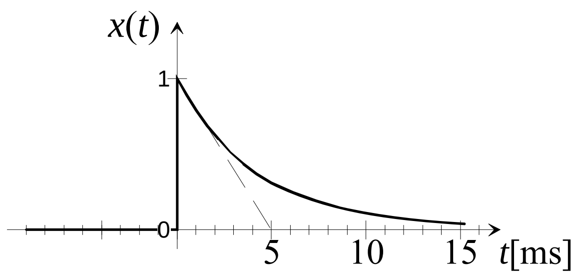
6. Vzorčenje in časovno diskretna Fouriereva transformacija (DtFT)

1. Naloga

Analogni signal je opisan z enačbo $x(t)=e^{-\frac{t}{\tau}}\cdot u(t)$, $\tau=5\text{ ms}$. Zapišite signal, ki ga dobimo z vzorčenjem signala $x(t)$ s frekvenco $f_S=1\text{ kHz}$.

Rešitev:

Narišimo analogni signal:



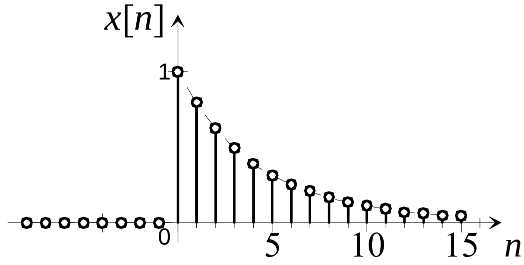
Iz frekvence vzorčenja izračunajmo čas med dvema vzorcema:

$$T_S=\frac{1}{f_S}=\frac{1}{1000\text{ Hz}}=0,001\text{ s}=1\text{ ms} \qquad (6.2.1)$$

Vzorčeni signal je niz vzorcev v trenutkih 0, T_S, $2T_S$, $3T_S$,...nT_S,...

$$x[n]=x(nT_S)=e^{-\frac{nT_S}{\tau}}\cdot u(nT_S) \qquad (6.2.2)$$

Dobljeni niz lahko ponazorimo s sliko:



Dobljeno enačbo lahko še malo uredimo:

$$x[n]=e^{-\frac{nT_S}{\tau}}=\left(e^{-\frac{T_S}{\tau}}\right)^n=\left(e^{-\frac{1\text{ ms}}{5\text{ ms}}}\right)^n=0,82^n\,;n\geq 0 \qquad (6.2.3)$$

2. Naloga

Med sabo primerjajte Fourierevo transformacijo za časovno zvezne signale in časovno diskretno Fourierevo transformacijo:

Rešitev:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt \tag{6.3.1}$$

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n] \cdot e^{-j\Omega n} \tag{6.3.2}$$

Spremembe:

$$\begin{gathered} t \to n \\ \omega \to \Omega \\ \int \to \sum \end{gathered} \tag{6.3.3}$$

3. Naloga

Izračunajte DtFT podanega časovno diskretnega signala:

$$x[n] = \begin{cases} a^n & ; \quad n \geq 0 \\ 0 & ; \quad n < 0 \end{cases}, a < 1$$

Rešitev:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} \tag{6.4.1}$$

$$X(\Omega) = \sum_{n=0}^{\infty} a^n \cdot e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(a \cdot e^{-j\Omega}\right)^n \tag{6.4.2}$$

$$q = \left(a \cdot e^{-j\Omega}\right) \tag{6.4.3}$$

$$\begin{aligned} \sum_{n=0}^{N} q^n &= q^0 + q^1 + q^2 + \ldots + q^N = \left(q^0 + q^1 + q^2 + \ldots + q^N\right) \frac{1-q}{1-q} = \\ &= \left(q^0 \cancel{+q^1} \cancel{+q^2} + \ldots \cancel{+q^N} \cancel{-q^1} \cancel{-q^2} - \ldots \cancel{-q^N} - q^{N+1}\right) \frac{1}{1-q} = \\ &= \frac{q^0 - q^{N+1}}{1-q} = \frac{1-q^{N+1}}{1-q} \end{aligned} \tag{6.4.4}$$

$$\lim_{N \to \infty} \left(\sum_{n=0}^{N} q^n \right) = \lim_{N \to \infty} \left(\frac{1-q^{N+1}}{1-q} \right) = \frac{1}{1-q} \tag{6.4.5}$$

$$X(\Omega) = \sum_{n=0}^{\infty} \left(a \cdot e^{-j\Omega}\right)^n = \frac{1}{1 - a \cdot e^{-j\Omega}} \tag{6.4.6}$$

4. Naloga

Izrazite amplitudni in fazni spekter izračunanega spektra časovno diskretnega signala:

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Rešitev:

$$\left|X(\Omega)\right| = \left|\frac{1}{1 - ae^{-j\Omega}}\right| = \left|\frac{1}{1 - \left(a\cos(\Omega) - ja\sin(\Omega)\right)}\right| = \left|\frac{1}{1 - a\cos(\Omega) + ja\sin(\Omega)}\right| \tag{6.5.1}$$

$$\left|X(\Omega)\right| = \frac{1}{\sqrt{\left(1 - a\cos(\Omega)\right)^2 + \left(a\sin(\Omega)\right)^2}} = \frac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2\cos^2(\Omega) + a^2\sin^2(\Omega)}} \tag{6.5.2}$$

$$\left|X(\Omega)\right| = \frac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2\left(\cos^2(\Omega) + \sin^2(\Omega)\right)}} = \frac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2}} \tag{6.5.3}$$

$$\phi(\Omega) = \arctan\left(\frac{\mathrm{Im}\left(X(\Omega)\right)}{\mathrm{Re}\left(X(\Omega)\right)}\right) \tag{6.5.4}$$

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}} = \frac{1}{1 - a\cos(\Omega) + ja\sin(\Omega)} \tag{6.5.5}$$

$$X(\Omega) = \frac{1 - a\cos(\Omega) + ja\sin(\Omega)}{\left(1 - a\cos(\Omega) + ja\sin(\Omega)\right)\left(1 - a\cos(\Omega) - ja\sin(\Omega)\right)} \tag{6.5.6}$$

$$X(\Omega) = \frac{1 - a\cos(\Omega) - ja\sin(\Omega)}{\left(1 - a\cos(\Omega)\right)^2 + \left(a\sin(\Omega)\right)^2} = \frac{1 - a\cos(\Omega) - ja\sin(\Omega)}{1 - 2a\cos(\Omega) + a^2} \tag{6.5.7}$$

$$\mathrm{Im}\left(X(\Omega)\right) = \frac{-a\sin(\Omega)}{1 - 2a\cos(\Omega) + a^2} \tag{6.5.8}$$

$$\mathrm{Re}\left(X(\Omega)\right) = \frac{1 - a\cos(\Omega)}{1 - 2a\cos(\Omega) + a^2} \tag{6.5.9}$$

$$\phi(\Omega) = \arctan\left(\frac{\dfrac{-a\sin(\Omega)}{1 - 2a\cos(\Omega) + a^2}}{\dfrac{1 - a\cos(\Omega)}{1 - 2a\cos(\Omega) + a^2}}\right) = \arctan\left(\frac{-a\sin(\Omega)}{1 - a\cos(\Omega)}\right) \tag{6.5.10}$$

$$\phi(\Omega) = -\arctan\left(\frac{a\sin(\Omega)}{1 - a\cos(\Omega)}\right) \tag{6.5.11}$$

5. Naloga

Izračunajte časovno diskretno Fourierevo transformacijo podanega signala. Narišite amplitudni in fazni spekter.

$$x[n]=\begin{cases}1 & ; \quad 0\le n\le 5\\ 0 & ; \quad sicer\end{cases}$$

Rešitev:

$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]\cdot e^{-j\Omega n}=\sum_{n=0}^{5}1\cdot e^{-j\Omega n}=\sum_{n=0}^{5}e^{-j\Omega n} \tag{6.6.1}$$

$$X(\Omega)=\frac{1-e^{-j\Omega 6}}{1-e^{-j\Omega}} \quad \text{(glej (6.4.4))} \tag{6.6.2}$$

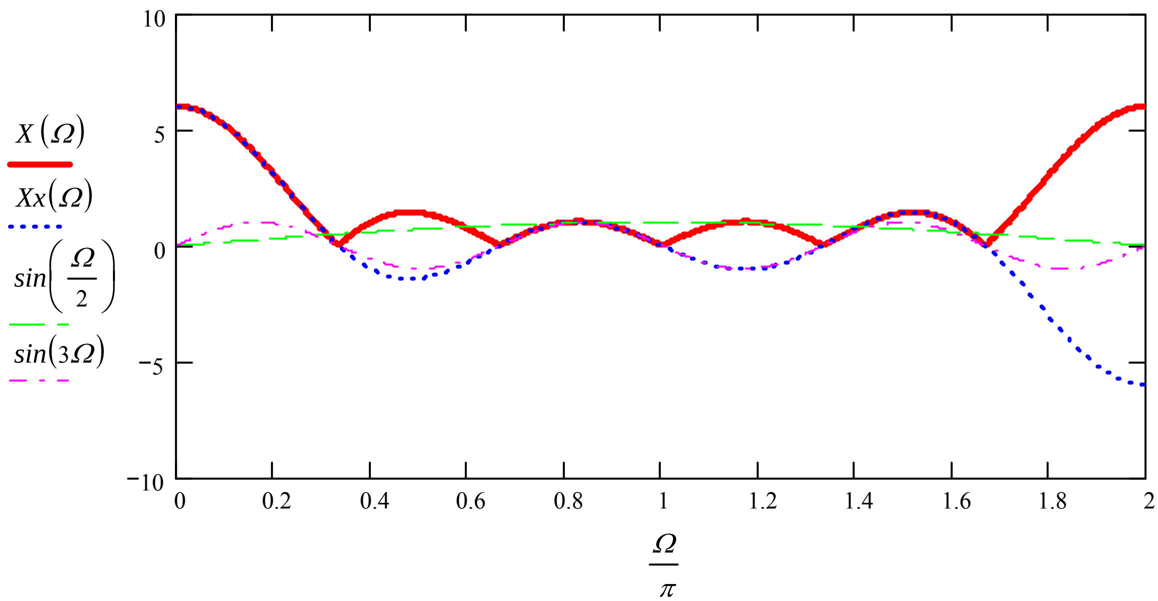
$$X(\Omega)=\frac{1-e^{-j\Omega 6}}{1-e^{-j\Omega}}=\frac{e^{-j\Omega 3}}{e^{-j\frac{\Omega}{2}}}\cdot\frac{e^{+j\Omega 3}-e^{-j\Omega 3}}{e^{+j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}=e^{-j\Omega\frac{5}{2}}\cdot\frac{j2\sin(3\Omega)}{j2\sin\left(\frac{\Omega}{2}\right)} \tag{6.6.3}$$

$$X(\Omega)=e^{-j\Omega\frac{5}{2}}\cdot\frac{\sin(3\Omega)}{\sin\left(\frac{\Omega}{2}\right)} \tag{6.6.4}$$

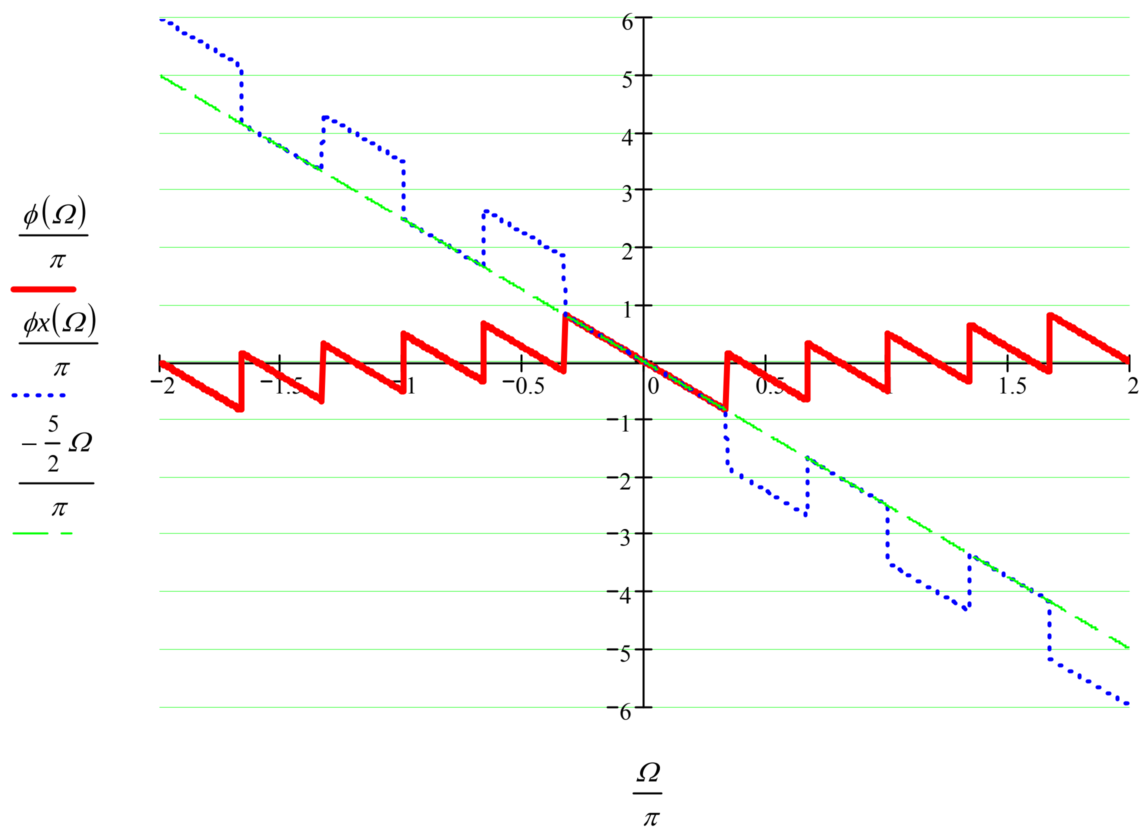
$$|X(\Omega)|=\left|\frac{\sin(3\Omega)}{\sin\left(\frac{\Omega}{2}\right)}\right| \tag{6.6.5}$$

$$\phi(\Omega)=-\frac{5}{2}\Omega+p\pi \tag{6.6.6}$$

Amplitudni spekter:



Fazni spekter:

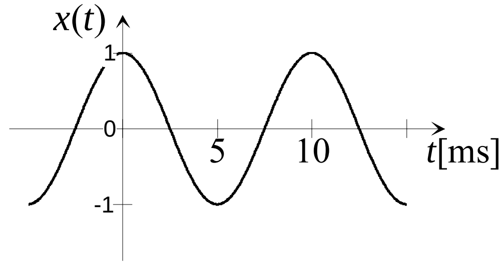


6. Naloga

Analogni signal je opisan z enačbo $x(t)=\cos(200\pi t)$. Zapišite signal, ki ga dobimo z vzorčenjem signala $x(t)$ s frekvenco $f_S = 1\,\text{kHz}$. Kolikšna je diskretna frekvenca Ω_0 dobljenega diskretnega signala?

Rešitev:

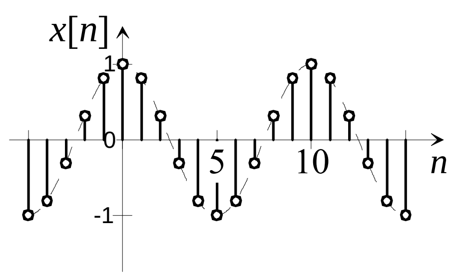
Narišimo analogni signal:



Iz frekvence vzorčenja izračunajmo čas med dvema vzorcema $T_S = 1\,\text{ms}$ in zapišimo vzorčeni signal.

$$x[n] = x(nT_S) = \cos(200\pi n T_S) = \cos(\Omega_0 n) \tag{6.7.1}$$

$$\Omega_0 = 200\pi \cdot 0{,}001 = \frac{2\pi}{10} \tag{6.7.2}$$



Diskretna frekvenca podaja razmerje med frekvenco vzorčenega signala in vzorčevalno frekvenco. V tem primeru je imel vzorčeni signal 10-krat nižjo frekvenco od frekvence vzorčenja, zato je $\Omega_0 = \frac{2\pi}{10}$.