Deveta vaja iz matematike 1

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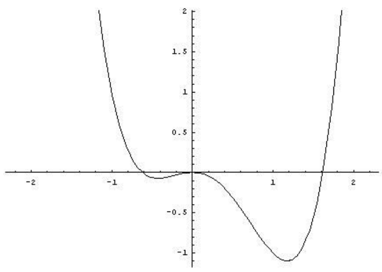
1. Čimbolj natančno nariši grafe naslednjih funkcij!

a $f(x) = x^4 - x^3 - x^2$

To je polinomska funkcija. Ničle: $x^4 - x^3 - x^2 = x^2(x^2 - x - 1) = 0$. Dvojno ničlo imamo v $x_{1,2} = 0$, drugi dve ničli pa dobimo po formuli za kvadratno enačbo: $x_{3,4} = \frac{1 \pm \sqrt{5}}{2}$.
Ekstremi: $f'(x) = 4x^3 - 3x^2 - 2x = x(4x^2 - 3x - 2) = 0$. Eno stacionarno točko dobimo v $x_1 = 0$, drugi dve pa iz kvadratne enačbe: $x_{2,3} = \frac{3 \pm \sqrt{41}}{8}$. Drugi odvod: $f''(x) = 12x^2 - 6x - 2$. Ker je $f''(0) < 0$, imamo v tej točki lokalni maksimum, v ostalih dveh stacionarnih točkah pa lokalni minimum, saj je tam drugi odvod pozitiven.



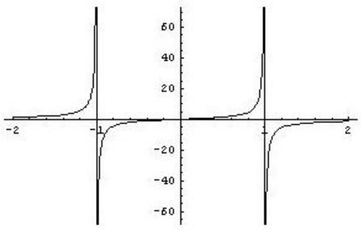
Slika 1: Graf funkcije $f(x) = x^4 - x^3 - x^2$

b $f(x) = \frac{2x}{1-x^2}$

To je racionalna funkcija. Ničla: $x = 0$. Poli: $x_1 = -1$, $x_2 = 1$. Vodoravna asimptota: $y = 0$. Odvod:

$$f'(x) = \frac{2(1-x^2) - 2x \cdot (-2x)}{(1-x^2)^2} = \frac{2 + 2x^2}{(1-x^2)^2} > 0.$$

Odvod je povsod strogo pozitiven. To pomeni, da stacionarnih točk ni, torej tudi ne ekstremov, funkcija pa je povsod strogo naraščajoča.



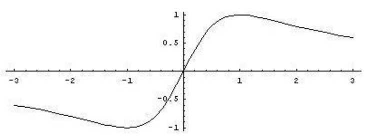
Slika 2: Graf funkcije $f(x) = \frac{2x}{1-x^2}$

c $f(x) = \frac{2x}{1+x^2}$

To je racionalna funkcija. Ničla: $x = 0$. Polov ni. Vodoravna asimptota: $y = 0$. Odvod:

$$f'(x) = \frac{2(1+x^2) - 2x \cdot 2x}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}.$$

Odvod je enak 0 v točkah $x_1 = -1$ in $x_2 = 1$. Ker je $f'(-2) < 0$, $f'(0) > 0$ in $f'(2) < 0$, je v točki $x_1 = -1$ minimum, v točki $x_2 = 1$ pa maksimum. Velja še: $f(-1) = -1$ in $f(1) = 1$.
Funkcija je naraščajoča na intervalu $(-1, 1)$ in padajoča na intervalu $(-\infty, -1) \cup (1, \infty)$.



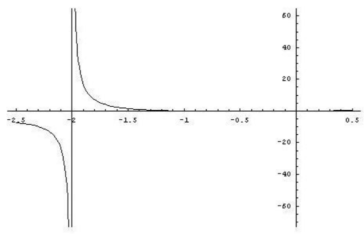
Slika 3: Graf funkcije $f(x) = \frac{2x}{1+x^2}$

d $f(x) = \frac{x^2+x}{x+2}$

To je racionalna funkcija. Ničle: $x(x+1) = 0$, torej $x_1 = 0$ in $x_2 = -1$. Pol: $x = -2$. Poševna asimptota: $y = x - 1$ (delimo polinoma in velja $x^2 + x = (x-1)(x+2) + 2$). Odvod:

$$f'(x) = \frac{(2x+1)(x+2) - (x^2+x)}{(x+2)^2} = \frac{x^2+4x+2}{(x+2)^2}.$$

Odvod je enak 0 v točkah $x_{1,2} = \frac{-4\pm\sqrt{8}}{2} = -2 \pm \sqrt{2}$. Ker je $f'(-4) > 0$ in $f'(-3) < 0$, je v točki $x_1 = -2-\sqrt{2}$ lokalni maksimum. Ker je $f'(-1) < 0$ in $f'(0) > 0$, je v točki $x_2 = -2+\sqrt{2}$ lokalni minimum.



Slika 4: Graf funkcije $f(x) = \frac{x^2+x}{x+2}$

e $f(x) = 2\sin\left(\frac{x}{2} - \frac{2\pi}{3}\right)$

Ničle:

$$\begin{aligned} \frac{x}{2} - \frac{2\pi}{3} &= k\pi, \qquad k \in \mathbb{Z} \\ x_k &= \frac{4\pi}{3} + 2k\pi \end{aligned}$$

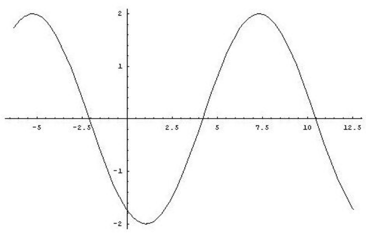
Nekaj ničel: $x_{-1} = -\frac{2\pi}{3}$, $x_0 = \frac{4\pi}{3}$, $x_1 = \frac{10\pi}{3}$.
Maksimumi:

$$\begin{aligned} \frac{x}{2} - \frac{2\pi}{3} &= \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z} \\ x_k &= \frac{7\pi}{3} + 4k\pi \end{aligned}$$

Nekaj maksimumov: $x_{-1} = -\frac{5\pi}{3}$, $x_0 = \frac{7\pi}{3}$.
Minimumi:

$$\begin{aligned} \frac{x}{2} - \frac{2\pi}{3} &= -\frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z} \\ x_k &= \frac{\pi}{3} + 4k\pi \end{aligned}$$

Nekaj minimumov: $x_0 = \frac{\pi}{3}$.



Slika 5: Graf funkcije $f(x) = 2\sin\left(\frac{x}{2} - \frac{2\pi}{3}\right)$

INTEGRALI

Pravila za integriranje:

$$\begin{aligned}
\int (f(x) \pm g(x))\mathrm{d}x &= \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x \\
\int (C \cdot f(x))\mathrm{d}x &= C \cdot \int f(x)\mathrm{d}x \quad (C = \text{konst.}) \\
\int u(x)\mathrm{d}v(x) &= u(x)v(x) - \int v(x)\mathrm{d}u(x) \quad (\text{per partes}) \\
\int f(x)\mathrm{d}x &= \int f(g(t))g'(t)\mathrm{d}t \quad (\text{uvedba nove spremenljivke}) \\
& \quad (x = g(t), \quad \mathrm{d}x = g'(t)\mathrm{d}t)
\end{aligned}$$

Tabela elementarnih integralov: ($a \in \mathbb{R}$, $C = \text{konst.}$)

$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \sin x \mathrm{d}x = -\cos x + C$
$\int \frac{\mathrm{d}x}{x} = \ln\lvert x\rvert + C$	$\int \cos x \mathrm{d}x = \sin x + C$
$\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + C$	$\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C$
$\int \frac{\mathrm{d}x}{\cos^2 x} = \mathrm{tg}x + C$	$\int \mathrm{sh}x \mathrm{d}x = \mathrm{ch}x + C$
$\int \frac{\mathrm{d}x}{\sin^2 x} = -\mathrm{ctg}x + C$	$\int \mathrm{ch}x \mathrm{d}x = \mathrm{sh}x + C$
$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\mathrm{arctg}\frac{x}{a} + C$	$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$
$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln x + \sqrt{x^2 \pm a^2} + C$

2. S pomočjo tabele elementarnih integralov izračunaj naslednje integrale!

a

$$\int (1-x^2)(1-x)\mathrm{d}x = \int (1-x-x^2+x^3)\mathrm{d}x = \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x + C$$

b

$$\begin{aligned}\int \mathrm{tg}^2 x \mathrm{d}x &= \int \frac{\sin^2 x}{\cos^2 x}\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x}\mathrm{d}x \\ &= \int \left(\frac{1}{\cos^2 x} - 1\right)\mathrm{d}x = \mathrm{tg}x - x + C\end{aligned}$$

c

$$\int \frac{x^4}{1+x^2}\mathrm{d}x = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right)\mathrm{d}x = \frac{x^3}{3} - x - \mathrm{arctg}x + C$$

V prvem koraku smo delili polinoma in dobili: $x^4 = (x^2+1)(x^2-1)+1$.

3. Izračunaj naslednje integrale z uvedbo nove spremenljivke!

a

$$\begin{aligned}\int \frac{x}{3x+2}\mathrm{d}x &= \int \frac{t-2}{3t}\frac{\mathrm{d}t}{3} = \frac{1}{9}\int \left(1 - \frac{2}{t}\right)\mathrm{d}t \\ &= \frac{1}{9}\left(t - 2\ln|t|\right) = \frac{1}{9}\left(3x+2-2\ln|3x+2|\right) + C\end{aligned}$$

Uvedli smo novo spremenljivko $t = 3x+2$ z diferencialom $\mathrm{d}t = 3\mathrm{d}x$, oz. $\mathrm{d}x = \frac{\mathrm{d}t}{3}$.

b

$$\int \sqrt[3]{1-3x}\mathrm{d}x = -\frac{1}{3}\int t^{\frac{1}{3}}\mathrm{d}t = -\frac{1}{3}\frac{t^{\frac{4}{3}}}{\frac{4}{3}} = -\frac{1}{4}(1-3x)^{\frac{4}{3}} + C$$

Uvedli smo novo spremenljivko $t = 1-3x$ z diferencialom $\mathrm{d}t = -3\mathrm{d}x$, oz. $\mathrm{d}x = -\frac{\mathrm{d}t}{3}$.

c

$$\begin{aligned}\int \left(\mathrm{e}^{-2x} + 3\mathrm{e}^{3x}\right)\mathrm{d}x &= \int \mathrm{e}^{-2x}\mathrm{d}x + 3\int \mathrm{e}^{3x}\mathrm{d}x \\ &= -\frac{1}{2}\int \mathrm{e}^t\mathrm{d}t + \int \mathrm{e}^u\mathrm{d}u = -\frac{1}{2}\mathrm{e}^t + \mathrm{e}^u \\ &= -\frac{1}{2}\mathrm{e}^{-2x} + \mathrm{e}^{3x} + C\end{aligned}$$

Uvedli smo novi spremenljivki $t = -2x$ z diferencialom $\mathrm{d}t = -2\mathrm{d}x$, oz. $\mathrm{d}x = -\frac{\mathrm{d}t}{2}$ in $u = 3x$ z diferencialom $\mathrm{d}u = 3\mathrm{d}x$, oz. $\mathrm{d}x = \frac{\mathrm{d}u}{3}$.

d

$$\int \frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}\mathrm{d}x = \int \frac{\mathrm{d}t}{1+t^2} = \mathrm{arctg}t = \mathrm{arctge}^x + C$$

Uvedli smo novo spremenljivko $t = \mathrm{e}^x$ z diferencialom $\mathrm{d}t = \mathrm{e}^x\mathrm{d}x$.

e

$$\int \mathrm{tg}x\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}t}{t} = -\ln|t| = -\ln|\cos x| + C$$

Uvedli smo novo spremenljivko $t = \cos x$ z diferencialom $\mathrm{d}t = -\sin x\mathrm{d}x$.

f

$$\begin{aligned}\int \sin^2 x\mathrm{d}x &= \int \frac{1-\cos 2x}{2}\mathrm{d}x = \frac{1}{4}\int(1-\cos t)\mathrm{d}t\\ &= \frac{1}{4}(t-\sin t) = \frac{1}{4}(2x-\sin 2x) + C\end{aligned}$$

Uporabimo formulo za sinus polovičnega kota: $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$. Uvedli smo novo spremenljivko $t = 2x$ z diferencialom $\mathrm{d}t = 2\mathrm{d}x$, oz. $\mathrm{d}x = \frac{\mathrm{d}t}{2}$.

g

$$\int \mathrm{e}^{\sin x}\cos x\mathrm{d}x = \int e^t\mathrm{d}t = e^t = \mathrm{e}^{\sin x} + C$$

Uvedli smo novo spremenljivko $t = \sin x$ z diferencialom $\mathrm{d}t = \cos x\mathrm{d}x$.

h

$$\int \frac{\ln^2 x}{x}\mathrm{d}x = \int t^2\mathrm{d}t = \frac{t^3}{3} = \frac{\ln^3 x}{3} + C$$

Uvedli smo novo spremenljivko $t = \ln x$ z diferencialom $\mathrm{d}t = \frac{\mathrm{d}x}{x}$.

i

$$\begin{aligned}\int \frac{\mathrm{d}x}{x\ln x\ln(\ln x)} &= \int \frac{\mathrm{d}t}{t\ln t} = \int \frac{\mathrm{d}u}{u}\\ &= \ln|u| = \ln|\ln t| = \ln|\ln(\ln x)| + C\end{aligned}$$

Uvedli smo novi spremenljivki $t = \ln x$ z diferencialom $\mathrm{d}t = \frac{\mathrm{d}x}{x}$ in $u = \ln t$ z diferencialom $\mathrm{d}u = \frac{\mathrm{d}t}{t}$.

j

$$\int \frac{\mathrm{arctg}\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x = \int 2t\mathrm{d}t = 2\cdot\frac{t^2}{2} = \mathrm{arctg}^2\sqrt{x} + C$$

Uvedli smo novo spremenljivko $t = \mathrm{arctg}\sqrt{x}$ z diferencialom $\mathrm{d}t = \frac{\mathrm{d}x}{2\sqrt{x}(1+x)}$.

k

$$\begin{aligned}\int x^3\sqrt{1-x^2}\mathrm{d}x &= \int (1-t^2)t(-t)\mathrm{d}t = \int (t^4-t^2)\mathrm{d}t\\ &= \frac{t^5}{5}-\frac{t^3}{3} = \frac{\sqrt{(1-x^2)^5}}{5}-\frac{\sqrt{(1-x^2)^3}}{3}+C\end{aligned}$$

Uvedli smo novo spremenljivko $t^2 = 1-x^2$ z diferencialom $2t\mathrm{d}t = -2x\mathrm{d}x$, oz. $x\mathrm{d}x = -t\mathrm{d}t$ in $x^2 = 1-t^2$.

4. S pomočjo integracije po delih (per partes) izračunaj naslednje integrale!

a

$$\begin{aligned}\int x^3\mathrm{e}^x\mathrm{d}x &= x^3\mathrm{e}^x - 3\int x^2\mathrm{e}^x\mathrm{d}x\\ &= x^3\mathrm{e}^x - 3x^2\mathrm{e}^x + 6\int x\mathrm{e}^x\mathrm{d}x\\ &= x^3\mathrm{e}^x - 3x^2\mathrm{e}^x + 6x\mathrm{e}^x - 6\int \mathrm{e}^x\mathrm{d}x\\ &= x^3\mathrm{e}^x - 3x^2\mathrm{e}^x + 6x\mathrm{e}^x - 6\mathrm{e}^x + C\end{aligned}$$

Integracijo per partes smo uporabili trikrat. Prvi enačaj: $u = x^3$, $\mathrm{d}v = \mathrm{e}^x\mathrm{d}x$ in zato $\mathrm{d}u = 3x^2\mathrm{d}x$, $v = \mathrm{e}^x$. Drugi enačaj: $u = x^2$, $\mathrm{d}v = \mathrm{e}^x\mathrm{d}x$ in zato $\mathrm{d}u = 2x\mathrm{d}x$, $v = \mathrm{e}^x$. Tretji enačaj: $u = x$, $\mathrm{d}v = \mathrm{e}^x\mathrm{d}x$ in zato $\mathrm{d}u = \mathrm{d}x$, $v = \mathrm{e}^x$.

b

$$\begin{aligned}\int x\ln(x-1)\mathrm{d}x &= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int\frac{x^2}{x-1}\mathrm{d}x\\ &= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int\left(x+1+\frac{1}{x-1}\right)\mathrm{d}x\\ &= \frac{x^2}{2}\ln(x-1) - \frac{x^2}{4} - \frac{x}{2} - \frac{1}{2}\ln|x-1| + C\end{aligned}$$

Integral smo rešili per partes: $u = \ln(x-1)$, $\mathrm{d}v = x\mathrm{d}x$ in zato $\mathrm{d}u = \frac{\mathrm{d}x}{x-1}$, $v = \frac{x^2}{2}$. Poleg tega smo delili polinoma in dobili $\frac{x^2}{x-1} = x+1+\frac{1}{x-1}$.

c

$$\begin{aligned}\int \arcsin x \mathrm{d}x &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x \\ &= x\arcsin x + \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} \\ &= x\arcsin x + t^{\frac{1}{2}} \\ &= x\arcsin x + \sqrt{1-x^2} + C\end{aligned}$$

Začetni integral rešimo z metodo per partes: $u = \arcsin x$, $\mathrm{d}v = \mathrm{d}x$ in zato $\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}$, $v = x$. Integral, ki ga tako dobimo, rešimo s substitucijo $t = 1 - x^2$ z diferencialom $\mathrm{d}t = -2x\mathrm{d}x$.

d $I = \int \mathrm{e}^{ax}\cos bx\mathrm{d}x, \quad a, b \in \mathbb{R}$

Integral izračunamo z dvakratno uporabo integracije po delih:

$$\begin{aligned}\int \mathrm{e}^{ax}\cos bx\mathrm{d}x &= \frac{1}{a}\mathrm{e}^{ax}\cos bx + \frac{b}{a}\int \mathrm{e}^{ax}\sin bx\mathrm{d}x \\ &= \frac{1}{a}\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\mathrm{e}^{ax}\sin bx - \frac{b^2}{a^2}\int \mathrm{e}^{ax}\cos bx\mathrm{d}x\end{aligned}$$

Prvi enačaj: $u = \cos bx$, $\mathrm{d}v = \mathrm{e}^{ax}\mathrm{d}x$ in zato $\mathrm{d}u = -b\sin bx\mathrm{d}x$, $v = \frac{1}{a}\mathrm{e}^{ax}$. Drugi enačaj: $u = \sin bx$, $\mathrm{d}v = \mathrm{e}^{ax}\mathrm{d}x$ in zato $\mathrm{d}u = b\cos bx\mathrm{d}x$, $v = \frac{1}{a}\mathrm{e}^{ax}$.
Sedaj rešimo enačbo:

$$\begin{aligned}I &= \frac{1}{a}\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\mathrm{e}^{ax}\sin bx - \frac{b^2}{a^2}I \\ \left(1+\frac{b^2}{a^2}\right)I &= \frac{1}{a}\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\mathrm{e}^{ax}\sin bx \\ I &= \frac{\frac{1}{a}\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\mathrm{e}^{ax}\sin bx}{1+\frac{b^2}{a^2}} + C\end{aligned}$$