IZPIT IZ MATEMATIKE 1
Univerzitetni študij
4. junij 2007

1. Reši enačbo
$$z^3 = -2 + 2\mathrm{i}.$$

Rešitev:

Zapišemo levo in desno stran enačbe v polarni obliki.
Velja: $z = |z|(\cos\varphi + \mathrm{i}\sin\varphi)$ oz. $z^3 = |z|^3(\cos 3\varphi + \mathrm{i}\sin 3\varphi)$.
Ker je $r = \sqrt{4+4} = 2\sqrt{2}$ in $\psi = \operatorname{arctg}\frac{-2}{2} = \operatorname{arctg}(-1) = \frac{3\pi}{4}$, je $-2 + 2\mathrm{i} = 2\sqrt{2}(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4})$.
Sledi:
$$|z|^3(\cos 3\varphi + \mathrm{i}\sin 3\varphi) = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right).$$
Dve kompleksni števili sta enaki, ko imata enak radij in enak kot. Zato rešimo enačbi: $|z|^3 = 2\sqrt{2}$ in $\cos 3\varphi = \cos\frac{3\pi}{4}$. Prva ima rešitev $|z| = \sqrt{2}$, druga pa $3\varphi = \frac{3\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$, torej $\varphi_k = \frac{\pi}{4} + \frac{2k\pi}{3}$, $k = 0, 1, 2$.
Dobimo tri kote: $\varphi_0 = \frac{\pi}{4}$, $\varphi_1 = \frac{11\pi}{12}$ in $\varphi_2 = \frac{19\pi}{12}$. Ti nam dajo tri rešitve po formuli $w_k = |z|(\cos\varphi_k + \mathrm{i}\sin\varphi_k)$, $k = 0, 1, 2, 3$:
$$\begin{aligned} w_0 &= \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = 1 + \mathrm{i}, \\ w_1 &= \sqrt{2}\left(\cos\frac{11\pi}{12} + \mathrm{i}\sin\frac{11\pi}{12}\right), \\ w_2 &= \sqrt{2}\left(\cos\frac{19\pi}{12} + \mathrm{i}\sin\frac{19\pi}{12}\right). \end{aligned}$$

2. Izračunaj limito
$$\lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{1-x}}{x}.$$

Rešitev:
$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{1-x}}{x} &= \lim_{x\to 0}\frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{x(\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x\to 0}\frac{1 + x - 1 + x}{x(\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x\to 0}\frac{2}{\sqrt{1+x} + \sqrt{1-x}} = 1 \end{aligned}$$

3. Določi parameter a tako, da se bosta tangenti v točki $x = 1$ na grafa polinomov $y = x^3 + ax^2 + 2x - 3$ in $y = 2x^2 - x - 5$ sekali pod kotom $\frac{\pi}{4}$.

   Rešitev:

   Odvajamo polinoma in izračunamo smerna koeficienta v dani točki:

$$\begin{aligned} y' = 3x^2 + 2ax + 2 \quad &\Rightarrow \quad k_1 = 5 + 2a, \\ y' = 4x - 1 \quad &\Rightarrow \quad k_2 = 3. \end{aligned}$$

   Kot med premicama (tangentama) izračunamo po formuli:

$$\mathrm{tg}\varphi = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|.$$

   Torej:

$$1 = \left| \frac{3 - 5 - 2a}{1 + 3(5 + 2a)} \right|.$$

   Zato je $|16 + 6a| = |-2 - 2a|$, oz. $|8 + 3a| = |1 + a|$.
   Rešujemo enačbo z absolutnimi vrednostmi, ki ima dve rešitvi: $a_1 = -\frac{9}{4}$, $a_2 = -\frac{7}{2}$.

4. Izračunaj integral

$$\int \frac{1 + \mathrm{tg}x}{1 - \mathrm{tg}x} \, \mathrm{d}x.$$

   Rešitev:

$$\begin{aligned} \int \frac{1 + \mathrm{tg}x}{1 - \mathrm{tg}x} \, \mathrm{d}x &= \int \frac{\cos x + \sin x}{\cos x - \sin x} \, \mathrm{d}x \\ &= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x - \sin x)^2} \, \mathrm{d}x \\ &= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x - 2 \sin x \cos x + \sin^2 x} \, \mathrm{d}x \\ &= \int \frac{\cos 2x}{1 - \sin 2x} \, \mathrm{d}x \\ &= -\frac{1}{2} \int \frac{\mathrm{d}t}{t} = -\frac{1}{2} \ln |t| = -\frac{1}{2} \ln |1 - \sin 2x| + C \end{aligned}$$

   Uvedli smo novo spremenljivko $t = 1 - \sin 2x$ z diferencialom $\mathrm{d}t = -2 \cos 2x \mathrm{d}x$.

5. Izračunaj dolžino loka krivulje

$$y = \frac{x^2 - 2\ln x}{4}$$

za $1 \leq x \leq \mathrm{e}$.

Rešitev:
Dolžino loka izračunamo po formuli:

$$s = \int_a^b \sqrt{1 + (y')^2}\mathrm{d}x.$$

Odvajamo:

$$y' = \frac{x^2 - 1}{2x}$$

in dobimo

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left(\frac{x^2 - 1}{2x}\right)^2} = \sqrt{\frac{(x^2+1)^2}{4x^2}} = \frac{x^2 + 1}{2x}.$$

Torej:

$$\begin{aligned} s &= \int_1^{\mathrm{e}} \frac{x^2 + 1}{2x}\mathrm{d}x \\ &= \frac{1}{2}\int_1^{\mathrm{e}} \left(x + \frac{1}{x}\right)\mathrm{d}x \\ &= \frac{1}{2}\left(\frac{x^2}{2}\Big|_1^{\mathrm{e}} + \ln x\Big|_1^{\mathrm{e}}\right) \\ &= \frac{\mathrm{e}^2 + 1}{4} \end{aligned}$$