx}(\tau) e^{-j\omega t}\, d\tau &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x_a^*(t) x_a(t+\tau)\, dt \right) e^{j\omega\tau}\, d\tau = \\
&= \int_{-\infty}^{+\infty} x_a^*(t) \left(\int_{-\infty}^{+\infty} x_a(t+\tau) e^{-j\omega(t+\tau)}\, d\tau \right) e^{j\omega t}\, dt = \\
&= \int_{-\infty}^{+\infty} x_a^*(t) X_a(\omega) e^{j\omega t}\, dt = X_a^*(\omega) X_a(\omega) = |X_a(\omega)|^2
\end{aligned}$$

Izhodišče: $x_a(t)$ ali $r_{xx}(\tau)$.
Zaradi enostavnosti vzamemo $x_a(t)$.

$$X_a(\omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\omega t}\, dt$$

$X_a(\omega)$ bomo določali na digitalni način.

$$x(n) \stackrel{\text{def}}{=} X_a(nT), \quad n = 0, \pm 1, \pm 2, \ldots$$

Če ni aliasinga (najmanj $2\times$ višja frekvenca vzorčenja od največje frekvence signala)(antialiasing filter):

$$X(\omega T) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega T n} = \frac{q}{T} X_a(\omega), \quad |\omega| \leq \frac{\pi}{T}$$

$$\boxed{X_a(\omega) = T \cdot X(\omega T), \quad |\omega| \leq \frac{\pi}{T}}$$

Spekter lahko izračunamo na digitalen način, če ni aliasinga.

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

Problem: n od $-\infty$ do $+\infty$.
Rešitev: namesto neskončnega signala uporabimo končen del signala.

$$x_w(n) = x(n)w(n), \quad w(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{drugod} \end{cases} \qquad \leftarrow \text{pravokotno okno}$$

$$X_w(\omega) = \frac{1}{2\pi}\int_0^{2\pi} X(\Theta W(\omega - \Theta)\, d\Theta$$

$$W(\omega) = \sum_{n=0}^{N-1} w(n)e^{-j\omega n}$$

Namesto $X(\omega)$ izračunamo $X_w(\omega)$.
Vprašanje: $X_w(\omega) \approx X(\omega)$? $\leftarrow$ Problem okna (*windowing problem*).

7.1 Problem okna

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

okno dolžine N_w

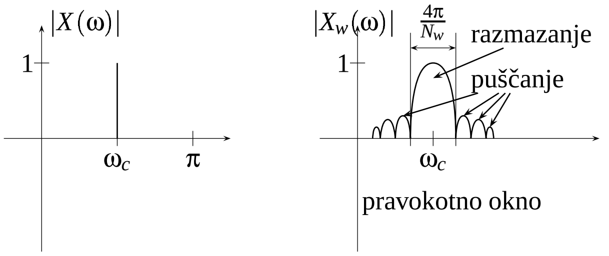
$$x_w(n) = x(n)w(n)$$

$$X_w(\omega) = \frac{1}{2\pi}\int_0^{2\pi} X(\Theta)W(\omega - \Theta)\, d\Theta$$

$$W(\omega) = \sum_{n=0}^{N_w-1} w(n)e^{-j\omega n}$$

$$\text{Pravokotno okno: } |W(\omega)| = \left| \frac{\sin \frac{\omega N_w}{2}}{\sin \frac{\omega}{2}} \right|$$

1. razmazanje (*spreading*)
2. puščanje (*leaking*)



Druga okna: Hann, Hamming, ...: zmanjša se puščanje, ampak se poveča razmazanje.

7.2 Neparametrične metode za določanje spektra

Uporaba Fourierove transformacije

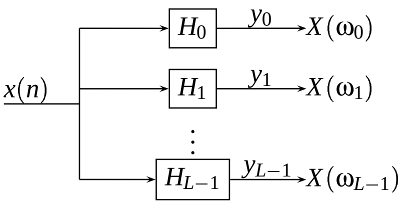
računamo z DFT in FFT algoritmom

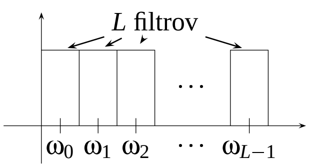
$$X_w(\omega) = \sum_{n=-\infty}^{+\infty} x(n)w(n)e^{-j\omega n} = \sum_{n=0}^{N_w-1} x(n)w(n)e^{-j\omega n}, \quad X_w(k) = \sum_{n=0}^{N_w-1} x(n)w(n)e^{-j\frac{2\pi nk}{N_w}}$$

Predpostavka: $X(\omega) \approx X_w(\omega)$
Pri analognih (zveznih) signalih je še ena predpostavka:

$$X_a(\omega_k) \approx T X(k), \quad \omega_k = \frac{2\pi}{N_w} \cdot \frac{k}{T}, \quad k = 0, 1, \ldots, N_w - 1 \quad \leftarrow \text{velja, če ni aliasinga}$$

Namesto DFT lahko uporabimo množico filtrov:





Kakšna je zveza med na ta način izmerjenim spektrom in tistim z DFT?
DFT:

$$X_w(k) = \sum_{l=0}^{N_w-1} x(l)w(l)e^{-\frac{2\pi jkl}{N_w}}$$

Filtri: $x(n) \longrightarrow \boxed{H_k} \longrightarrow y_k(n)$ (KEO filtri dolžine N_w) primerjava

$$y_k(n) = \sum_{l=0}^{N_w-1} x(l)h_k(n-l)$$

$$n = N_w - 1: \quad y_k(N_w-1) = \sum_{l=0}^{N_w-1} x(l)h_k(N_w-1-l)$$

Da je enako: mora biti $h_k(N_w-1-l) = w(l)e^{-\frac{2\pi jkl}{N_w}}$

$$H_k(\omega) = \sum_{n=0}^{N_w-1} h_k(n)e^{-j\omega n}$$

$$h_k(n) = w(N_w-1-n)e^{\frac{2\pi jk(n+1)}{N_w}}$$

$$H_k(\omega) = \sum_{n=0}^{N_w-1} h_k(n)e^{-j\omega n} = \sum_{n=0}^{N_w-1} w(N_w-1-n)e^{-jn\left(\omega - \frac{2\pi}{N_w}\cdot k\right)}e^{2\pi j\frac{k}{N_w}} =$$

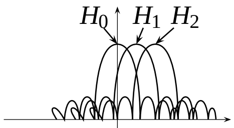
$$= e^{-j(N_w-1)\omega} \sum_{n=0}^{N_w-1} w(n)e^{-jn\left(\frac{2\pi k}{N_w} - \omega\right)}$$

$$W(\omega) = \sum_{n=0}^{N_w-1} w(n) e^{-j\omega n}$$

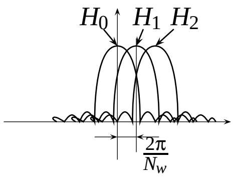
$$\boxed{H_k(\omega) = e^{-j(N_w-1)\omega} W\left(\frac{2\pi k}{N_w} - \omega\right)}$$

Pri realnih oknih $w(n)$: $|W(\omega)| = |W(-\omega)|$

$$\boxed{|H_k(\omega)| = \left|W\left(\omega - \frac{2\pi}{N_w} \cdot k\right)\right|}$$



pravokotno okno



Hannovo okno

Ugotovitev: DFT je množica KEO filtrov, katerih frekvenčni odziv je enak odzivu okna ($W(\omega)$).
razmazanje $\leftarrow$ širina filtra
puščanje $\leftarrow$ slab filter, s prisluhom

7.3 Parametrične metode za določanje spektra

Model procesa, ki generira signal:
$u(n) \longrightarrow \boxed{H(z)} \longrightarrow x(n)$

$$X(\omega) = H(\omega) \cdot U(\omega)$$

- če poznamo paramtre od $H(z) \Rightarrow$ poznamo spekter
- določevanje parametrov $\Leftrightarrow$ določevanje spektra

$$H(z) = \frac{\sum_{l=0}^{q} a_l z^{-l}}{1 + \sum_{l=1}^{p} b_l z^{-l}}$$

$$x(n) = \sum_{l=0}^{q} a_l u(n-l) - \sum_{l=1}^{p} b_l x(n-l)$$

1. pri danem zaporedju $x(n)$, $n = 0, 1, \ldots, N_w - 1$ izračunamo parametre a_l, b_l
2. iz parametrov a_l, b_l izračunamo $H(\omega)$ in naredimo predpostavko $X(\omega) = H(\omega)(\cdot c) \leftarrow$ konstanta

Poseben primer modela je model s samimi poli:

$$H(z) = \frac{a_0}{1 - \sum_{l=1}^{p} b_l z^{-l}}$$

Druga imena za ta model:

1. linearna predikcija
2. avtoregresivni model

Ta model ustreza enačbi

$$x(n) = \underbrace{\sum_{l=1}^{p} b_l x(n-l)}_{\tilde{x}(n)} + a_0 u(n)$$

$$\tilde{x}(n) = \sum_{l=1}^{p} b_l x(n-l)$$

linearna predikcija ali avtoregresijska metoda

$$x(n) = \tilde{x}(n) + \underbrace{a_0 u(n)}_{\text{lahko damo 0}}$$

$$e(n) = x(n) - \tilde{x}(n) \quad \text{– napaka}$$

$$E = \sum_{n=0}^{L-1} e^2(n), \qquad \min_{\{b_1, b_2, \ldots, b_p\}} E; \quad \text{optimizacijski postopek:}$$

$$E = \sum_{n=0}^{L-1} (x(n) - \tilde{x}(n))^2 = \sum_{n=0}^{L-1} \left(x(n) - \sum_{l=1}^{p} b_l x(n-l) \right)^2$$

$$\frac{\partial E}{\partial b_l} = 0, \quad l = 1, 2, \ldots, p$$

$$\text{p enačb: } \sum_{n=0}^{L-1} x(n) x(n-l) = \sum_{i=1}^{p} b_i \sum_{n=0}^{L-1} x(n-1) x(n-l), \quad l = 1, 2, \ldots, p$$

Vprašanje: koliko je L?

1. avtokorelacijska metoda (LP (linearno-prediktivni) oz. AR (avtoregresijski) spekter)
   - postopamo podobno, kot pri neparametrični metodi
   - $x(n) = \begin{cases} 0 & \text{za } n \geq N_w \text{ in } n < 0 \text{ (okno dolžine } N_w\text{)} \\ \neq 0 & \text{za } 0 < n \leq N_w - 1 \end{cases}$
   - $e(n) \neq 0$ za $n = 0, 1, \ldots, N_w - 1, N_w, \ldots, N_w + p - 1 \Rightarrow L = N_w + p$
   - potrebno je uporabiti okensko funkcijo (Hann, Hamming,...)

$$\sum_{n=0}^{N_w+p-1} x(n) x(n-l) = \sum_{i=1}^{p} b_i \sum_{n=0}^{N_w+p-1} x(n-i) x(n-l), \quad l = 1, 2, \ldots, p$$

$$r_{xx}(l) = \sum_{n=-\infty}^{+\infty} x(n)x(n+l) = \sum_{n=0}^{N_w-1-l} x(n)x(n+l) = \sum_{n=0}^{N_w-1} x(n)x(n-l) =$$
$$= \sum_{n=0}^{N_w+p-1} x(n)x(n-l)$$

$$r_{xx}(|i-l|) = \sum_{n=-\infty}^{+\infty} x(n-i)x(n-l) = \sum_{n=0}^{N_w+p-1} x(n-i)x(n-l)$$

$$\boxed{r_{xx}(l) = \sum_{i=1}^{p} b_i r_{xx}(|i-l|), \quad l = 1,2,\dots,p} \quad \leftarrow \text{Toeplitz matrika}$$

Obstaja zelo enostaven algoritem za računanje $b_1, b_2, \dots, b_p$

2. kovariančna metoda (velja za LP oz. AR spekter) – bolj zapletena od prejšnje
   - ne predpostavljamo, da je $x(n) = 0$ izven okna!
   - množenje z okensko funkcijo ni potrebno

$$N_w \text{ vzorcev od } x(n)$$

$$L = N_w$$

   - pri računanju so potrebne vrednosti $x(n)$ za $n = -p, -p+1, \dots, 0, 1, \dots, N_w - 1$

$$\Phi(i,l) = \sum_{n=0}^{N_w-1} x(n-i)x(n-l) = \sum_{n=-i}^{N_w-i-1} x(n)x(n+i-l) = \sum_{n=-l}^{N_w-l-1} x(N)x(n+l-i)$$

$$\boxed{\Phi(0,l) = \sum_{i=1}^{p} b_i \Phi(i,l), \quad l = 1,2,\dots,p} \quad \text{simetrična matrika (ni Toeplitzova)}$$

$$E = \sum_{n=0}^{L-1} e^2(n) = a_0 \sum_{n=0}^{L-1} u^2(n) \quad \Rightarrow \boxed{a_0 = \sqrt{E}}$$

Vedno lahko izberemo vzbujanje, da velja: $\sum_{n=0}^{L-1} u^2(n) = 1$

$$X(\omega) = H(\omega)U(\omega)$$

$$U(\omega) = 1$$

$$X(\omega) = H(\omega)$$

7.4 Primerjava LP spektra z neparametričnim spektrom

$$E = \sum_{n=0}^{L-1} e(n)^2 = \sum_{n=0}^{L-1} \left(x(n) - \sum_{l=1}^{p} b_l x(n-l) \right)^2$$

Parsevalov izrek

$$E = \frac{1}{2\pi} \int_0^{2\pi} |X_w(\omega)|^2 \left| 1 - \sum_{l=1}^{p} b_l e^{-j\omega l} \right|^2 d\omega$$

$$X_W(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

Pomeni okno dolžine L

$$|H(\omega)|^2 = \frac{a_0^2}{\left| 1 - \sum_{l=1}^{p} b_l e^{-j\omega l} \right|^2}$$

$$\boxed{E = \frac{a_0}{2\pi} \int_0^{2\pi} \frac{|X_w(\omega)|^2}{|H(\omega)|^2} d\omega} \qquad p \to \infty : H(\omega) \to X_w(\omega)$$

$H(\omega)$ dobimo z minimizacijo E.
Z minimizacijo E povzročimo, da sta X_w in H zelo podobna v "vrhovih" spektra, manj podobna sta v "dolinah" spektra.