Deseta vaja iz matematike 1

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1. Izračunaj naslednje integrale racionalnih funkcij!

a

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2-x-2} &= \int \left(\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\right)\mathrm{d}x \\ &= \frac{1}{3}\int \frac{\mathrm{d}x}{x-2} - \frac{1}{3}\int \frac{\mathrm{d}x}{x+1} \\ &= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C\end{aligned}$$

V prvem koraku smo dali prvotni ulomek na parcialne ulomke:

$$\frac{1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{(A+B)x + A - 2B}{x^2-x-2}.$$

Dobimo sistem enačb $A+B=0$, $A-2B=1$, ki ima rešitev $A=\frac{1}{3}$ in $B=-\frac{1}{3}$.

b

$$\begin{aligned}&\int \frac{x}{(x+1)(x+2)(x+3)}\mathrm{d}x \\ &= \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{2}{x+2} + \frac{-\frac{3}{2}}{x+3}\right)\mathrm{d}x \\ &= -\frac{1}{2}\int \frac{\mathrm{d}x}{x+1} + 2\int \frac{\mathrm{d}x}{x+2} - \frac{3}{2}\int \frac{\mathrm{d}x}{x+3} \\ &= -\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3| + C\end{aligned}$$

V prvem koraku smo dali prvotni ulomek na parcialne ulomke:

$$\begin{aligned}&\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\ =\ &\frac{(A+B+C)x^2 + (5A+4B+3C)x + 6A+3B+2C}{(x+1)(x+2)(x+3)}.\end{aligned}$$

Dobimo sistem enačb $A+B+C=0$, $5A+4B+3C=1$, $6A+3B+2C=0$, ki ima rešitev $A=-\frac{1}{2}$, $B=2$ in $C=-\frac{3}{2}$.

c

$$\begin{aligned}
&\int \frac{1}{x^3+x^2+2x+2}\mathrm{d}x \\
= &\int \left(\frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+2}\right)\mathrm{d}x \\
= &\frac{1}{3}\int \frac{\mathrm{d}x}{x+1} - \frac{1}{3}\int \frac{x\mathrm{d}x}{x^2+2} + \frac{1}{3}\int \frac{\mathrm{d}x}{x^2+2} \\
= &\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+2| + \frac{1}{3\sqrt{2}}\mathrm{arctg}\frac{x}{\sqrt{2}} + C
\end{aligned}$$

V prvem koraku smo dali prvotni ulomek na parcialne ulomke:

$$\begin{aligned}
\frac{1}{x^3+x^2+2x+2} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+2} \\
&= \frac{(A+B)x^2+(B+C)x+2A+C}{x^3+x^2+2x+2}.
\end{aligned}$$

Dobimo sistem enačb $A+B=0$, $B+C=0$, $2A+C=1$, ki ima rešitev $A=\frac{1}{3}$, $B=-\frac{1}{3}$ in $C=\frac{1}{3}$.
Integral $\int \frac{x\mathrm{d}x}{x^2+2}$ rešimo posebej z uvedbo nove spremenljivke $t=x^2+2$ in zato $\mathrm{d}t=2x\mathrm{d}x$.

$$\int \frac{x\mathrm{d}x}{x^2+2} = \frac{1}{2}\int \frac{\mathrm{d}t}{t} = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|x^2+2|$$

d $\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x$

Integral rešimo z nastavkom za rešitev:

$$\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x = \frac{A}{x+1} + B\ln|x+1| + C\ln|x-1|$$

V nastavku imamo dva logaritemska člena; za vsak linearni faktor po enega, prvi člen pa ustreza drugi potenci enega izmed faktorjev. Odvajamo nastavek in dobimo:

$$\begin{aligned}
\frac{x^2+1}{(x+1)^2(x-1)} &= \frac{-A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-1} \\
&= \frac{-A(x-1)+B(x+1)(x-1)+C(x+1)^2}{(x+1)^2(x-1)} \\
&= \frac{(B+C)x^2+(2C-A)x+A-B+C}{(x+1)^2(x-1)}
\end{aligned}$$

Dobimo sistem enačb $B+C=1$, $2C-A=0$, $A-B+C=1$, ki ima rešitev $A=1$, $B=\frac{1}{2}$ in in $C=\frac{1}{2}$.
Integral je torej enak:

$$\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x = \frac{1}{x+1} + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C.$$

DOLOČENI INTEGRALI

<u>Newton-Leibnitzova formula:</u>

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \qquad F(x) = \int f(x)\mathrm{d}x$$

2. Izračunaj naslednje določene integrale!

a

$$\begin{aligned}\int_1^2 \left(x^2 + \frac{1}{x^4}\right)\mathrm{d}x &= \int_1^2 \left(x^2 + x^{-4}\right)\mathrm{d}x = \left(\frac{x^3}{3} + \frac{x^{-3}}{-3}\right)\Bigg|_{x=1}^{2} \\ &= \frac{8}{3} - \frac{1}{3} - \frac{1}{24} + \frac{1}{3} = \frac{21}{8}\end{aligned}$$

b

$$\int_0^{\frac{\pi}{4}} \sin(4x)\mathrm{d}x = \frac{1}{4}\int_0^{\pi} \sin t\mathrm{d}t = -\frac{1}{4}\cos t\Big|_0^{\pi} = -\frac{1}{4}(-1-1) = \frac{1}{2}$$

Uporabili smo substitucijo: $t = 4x$, $\mathrm{d}t = 4\mathrm{d}x$. Nove meje: $x = 0 \Rightarrow t = 0$ in $x = \frac{\pi}{4} \Rightarrow t = \pi$.

c

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{\cos^4 x}\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{2\sin x\cos x}{\cos^4 x}\mathrm{d}x = 2\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\mathrm{d}x \\ &= -2\int_1^{\frac{\sqrt{2}}{2}} \frac{\mathrm{d}t}{t^3} = 2\int_{\frac{\sqrt{2}}{2}}^1 t^{-3}\mathrm{d}t \\ &= 2\frac{t^{-2}}{-2}\Bigg|_{\frac{\sqrt{2}}{2}}^1 = -1 + 2 = 1\end{aligned}$$

Uporabili smo substitucijo: $t = \cos x$, $\mathrm{d}t = -\sin x\mathrm{d}x$. Nove meje: $x = 0 \Rightarrow t = 1$ in $x = \frac{\pi}{4} \Rightarrow t = \frac{\sqrt{2}}{2}$.
Upoštevali smo tudi pravilo: $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$.

d

$$\begin{aligned}
\int_{-2}^{2} x^2 \sin x \mathrm{d}x &= -x^2 \cos x\Big|_{-2}^{2} + 2\int_{-2}^{2} x\cos x \mathrm{d}x \\
&= -4\cos 2 + 4\cos 2 + 2x\sin x\Big|_{-2}^{2} - 2\int_{-2}^{2} \sin x \mathrm{d}x \\
&= 4\sin 2 - 4\sin 2 + 2\cos x\Big|_{-2}^{2} \\
&= 2\cos 2 - 2\cos 2 = 0
\end{aligned}$$

Integral izračunamo dvakrat per partes po formuli:

$$\int_a^b u\mathrm{d}v = uv\Big|_a^b - \int_a^b v\mathrm{d}u.$$

Prvič vzamemo $u = x^2$, $\mathrm{d}v = \sin x\mathrm{d}x$ in zato $\mathrm{d}u = 2x\mathrm{d}x$, $v = -\cos x$, drugič pa $u = x$, $\mathrm{d}v = \cos x\mathrm{d}x$ in zato $\mathrm{d}u = \mathrm{d}x$, $v = \sin x$. Opazimo še, da je integral lihe funkcije na simetričnem intervalu enak 0.

e

$$\begin{aligned}
\int_1^{\mathrm{e}} x^2 \ln x \mathrm{d}x &= \frac{1}{3}x^3 \ln x\Big|_1^{\mathrm{e}} - \frac{1}{3}\int_1^{\mathrm{e}} x^2 \mathrm{d}x \\
&= \frac{1}{3}\mathrm{e}^3 \underbrace{\ln \mathrm{e}}_{=1} - \frac{1}{3}\underbrace{\ln 1}_{=0} - \frac{1}{9}x^3\Big|_1^{\mathrm{e}} \\
&= \frac{1}{3}\mathrm{e}^3 - \frac{1}{9}\mathrm{e}^3 + \frac{1}{9} = \frac{1}{9}(2\mathrm{e}^3 + 1)
\end{aligned}$$

Integral smo izračunali per partes, kjer vzamemo $u = \ln x$, $\mathrm{d}v = x^2\mathrm{d}x$ in zato $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, $v = \frac{x^3}{3}$.

f $I = \int_0^{\ln 5} \frac{\sqrt{\mathrm{e}^x - 1}}{1 + 3\mathrm{e}^{-x}}\mathrm{d}x$

Izračunajmo najprej nedoločeni integral:

$$\begin{aligned}
\int \frac{\sqrt{\mathrm{e}^x - 1}}{1 + 3\mathrm{e}^{-x}}\mathrm{d}x &= \int \frac{\mathrm{e}^x\sqrt{\mathrm{e}^x - 1}}{\mathrm{e}^x + 3}\mathrm{d}x = 2\int \frac{t^2}{t^2 + 4}\mathrm{d}t \\
&= 2\int \left(1 - \frac{4}{t^2 + 4}\right)\mathrm{d}t = 2t - 4\mathrm{arctg}\frac{t}{2} \\
&= 2\sqrt{\mathrm{e}^x - 1} - 4\mathrm{arctg}\frac{\sqrt{\mathrm{e}^x - 1}}{2} + C
\end{aligned}$$

Napravili smo substitucijo: $t^2 = \mathrm{e}^x - 1$, $2t\mathrm{d}t = \mathrm{e}^x\mathrm{d}x$.
Sedaj vstavimo meje:

$$I = \left(2\sqrt{\mathrm{e}^x - 1} - 4\mathrm{arctg}\frac{\sqrt{\mathrm{e}^x - 1}}{2}\right)\Bigg|_0^{\ln 5} = 4 - 4\mathrm{arctg}1 = 4 - \pi$$

g $I = \int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\mathrm{d}x$

Izračunajmo najprej nedoločeni integral.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{1+x^2}}\mathrm{d}x &= \int \frac{\frac{1}{\cos^2 t}}{\mathrm{tg}^2 t \cdot \frac{1}{\cos t}}\mathrm{d}t = \int \frac{\cos t}{\sin^2 t}\mathrm{d}t \\ &= \int \frac{\mathrm{d}u}{u^2} = \int u^{-2}\mathrm{d}u \\ &= -u^{-1} = -\frac{1}{\sin t} = -\frac{\sqrt{1+x^2}}{x}\end{aligned}$$

Uporabili smo dve substituciji, in sicer najprej $x = \mathrm{tg}t$, torej $\mathrm{d}x = \frac{1}{\cos^2 t}\mathrm{d}t$ in

$$\sqrt{1+x^2} = \sqrt{1+\mathrm{tg}^2 t} = \sqrt{\frac{1}{\cos^2 t}} = \frac{1}{\cos t}.$$

Nato pa še substitucijo $u = \sin t$, torej $\mathrm{d}u = \cos t\mathrm{d}t$. Nazadnje opazimo še, da velja $\sin t = \mathrm{tg}t \cdot \cos t = \frac{x}{\sqrt{1+x^2}}$.
Vstavimo še meje in dobimo:

$$I = \left(-\frac{\sqrt{1+x^2}}{x}\right)\Bigg|_1^2 = \sqrt{2} - \frac{\sqrt{5}}{2}.$$

Opomba: Integrali z $\sqrt{1-x^2}$, $\sqrt{x^2-1}$ in $\sqrt{x^2+1}$ se prevedejo na integrale trigonometrijskih funkcij s substitucijami:

$$\begin{aligned} R(x, \sqrt{1-x^2}) &\Rightarrow x = \sin t \,\text{ali}\, x = \cos t, \\ R(x, \sqrt{x^2+1}) &\Rightarrow x = \mathrm{tg}t \,\text{ali}\, x = \mathrm{sh}t, \\ R(x, \sqrt{x^2-1}) &\Rightarrow x = \frac{1}{\cos t}.\end{aligned}$$

3. Izračunaj nepravi integral $I = \int_1^\infty \frac{1}{(x^2+1)(x^2+3)}\mathrm{d}x$!

   Najprej izračunajmo nedoločeni integral. To je integral racionalne funkcije.

$$\begin{aligned}\int \frac{1}{(x^2+1)(x^2+3)}\mathrm{d}x &= \int \left(\frac{\frac{1}{2}}{x^2+1} - \frac{\frac{1}{2}}{x^2+3}\right)\mathrm{d}x \\ &= \frac{1}{2}\int \frac{\mathrm{d}x}{x^2+1} - \frac{1}{2}\int \frac{\mathrm{d}x}{x^2+3} \\ &= \frac{1}{2}\mathrm{arctg}x - \frac{1}{2\sqrt{3}}\mathrm{arctg}\frac{x}{\sqrt{3}} + C\end{aligned}$$

V prvem koraku smo dali prvotni ulomek na parcialne ulomke:

$$\begin{aligned}
\frac{1}{(x^2+1)(x^2+3)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3} \\
&= \frac{(Ax+B)(x^2+3)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+3)} \\
&= \frac{(A+C)x^3+(B+D)x^2+(3A+C)x+3B+D}{(x^2+1)(x^2+3)}
\end{aligned}$$

Dobimo sistem enačb $A+C=0$, $B+D=0$, $3A+C=0$, $3B+D=1$, ki ima rešitev $A=0$, $C=0$, $B=\frac{1}{2}$ in $D=-\frac{1}{2}$.
Sedaj pa vstavimo še meje:

$$\begin{aligned}
I &= \left(\frac{1}{2}\mathrm{arctg}x - \frac{1}{2\sqrt{3}}\mathrm{arctg}\frac{x}{\sqrt{3}}\right)\Bigg|_1^\infty \\
&= \frac{1}{2}\mathrm{arctg}\infty - \frac{1}{2\sqrt{3}}\mathrm{arctg}\infty - \frac{1}{2}\mathrm{arctg}1 + \frac{1}{2\sqrt{3}}\mathrm{arctg}\frac{1}{\sqrt{3}} \\
&= \frac{\pi}{4} - \frac{\pi}{4\sqrt{3}} - \frac{\pi}{8} + \frac{\pi}{12\sqrt{3}} = \frac{\pi}{8} - \frac{\pi}{6\sqrt{3}}
\end{aligned}$$