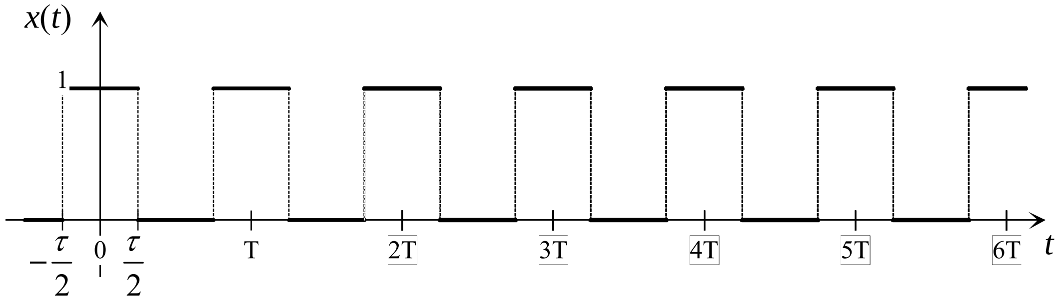
3. Fouriereva transformacija

UVOD

Fouriereva transformacija je nadgradnja Fouriereve vrste za aperiodične signale.

Iz Fouriereve vrste preidemo v Fourierevo transformacijo, če periodični signal tako raztegnemo, da postane aperiodični. Za primer si poglejmo pravokotni signal:

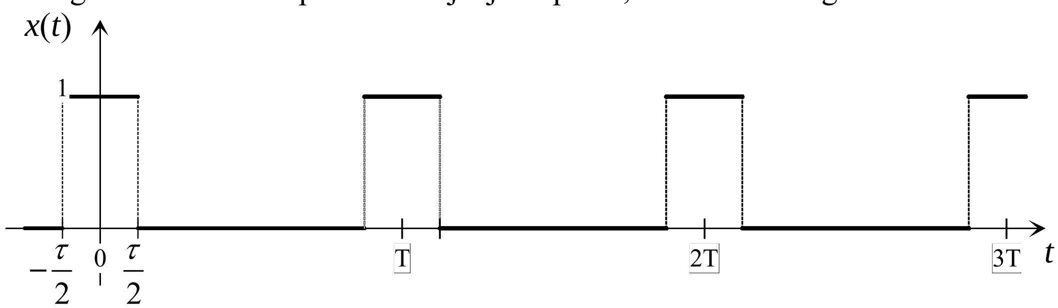


$$X_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0}dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jk\omega_0}dt = \frac{-1}{jk\omega_0 T}\left(e^{-jk\omega_0\frac{\tau}{2}} - e^{+jk\omega_0\frac{\tau}{2}}\right) = \frac{\tau}{T}\cdot\frac{\sin\left(k\pi\frac{\tau}{T}\right)}{k\pi\frac{\tau}{T}} \tag{3.0.1}$$

$$\tau = \frac{T}{2} \tag{3.0.2}$$

$$\left|X_k\right| = \frac{1}{2}\cdot\left|\frac{\sin\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}}\right| \tag{3.0.3}$$

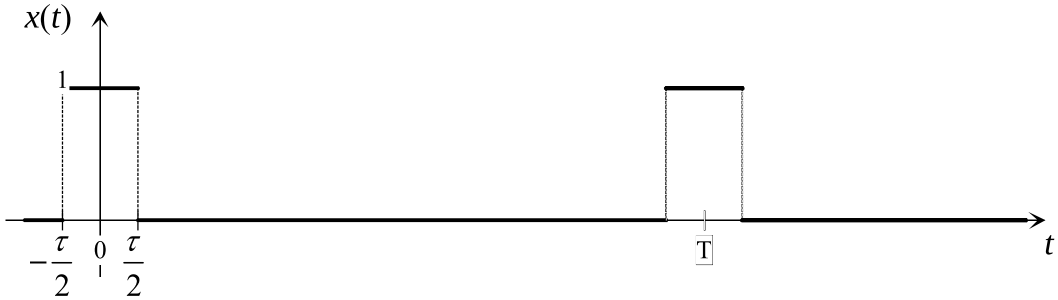
Če periodo raztegnemo brez da bi povečali trajanje impulza, dobimo nov signal:



Formula za Fourierevo vrsto ostane enaka, le τ je sedaj glede na periodo dvakrat manjši.

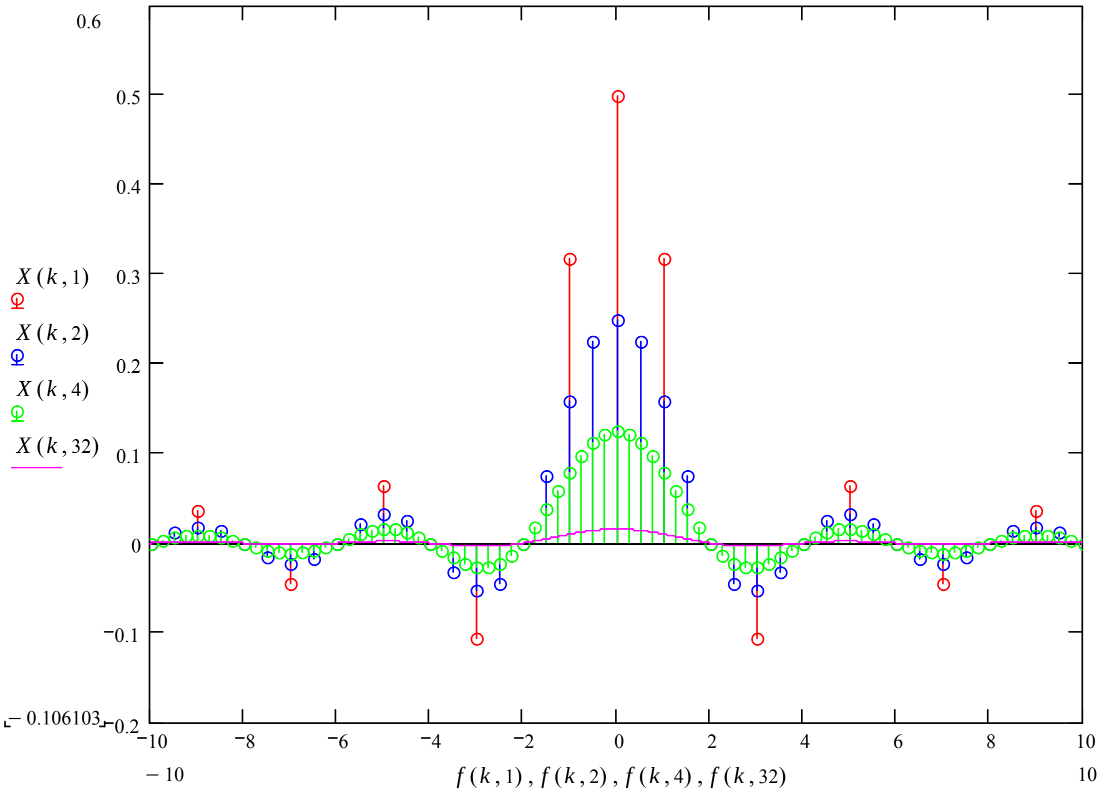
$$\tau = \frac{T}{4} \quad , \quad \left|X_k\right| = \frac{1}{4}\cdot\left|\frac{\sin\left(\frac{k\pi}{4}\right)}{\frac{k\pi}{4}}\right| \tag{3.0.4}$$

Če periodo še raztegnemo, se Fourierева vrsta še bolj spremeni:



$$\tau = \frac{T}{8} \quad , \quad |X_k| = \frac{1}{8} \cdot \left| \frac{\sin\left(\frac{k\pi}{8}\right)}{\frac{k\pi}{8}} \right| \tag{3.0.5}$$

Rezultate primerjajmo v grafu:



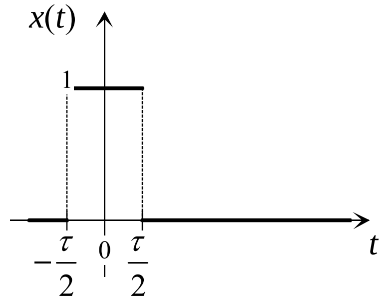
Komponente Fouriereve vrste signala z najmanjšo periodo so največje, vendar jih je najmanj.
Komponente Fouriereve vrste signala z dvakratno periodo so dvakrat nižje, vendar jih je dvakrat več.
Komponente Fouriereve vrste signala s štirikratno periodo so štirikrat nižje, vendar jih je štirikrat več.
Če zelo povečamo periodo, velikost komponent Fouriereve vrste zelo pade. To povzroči člen $\frac{1}{T}$, ki ga bomo zato pri aperiodičnih signalih spustili.
Iz slike vidimo, da se z večanjem periode spekter gosti – spektralnih črt je čedalje več. To je zato, ker osnovna frekvenca ω_0 upada, če povečujemo periodo $\left(\omega_0 = \frac{2\pi}{T}\right)$. Če gremo tako daleč, da periodo povečamo do neskončno, upade osnovna krožna frekvenca praktično na nič. Vpeljimo novo spremenljivko $\omega = k\omega_0$. Ker je ω_0 neskončno majhen, je korak spremenljivke ω neskončno majhen - ω je zvezna spremenljivka. Spremenljivke *k* smo se znebili in vpeljali ω zato je X_k postala $X(\omega)$.
Ker smo periodo signala povečali do neskončno, so se tudi meje integrala povečale do neskončno. Vse povedano povzemimo:

$$\cancel{\frac{1}{T}} \quad -\frac{T}{2} \to -\infty \quad \frac{T}{2} \to \infty \quad k\omega_0 \to \omega \quad X_k \to X(\omega)$$

$$X_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt \Rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (3.0.6)$$

1. Naloga

Izračunajte Fourierevo transformacijo $X(\omega)$ za podani aperiodični signal.



Rešitev:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \tag{3.1.1}$$

$$X(\omega) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1 \cdot e^{-j\omega t} dt = \frac{1}{-j\omega} \cdot e^{-j\omega t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = -\frac{1}{j\omega} \cdot \left(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}} \right) \tag{3.1.2}$$

$$X(\omega) = -\frac{1}{j\omega} \cdot \left(\cos\left(-\omega\frac{\tau}{2}\right) + j\sin\left(-\omega\frac{\tau}{2}\right) - \cos\left(\omega\frac{\tau}{2}\right) - j\sin\left(\omega\frac{\tau}{2}\right) \right) \tag{3.1.3}$$

$$X(\omega) = -\frac{1}{j\omega} \cdot \left(-j2\sin\left(\omega\frac{\tau}{2}\right) \right) = 2 \cdot \frac{\tau}{2} \cdot \frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}} \tag{3.1.4}$$

$$X(\omega) = \tau \cdot \frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}} \tag{3.1.5}$$

Izračunan izraz velja, če je ω različen od 0. Za $\omega = 0$ je potrebno izpeljati svoj izraz. To lahko naredimo tako, da v začetno formulo vstavimo $\omega = 0$, ali pa dobljeni izraz limitiramo. Naredimo na oba načina:

1. V začetni izraz vstavimo $\omega = 0$:

$$X(0) = \int_{-\infty}^{\infty} x(t) \cdot e^{0} dt = \int_{-\infty}^{\infty} x(t) dt \tag{3.1.6}$$

Dobljeni izraz predstavlja ploščino lika, ki ga opisuje signal.

$$X(0) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt = t\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{\tau}{2} - \left(-\frac{\tau}{2}\right) = \tau \tag{3.1.7}$$

2. V izrazu (3.1.5) limitiramo ω proti 0:

$$X(0)=\lim_{\omega\to 0}\left(\tau\cdot\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}}\right)=\tau\lim_{\omega\to 0}\left(\frac{\frac{d}{dk}\left(\sin\left(\omega\frac{\tau}{2}\right)\right)}{\frac{d}{dk}\left(\omega\frac{\tau}{2}\right)}\right)=\tau\lim_{k\to 0}\left(\frac{\frac{\tau}{2}\cos\left(\omega\frac{\tau}{2}\right)}{\frac{\tau}{2}}\right) \tag{3.1.8}$$

$$X(0)=\tau\left(\frac{\frac{\tau}{2}\cos(0)}{\frac{\tau}{2}}\right)=\tau \tag{3.1.9}$$

2. Naloga

Izpeljite pravila za Fourierevo transformacijo:

a) $x(t)=a\cdot f(t)+b\cdot g(t)\quad;\quad a\in\mathbb{R},b\in\mathbb{R},f(t)\leftrightarrow F(\omega),g(t)\leftrightarrow G(\omega)$

b) $x(t)=f(t-\tau)\quad;\quad \tau\in\mathbb{R},f(t)\leftrightarrow F(\omega)$

c) $x(t)=f(t+\tau)\quad;\quad \tau\in\mathbb{R},f(t)\leftrightarrow F(\omega)$

d) $x(t)=f\left(\frac{t}{a}\right)\quad;\quad a\in\mathbb{R},a>0,f(t)\leftrightarrow F(\omega)$

e) $x(t)=f(t)\cdot\sin(\omega_0 t)\quad;\quad \omega_0\in\mathbb{R},f(t)\leftrightarrow F(\omega)$

Rešitve:

a) $x(t)=a\cdot f(t)+b\cdot g(t)\quad;\quad a\in\mathbb{R},b\in\mathbb{R},f(t)\leftrightarrow F(\omega),g(t)\leftrightarrow G(\omega)$

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\left(a\cdot f(t)+b\cdot g(t)\right)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\left(a\cdot f(t)\cdot e^{-j\omega t}+b\cdot g(t)\cdot e^{-j\omega t}\right)dt \tag{3.2.1}$$

$$X(\omega)=\int_{-\infty}^{\infty}a\cdot f(t)\cdot e^{-j\omega t}dt+\int_{-\infty}^{\infty}b\cdot g(t)\cdot e^{-j\omega t}dt=a\int_{-\infty}^{\infty}f(t)\cdot e^{-j\omega t}dt+b\int_{-\infty}^{\infty}g(t)\cdot e^{-j\omega t}dt \tag{3.2.2}$$

$$X(\omega)=a\cdot F(\omega)+b\cdot G(\omega) \tag{3.2.3}$$

$$\boxed{x(t)=a\cdot f(t)+b\cdot g(t)\quad\leftrightarrow\quad X(\omega)=a\cdot F(\omega)+b\cdot G(\omega)} \tag{3.2.4}$$

b) $x(t) = f(t-\tau) \quad ; \quad \tau \in \mathbb{R}, f(t) \leftrightarrow F(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t-\tau) e^{-j\omega t} dt \tag{3.2.5}$$

$$\begin{array}{ll} x = t - \tau & \\ t = x + \tau & x(t=\infty) = \infty \\ & x(t=-\infty) = -\infty \\ dx = dt & \end{array} \tag{3.2.6}$$

$$X(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega(x+\tau)} dx = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} e^{-j\omega\tau} dx = e^{-j\omega\tau} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \tag{3.2.7}$$

$$X(\omega) = e^{-j\omega\tau} F(\omega) \tag{3.2.8}$$

$$\boxed{x(t) = f(t-\tau) \quad \leftrightarrow \quad X(\omega) = F(\omega) e^{-j\omega\tau}} \tag{3.2.9}$$

c) $x(t) = f(t+\tau) \quad ; \quad \tau \in \mathbb{R}, f(t) \leftrightarrow F(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t+\tau) e^{-j\omega t} dt \tag{3.2.10}$$

$$\begin{array}{ll} x = t + \tau & \\ t = x - \tau & x(t=\infty) = \infty \\ & x(t=-\infty) = -\infty \\ dx = dt & \end{array} \tag{3.2.11}$$

$$X(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega(x-\tau)} dx = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} e^{+j\omega\tau} dx = e^{+j\omega\tau} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \tag{3.2.12}$$

$$X(\omega) = e^{+j\omega\tau} F(\omega) \tag{3.2.13}$$

$$\boxed{x(t) = f(t+\tau) \quad \leftrightarrow \quad X(\omega) = F(\omega) e^{+j\omega\tau}} \tag{3.2.14}$$

Rezultata nalog b) in c) lahko združimo:

$$\boxed{x(t) = f(t\pm\tau) \quad \leftrightarrow \quad X(\omega) = F(\omega) e^{\pm j\omega\tau}} \tag{3.2.15}$$

d) $x(t) = f\left(\frac{t}{a}\right) \quad ; \quad a \in \mathbb{R}, a > 0, f(t) \leftrightarrow F(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f\left(\frac{t}{a}\right) e^{-j\omega t} dt \tag{3.2.16}$$

$$\begin{aligned} & x = \frac{t}{a} \\ & t = ax \qquad \begin{aligned} & x(t=\infty) = \infty \\ & x(t=-\infty) = -\infty \end{aligned} \\ & dx = \frac{dt}{a} \end{aligned} \tag{3.2.17}$$

$$X(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega a x} \cdot a \cdot dx = a \int_{-\infty}^{\infty} f(x) e^{-j\omega a x} dx = a \cdot F(a\omega) \tag{3.2.18}$$

$$\boxed{x(t) = f\left(\frac{t}{a}\right) \quad \leftrightarrow \quad X(\omega) = a \cdot F(a\omega)} \tag{3.2.19}$$

e) $x(t) = f(t) \cdot \sin(\omega_0 t) \quad ; \quad \omega_0 \in \mathbb{R}, f(t) \leftrightarrow F(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) \cdot \sin(\omega_0 t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) \cdot \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-j\omega t} dt \tag{3.2.20}$$

$$X(\omega) = \frac{1}{2j}\left(\int_{-\infty}^{\infty} f(t) \cdot e^{j\omega_0 t} \cdot e^{-j\omega t} dt - \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega_0 t} \cdot e^{-j\omega t} dt \right) \tag{3.2.21}$$

$$X(\omega) = \frac{1}{2j}\left(\int_{-\infty}^{\infty} f(t) \cdot e^{-j(\omega - \omega_0)t} dt - \int_{-\infty}^{\infty} f(t) \cdot e^{-j(\omega + \omega_0)t} dt \right) \tag{3.2.22}$$

$$X(\omega) = \frac{1}{2j}\left(F(\omega - \omega_0) - F(\omega + \omega_0)\right) \tag{3.2.23}$$

$$\boxed{x(t) = f(t) \cdot \sin(\omega_0 t) \quad \leftrightarrow \quad X(\omega) = \frac{1}{2j}\left(F(\omega - \omega_0) - F(\omega + \omega_0)\right)} \tag{3.2.24}$$

Povzetek:

$$x(t) = a \cdot f(t) + b \cdot g(t) \quad \leftrightarrow \quad X(\omega) = a \cdot F(\omega) + b \cdot G(\omega) \tag{3.2.25}$$

$$x(t) = f(t \pm \tau) \quad \leftrightarrow \quad X(\omega) = F(\omega) e^{\pm j\omega\tau} \tag{3.2.26}$$

$$x(t) = f\left(\frac{t}{a}\right) \quad \leftrightarrow \quad X(\omega) = a \cdot F(a\omega) \tag{3.2.27}$$

$$x(t) = f(t) \cdot \sin(\omega_0 t) \quad \leftrightarrow \quad X(\omega) = \frac{1}{2j}\left(F(\omega - \omega_0) - F(\omega + \omega_0)\right) \tag{3.2.28}$$

$$x(t) = f(t) \cdot \cos(\omega_0 t) \quad \leftrightarrow \quad X(\omega) = \frac{1}{2}\left(F(\omega - \omega_0) + F(\omega + \omega_0)\right) \tag{3.2.29}$$

Skorajda enaka pravila veljajo tudi za periodične signale:

$$x(t) = a \cdot f(t) + b \cdot g(t) \quad \leftrightarrow \quad X_k = a \cdot F_k + b \cdot G_k \tag{3.2.30}$$

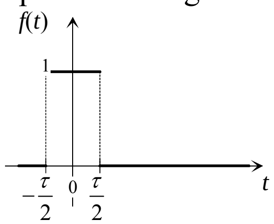
$$x(t) = f(t \pm \tau) \quad \leftrightarrow \quad X_k = F_k e^{\mp jk\omega_0\tau} \tag{3.2.31}$$

$$x(t) = f(t) \cdot \sin(n\omega_0 t) \quad \leftrightarrow \quad X_k = \frac{1}{2j}\left(F_{k-n} - F_{k+n}\right) \tag{3.2.32}$$

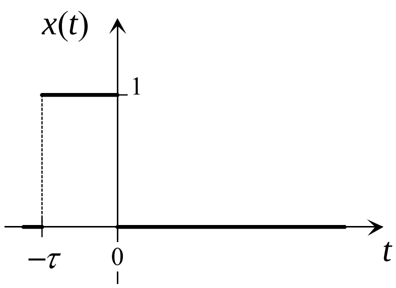
$$x(t) = f(t) \cdot \cos(n\omega_0 t) \quad \leftrightarrow \quad X_k = \frac{1}{2}\left(F_{k-n} + F_{k+n}\right) \tag{3.2.33}$$

3. Naloga

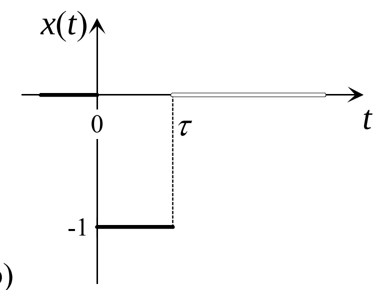
S pomočjo pravil in rezultata prve naloge izračunajte Fourierevo transformacijo $X(\omega)$ za podane aperiodične signale.



$$F(\omega) = \tau \cdot \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}}, \; F(0) = \tau$$



a)



b)

Rešitve:

a)

$$x(t) = f\left(t + \frac{\tau}{2}\right) \leftrightarrow X(\omega) = F(\omega) \cdot e^{-j\omega\frac{\tau}{2}} = \tau \cdot \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}} \cdot e^{-j\omega\frac{\tau}{2}} \tag{3.3.1}$$

$$X(0) = \tau \tag{3.3.2}$$

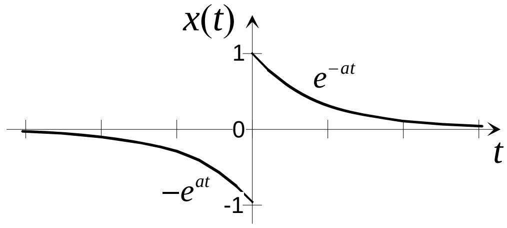
b)

$$x(t) = -f\left(t - \frac{\tau}{2}\right) \leftrightarrow X(\omega) = -F(\omega) \cdot e^{j\omega\frac{\tau}{2}} = -\tau \cdot \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}} \cdot e^{j\omega\frac{\tau}{2}} \tag{3.3.3}$$

$$X(0) = -\tau \tag{3.3.4}$$

4. Naloga

Izračunajte Fourierevo transformacijo signala $x(t)=\begin{cases} e^{-at} & ; \quad t\geq 0 \\ -e^{at} & ; \quad t<0 \end{cases}, a>0$. Iz rezultata izpeljite Fourierevo transformacijo za enotino stopnico. Pri reševanju uporabite pravila za Fourierevo transformacijo.



Rešitev:

Signal $x(t)$ lahko razstavimo v dva signala:

$$x(t)=f(t)+g(t) \tag{3.4.1}$$

$$f(t)=e^{-at}\cdot u(t) \tag{3.4.2}$$

$$g(t)=-e^{at}\cdot u(-t) \tag{3.4.3}$$

Izračunajmo Fourierevo transformacijo funkcije (3.4.2):

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} e^{-at}\cdot u(t)\cdot e^{-j\omega t}dt=\int_{0}^{\infty} e^{-at}\cdot e^{-j\omega t}dt=\int_{0}^{\infty} e^{-j\omega t-at}dt=\int_{0}^{\infty} e^{-(j\omega+a)t}dt \tag{3.4.4}$$

$$F(\omega)=\frac{1}{-(j\omega+a)}\cdot e^{-(j\omega+a)t}\Big|_0^{\infty}=-\frac{1}{(j\omega+a)}\cdot\left(e^{-(j\omega+a)\infty}-e^{-(j\omega+a)0}\right) \tag{3.4.5}$$

$$F(\omega)=-\frac{1}{(j\omega+a)}\cdot\left(\underbrace{e^{-j\omega\cdot\infty}}_{[-1,1]}\cdot\underbrace{e^{-a\cdot\infty}}_{0}-1\right)=\frac{1}{(j\omega+a)} \tag{3.4.6}$$

Funkcijo $g(t)$ lahko izrazimo s funkcijo $f(t)$:

$$g(t)=-f(-t) \tag{3.4.7}$$

Za Fourierevo transformacijo uporabimo pravilo o transformaciji prezrcaljene funkcije:

$$x(-t)\leftrightarrow X(-\omega) \tag{3.4.8}$$

$$g(t)\leftrightarrow G(\omega)=-F(-\omega)=-\frac{1}{a-j\omega} \tag{3.4.9}$$

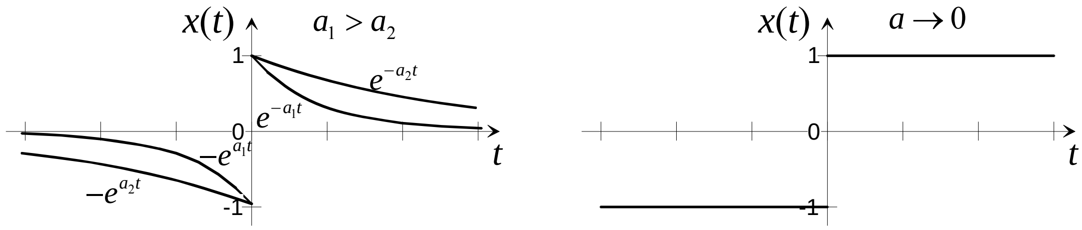
Fouriereva transformacija signala $x(t)$ je vsota Fourierevih transformacij signalov $f(t)$ in $g(t)$:

$$x(\omega)=F(\omega)+G(\omega)=\frac{1}{a+j\omega}-\frac{1}{a-j\omega} \tag{3.4.10}$$

$$X(\omega)=\frac{a-j\omega}{(a+j\omega)(a-j\omega)}-\frac{a+j\omega}{(a+j\omega)(a-j\omega)}=\frac{a-j\omega-a-j\omega}{a^2+\omega^2}=\frac{-2j\omega}{a^2+\omega^2} \tag{3.4.11}$$

$$X(\omega)=-j\frac{2\omega}{a^2+\omega^2} \tag{3.4.12}$$

Če *a* manjšamo, postaja funkcija čedalje bolj položna. Ko ga limitiramo proti 0, dobimo funkcijo $\operatorname{sign}(t)$:



Limito izvedimo tudi nad Fourierevo transformacijo:

$$\lim_{a\to 0}\left(X(\omega)\right)=\lim_{a\to 0}\left(-j\frac{2\omega}{a^2+\omega^2}\right)=-j\frac{2\omega}{\omega^2}=-j\frac{2}{\omega} \tag{3.4.13}$$

$$\operatorname{sign}(t)\leftrightarrow -j\frac{2}{\omega} \tag{3.4.14}$$

Iz funkcije $\operatorname{sign}(t)$ dobimo enotino stopnico, če ji prištejemo 1 in vsoto delimo z 2:

$$u(t)=\frac{1}{2}\left(1+\operatorname{sign}(t)\right)=\frac{1}{2}+\frac{1}{2}\operatorname{sign}(t) \tag{3.4.15}$$

Fouriereva transformacija enotine stopnice je torej vsota Fourierove transformacije konstante 1/2 in Fourierove transformacije funkcije $\frac{1}{2}\operatorname{sign}(t)$. Ker oboje poznamo

$$1\leftrightarrow 2\pi\delta(\omega) \tag{3.4.16}$$

$$\operatorname{sign}(t)\leftrightarrow -j\frac{2}{\omega} \tag{3.4.17}$$

Lahko direktno zapišemo Fourierevo transformacijo enotine stopnice:

$$u(t)\leftrightarrow \frac{1}{2}\cdot 2\pi\delta(\omega)+\frac{1}{2}\cdot\left(-j\frac{2}{\omega}\right)=\pi\delta(\omega)-\frac{j}{\omega} \tag{3.4.18}$$

$$u(t)\leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega} \tag{3.4.19}$$

5. Naloga

Izračunajte Fourierevo transformacijo signala $x(t)=\sin(\omega_0 t)$.

Rešitev:

Preden vstavimo funkcijo v Fourierevo transformacijo, jo preoblikujmo v priročnejšo obliko:

$$x(t)=\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j} \tag{3.5.1}$$

$$X(\omega)=\int_{-\infty}^{\infty}\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)\cdot e^{-j\omega t}dt=\frac{1}{2j}\left(\int_{-\infty}^{\infty}e^{j\omega_0 t}\cdot e^{-j\omega t}dt-\int_{-\infty}^{\infty}e^{-j\omega_0 t}\cdot e^{-j\omega t}dt\right) \tag{3.5.2}$$

$$X(\omega)=\frac{1}{2j}\left(\int_{-\infty}^{\infty}e^{-j(\omega-\omega_0)t}dt-\int_{-\infty}^{\infty}e^{-j(\omega+\omega_0)t}dt\right) \tag{3.5.3}$$

Primerjajmo integrala v izrazu (3.5.3) s Fourierevo transformacijo konstante:

$$1\leftrightarrow\int_{-\infty}^{\infty}1\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-j\omega t}dt=2\pi\delta(\omega) \tag{3.5.4}$$

Integrala se razlikujeta le v eksponentu, kjer je namesto ω izraz $(\omega-\omega_0)$ oziroma $(\omega+\omega_0)$. Torej velja:

$$\int_{-\infty}^{\infty}e^{-j(\omega-\omega_0)t}dt=2\pi\delta(\omega-\omega_0)\quad,\quad\int_{-\infty}^{\infty}e^{-j(\omega+\omega_0)t}dt=2\pi\delta(\omega+\omega_0) \tag{3.5.5}$$

$$X(\omega)=\frac{1}{2j}\left(2\pi\delta(\omega-\omega_0)-2\pi\delta(\omega+\omega_0)\right) \tag{3.5.6}$$

$$X(\omega)=\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)=-j\pi\delta(\omega-\omega_0)+j\pi\delta(\omega+\omega_0) \tag{3.5.7}$$

$$X(\omega)=j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0) \tag{3.5.8}$$