1. Kategorizacija signalov

1. Naloga

Narišite graf podanega periodičnega signala in izračunajte njegovo frekvenco, periodo in normalizirano moč:

a) $x(t) = 2\cos(200\pi t)$

b) $x(t) = \sin^2(500\pi t)$

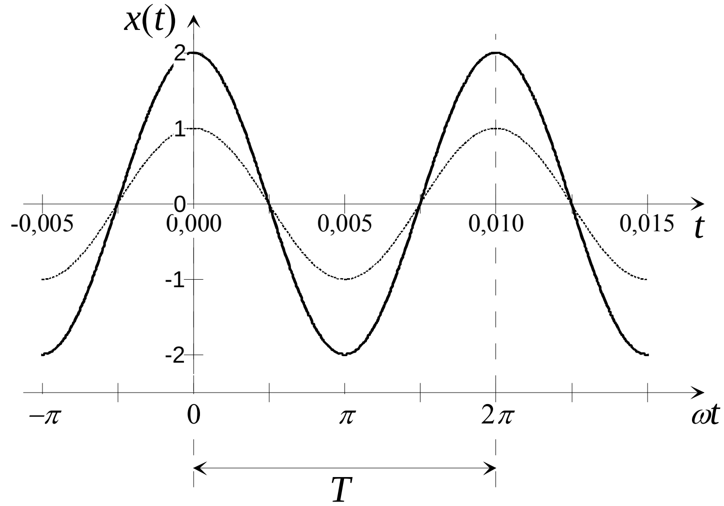
c) $x(t) = \cos(2\pi t) + \cos(4\pi t)$

d) $x(t) = \operatorname{sign}(\cos(\omega t))$

Rešitve:

a) $x(t) = 2\cos(200\pi t)$

Graf:



Frekvenca in perioda:

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$$T = \frac{1}{f} = 0{,}01\,\text{s} \tag{1.1.2}$$

Normirana moč:

$$p_x(t) = x^2(t) = \left(2\cos(\omega t)\right)^2 = 4\cos^2(\omega t) \tag{1.1.3}$$

$$P_x = \frac{1}{T}\int_0^T p_x(t)\,dt = \frac{1}{T}\int_0^T 4\cos^2(\omega t)\,dt \tag{1.1.4}$$

$$\begin{aligned} &\omega t = \varphi \quad && t = 0 \quad && \varphi = 0 \\ &\omega dt = d\varphi \quad && t = T \quad && \varphi = \omega T = \frac{2\pi}{T}T = 2\pi \\ &dt = \frac{d\varphi}{\omega} \end{aligned} \tag{1.1.5}$$

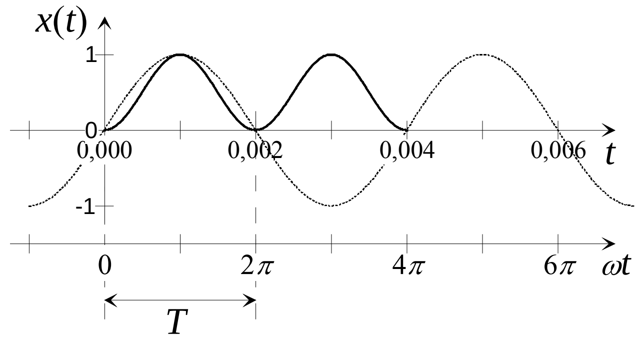
$$P_x = \frac{1}{T}\int_0^{2\pi} 4\cos^2(\varphi)\frac{d\varphi}{\omega} = \frac{4}{\omega T}\int_0^{2\pi}\cos^2(\varphi)\,d\varphi = \frac{4}{2\pi}\int_0^{2\pi}\cos^2(\varphi)\,d\varphi \tag{1.1.6}$$

$$P_x = \frac{2}{\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(2\varphi)\right)d\varphi = \frac{1}{\pi}\int_0^{2\pi} d\varphi - \frac{1}{\pi}\int_0^{2\pi}\cos(2\varphi)\,d\varphi \tag{1.1.7}$$

$$P_x = \frac{1}{\pi}\varphi\Big|_0^{2\pi} - \frac{1}{\pi}\frac{\sin(2\varphi)}{2}\Bigg|_0^{2\pi} = \frac{1}{\pi}2\pi - \frac{1}{\pi}\left(\sin(4\pi) - \sin(0)\right) = 2 \tag{1.1.8}$$

b) $x(t) = \sin^2(500\pi t)$

Graf:



$x(t) = \sin^2(500\pi t) = \frac{1}{2} - \frac{1}{2}\cos(1000\pi t)$

Frekvenca in perioda:

$$\omega_0 = 1000\pi,\ f = \frac{\omega_0}{2\pi} = 500\ \text{Hz} \tag{1.1.9}$$

$$T = \frac{1}{f} = 0,002\ \text{s} \tag{1.1.10}$$

Normirana moč:

$$p_x(t) = x^2(t) = \left(\frac{1}{2}\left(1-\cos(\omega_0 t)\right)\right)^2 = \frac{1}{4}\left(1 - 2\cos(\omega_0 t) + \cos^2(\omega_0 t)\right) \tag{1.1.11}$$

$$p_x(t) = \frac{1}{4}\left(1 - 2\cos(\omega_0 t) + \frac{1}{2}\left(1+\cos(2\omega_0 t)\right)\right) \tag{1.1.12}$$

$$p_x(t) = \frac{1}{8}\left(3 - 4\cos(\omega_0 t) + \cos(2\omega_0 t)\right) \tag{1.1.13}$$

$$P_x = \frac{1}{T}\int_0^T p_x(t)\,dt = \frac{1}{T}\int_0^T \frac{1}{8}\left(3 - 4\cos(\omega_0 t) + \cos(2\omega_0 t)\right)dt \tag{1.1.14}$$

$$\omega_0 t = \varphi \qquad t = 0 \qquad \varphi = 0$$

$$\omega_0 dt = d\varphi \quad t = T \quad \varphi = \omega_0 T = \frac{2\pi}{T}T = 2\pi \tag{1.1.15}$$

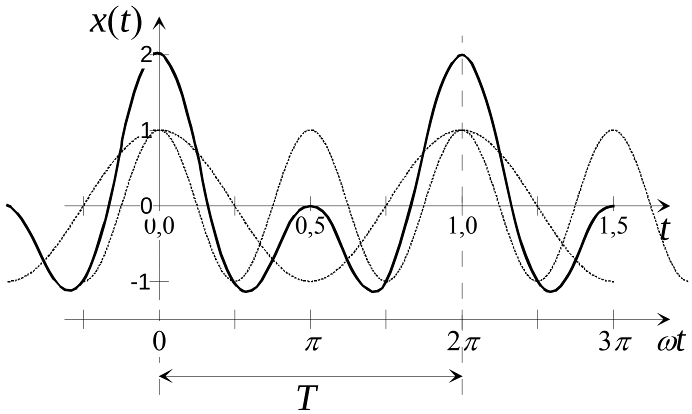
$$dt = \frac{d\varphi}{\omega_0}$$

$$P_x = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{8}\left(3 - 4\cos(\varphi) + \cos(2\varphi)\right)d\varphi \tag{1.1.16}$$

$$P_x = \frac{1}{16\pi}\left(3\cdot 2\pi - 4\cdot\left(\sin(2\pi) - \sin(0)\right) + \left(\sin(4\pi) - \sin(0)\right)\right) = \frac{3}{8} \tag{1.1.17}$$

c) $x(t)=\cos(2\pi t)+\cos(4\pi t)$

Graf:



Frekvenca in perioda:

$$\omega_0 = 2\pi,\ f = \frac{\omega_0}{2\pi} = 1\,\text{Hz} \tag{1.1.18}$$

$$T = \frac{1}{f} = 1\,\text{s} \tag{1.1.19}$$

Normirana moč:

$$p_x(t) = x^2(t) = \left(\cos(2\pi t)+\cos(4\pi t)\right)^2 \tag{1.1.20}$$

$$p_x(t) = \cos^2(2\pi t) + 2\cos(2\pi t)\cos(4\pi t) + \cos^2(4\pi t) \tag{1.1.21}$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right) \tag{1.1.22}$$

$$p_x(t) = \frac{1}{2}\left(1+\cos(4\pi t)\right) + \frac{1}{2}\left(\cos(6\pi t)+\cos(-2\pi t)\right) + \frac{1}{2}\left(1+\cos(8\pi t)\right) \tag{1.1.23}$$

$$p_x(t) = \frac{1}{2}\left(2+\cos(4\pi t)+\cos(6\pi t)+\cos(2\pi t)+\cos(8\pi t)\right) \tag{1.1.24}$$

$$P_x = \frac{1}{T}\int_0^T p_x(t)\,dt \tag{1.1.25}$$

$$P_x = \frac{1}{T}\int_0^T \frac{1}{2}\left(2+\cos(4\pi t)+\cos(6\pi t)+\cos(2\pi t)+\cos(8\pi t)\right)dt \tag{1.1.26}$$

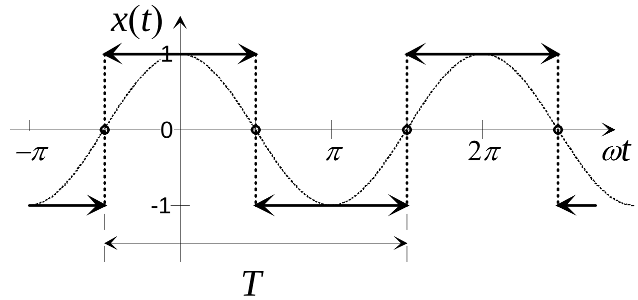
$$P_x = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left(2+\cos(2\varphi)+\cos(3\varphi)+\cos(\varphi)+\cos(4\varphi)\right)d\varphi \tag{1.1.27}$$

$$P_x = \frac{1}{4\pi}\left(2\cdot 2\pi + \left(\sin(4\pi)-\sin(0)\right)+\ldots+\left(\sin(8\pi)-\sin(0)\right)\right) = 1 \tag{1.1.28}$$

d) $x(t) = \mathrm{sign}(\cos(\omega t))$

$$\mathrm{sign}(x) = \begin{cases} -1 & ; \quad x < 0 \\ 0 & ; \quad x = 0 \\ 1 & ; \quad x > 0 \end{cases} \tag{1.1.29}$$

Graf:



Frekvenca in perioda:

$$f = \frac{\omega}{2\pi} \tag{1.1.30}$$

$$T = \frac{2\pi}{\omega} \tag{1.1.31}$$

Normirana moč:

$$p_x(t) = x^2(t) = \left(\mathrm{sign}(\cos(\omega t))\right)^2 = 1 \quad , \quad t \neq \frac{\pi}{2\omega} + k\frac{\pi}{\omega} \tag{1.1.32}$$

$$P_x = \frac{1}{T}\int_0^T p_x(t)\,dt = \frac{1}{2\pi}\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 \cdot dt + \int_{\frac{\pi}{4}}^{3\frac{\pi}{4}} 1 \cdot dt\right) = \frac{1}{2\pi}\left(\frac{\pi}{4} + \frac{\pi}{4} + 3\frac{\pi}{4} - \frac{\pi}{4}\right) = 1 \tag{1.1.33}$$

2. Naloga

Narišite graf in izračunajte normalizirano energijo podanih aperiodičnih signalov:

a) $x(t) = e^{-\frac{t}{\tau}} \cdot u(t)$

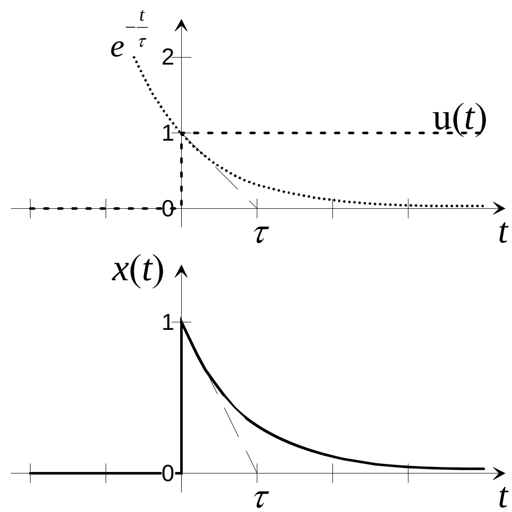
b) $x(t) = \text{sign}(t) \cdot e^{-\frac{|t|}{\tau}}$

c) $x(t) = u(t+2) - u(t-2)$

d) $x(t) = \dfrac{\sin\left(\dfrac{2\pi}{\tau}t\right)}{\dfrac{2\pi}{\tau}t}$

a) $x(t) = e^{-\frac{t}{\tau}} \cdot u(t)$

$$u(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases} \tag{1.2.1}$$

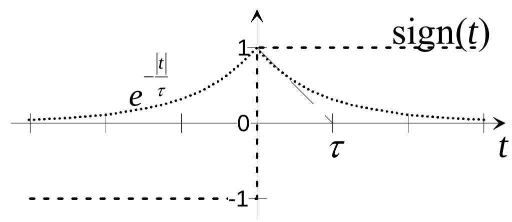


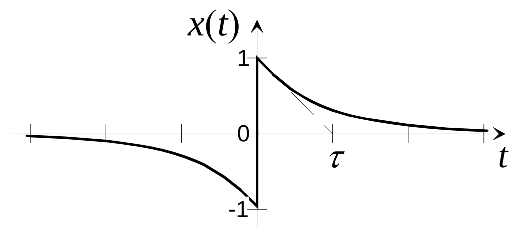
$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_0^{\infty} \left(e^{-\frac{t}{\tau}} \right)^2 dt = \int_0^{\infty} e^{-\frac{2t}{\tau}} dt = \frac{1}{-\frac{2}{\tau}} \cdot e^{-\frac{2t}{\tau}} \Bigg|_0^{\infty} \tag{1.2.2}$$

$$E_x = \frac{1}{-\frac{2}{\tau}} \cdot \left(e^{-\infty} - e^{0} \right) = -\frac{1}{-\frac{2}{\tau}} = \frac{\tau}{2} \tag{1.2.3}$$

b) $x(t)=\operatorname{sign}(t)\cdot e^{-\frac{|t|}{\tau}}$

$$\operatorname{sign}(x)=\begin{cases}-1 & ; \quad x<0\\ 0 & ; \quad x=0\\ 1 & ; \quad x>0\end{cases} \tag{1.2.4}$$



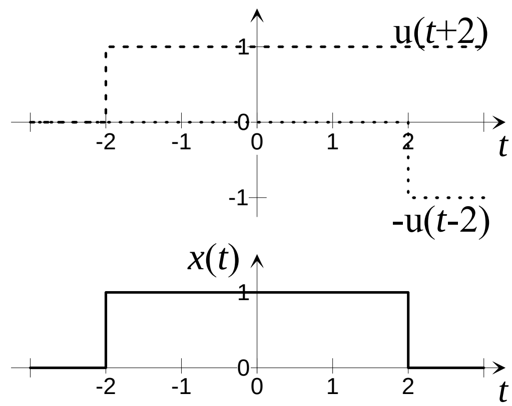


$$E_x=\int_{-\infty}^{\infty}x^2(t)\,dt=\int_{-\infty}^{\infty}\left(\operatorname{sign}(t)\cdot e^{-\frac{|t|}{\tau}}\right)^2dt=\int_{-\infty}^{\infty}e^{-\frac{2|t|}{\tau}}dt=\int_{-\infty}^{0}e^{\frac{2t}{\tau}}dt+\int_{0}^{\infty}e^{-\frac{2t}{\tau}}dt \tag{1.2.5}$$

Integral $\int_{-\infty}^{0}e^{\frac{2t}{\tau}}dt$ je ravno enak integralu $\int_{0}^{\infty}e^{-\frac{2t}{\tau}}dt$, saj je funkcija le prezrcaljena okoli ordinatne osi, sicer pa enaka. Integral $\int_{0}^{\infty}e^{-\frac{2t}{\tau}}dt$ smo že izračunali v prejšnji vaji:

$$E_x=2\cdot\int_{0}^{\infty}e^{-\frac{2t}{\tau}}dt=2\cdot\frac{\tau}{2}=\tau \tag{1.2.6}$$

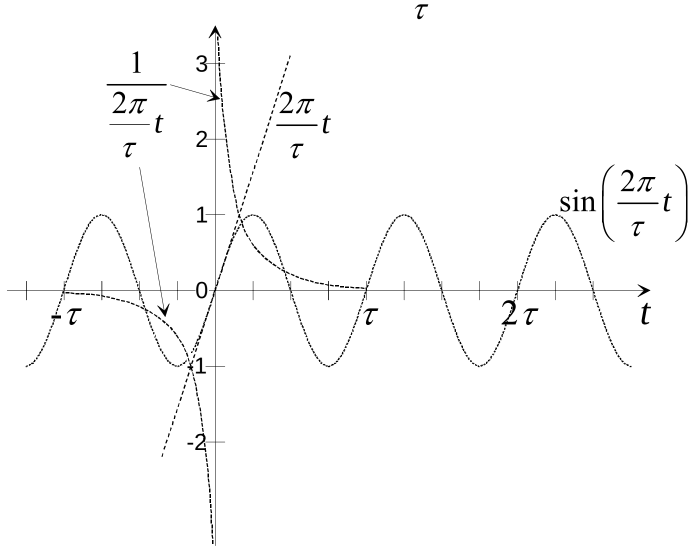
c) $x(t)=u(t+2)-u(t-2)$



$$E_x=\int_{-\infty}^{\infty}x^2(t)\,dt=\int_{-2}^{2}1^2\,dt=\int_{-2}^{2}1dt=t\Big|_{-2}^{2}=2-(-2)=4 \qquad (1.2.7)$$

d) $x(t)=\dfrac{\sin\left(\dfrac{2\pi}{\tau}t\right)}{\dfrac{2\pi}{\tau}t}$

Funkcijo $x(t)$ lahko razstavimo na člena $\sin\left(\dfrac{2\pi}{\tau}t\right)$ in $\dfrac{1}{\dfrac{2\pi}{\tau}t}$.



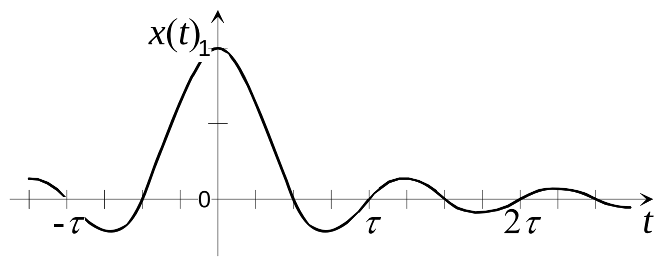
Rezultirajoča funkcija je sinus, ki z oddaljenostjo od časa $t = 0$ upada. Pri časih manjših od nič je člen $\frac{1}{\frac{2\pi}{\tau}t}$ negativen, zato je v tem delu sinus prezrcaljen okoli abscisne osi (osi x). Nejasno je le, kaj se zgodi v točki $t = 0$, ker sta $\sin\left(\frac{2\pi}{\tau}0\right) = 0$ in $\frac{1}{\frac{2\pi}{\tau}0} = \infty$, kar nam da produkt $0 \cdot \infty$, ki ni določen. Zato moramo izračunati limito ko gre čas proti 0:

$$\lim_{t\to 0} x(t) = \lim_{t\to 0} \frac{\sin\left(\frac{2\pi}{\tau}t\right)}{\frac{2\pi}{\tau}t} \tag{1.2.8}$$

Uporabimo Bernoulli-l'Hospitalovo pravilo (Matematični priročnik, str. 40):

$$\lim_{t\to 0} \frac{\frac{d}{dt}\left(\sin\left(\frac{2\pi}{\tau}t\right)\right)}{\frac{d}{dt}\left(\frac{2\pi}{\tau}t\right)} = \lim_{t\to 0} \frac{\frac{2\pi}{\tau}\cos\left(\frac{2\pi}{\tau}t\right)}{\frac{2\pi}{\tau}} = 1 \tag{1.2.9}$$

Funkcija $x(t)$ ima pri času $t = 0$ vrednost 1:



Iz slike vidimo, da sta integrala za levo in desno stran grafa enaka. Zato bomo računali integral le za eno stran in rezultat pomnožili z 2.

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{2\pi}{\tau}t\right)}{\left(\frac{2\pi}{\tau}t\right)^2}dt = 2\cdot\int_{0}^{\infty} \frac{\sin^2\left(\frac{2\pi}{\tau}t\right)}{\left(\frac{2\pi}{\tau}t\right)^2}dt \tag{1.2.10}$$

V matematičnem priročniku poiščemo rešitev integrala.

$$E_x = \frac{2}{\left(\frac{2\pi}{\tau}\right)^2}\cdot\int_{0}^{\infty} \frac{\sin^2\left(\frac{2\pi}{\tau}t\right)}{t^2}dt = \frac{2\cdot\tau^2}{4\pi^2}\cdot\frac{\pi}{2}\cdot\left|\frac{2\pi}{\tau}\right| = \frac{\tau}{2} \tag{1.2.11}$$