MATEMATIKA I
zapiski z avditornih vaj

Šolsko leto 2007 / 2008
Izvajalec Andrej Perne

Avtor dokumenta Žiga Bezjak
Sodelavci Blaž Potočnik, Aljoša Praznik

UREJANJE DOKUMENTA

VERZIJA	01	**REVIZIJA**	01
DATUM	1. 3. 2009		

ZADNJI POPRAVLJAL /
PREGLEDAL Blaž Potočnik, Aljoša Praznik

OPOMBE

POPRAVKI

MAT I- V
9. 10. 2007

MA 1- PERNE VAJE
07/08

MNOŽICE:

$A \subset B$, če za vsak $x \in A$ velja da je $x \in B$

$A = B \iff A \subset B$ in $B \subset A$

$A \cup B = \{x,\ x \in A$ ali $x \in B\}$ 

$A \cap B = \{x,\ x \in A$ in $x \in B\}$ 

$A - B = \{x,\ x \in A$ in $x \notin B\}$ 

$A^c = \bar{A} = \{x,\ x \notin A\}$

A in B sta **disjunktni**, če je $A \cap B = \emptyset$

$A - B = A^c \cap B^c \qquad (A^c)^c = A$

$A \cap B = B \cap A$

$A \cup B = B \cup A$

$A \cap (B \cap C) = (A \cap B) \cap C$

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cup A^c = U$

$A \cap A^c = \emptyset$

$A \cap \emptyset = \emptyset$

$A \cup \emptyset = A$

$A \cup U = U$

$A \cap U = A$

De Morgan:

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

Kartezični produkt:

$A \times B \{(x,y),\ x \in A$ in $y \in B\}$ $\quad |A| = m,\ |B| = n \quad |A \times B| = m \cdot n$

Potenčna množica

$PA = \{X,\ X \subseteq A\}$

$m A = |A| = n \implies |P(A)| = 2^n$

1. $U = \mathbb{N}$ zapišemo 3 množice

$A = \{2n,\ n \in \mathbb{N}\},\ n \leq 4\}$

$A = \{2, 4, 6, 8\}$

$B = \{n,\ n \in \mathbb{N}\}$ n-praštevilo, $n < 7\}$

$B = \{2, 3, 5,\}$

$C = \{n,\ n$-liho, $n < 10\}$

$C = \{1, 3, 5, 7, 9\}$

2. Zapiši naslednje množice

$A \cap B = \{2\}$

$B \cup C = \{1, 2, 3, 5, 7, 9\}$

$B \setminus C = \{2\}$

$A \times B = \{(2,2); (2,3); (2,5); (4,2); (4,3); (4,5); (6,2); (6,3); (6,5); (8,2); (8,3); (8,5)\}$

$P(A) = \{\emptyset, \{2\}, \{4\}, \{6\}, \{8\}, \{2,4\}, \{2,6\}, \{2,8\}, \{4,6\}, \{4,8\}, \{6,8\}, \{2,4,6\}, \{2,4,8\}; \{2,6,8\}; \{4,6,8\}, \{2,4,6,8\}\}$

3. $(A \cup B)^c = A^c \cap B^c$

$(\subseteq): x \in (A \cup B)^c \Rightarrow x \notin A \cup B \Rightarrow x \notin A$ in $x \notin B \Rightarrow$
$x \in A^c$ in $x \in B^c$
$\Leftrightarrow x \in A^c \cap B^c$

4. Poenostavi izraz:

$$A \setminus (A \setminus B) = A \setminus (A \cap B^c) = A \cap (A \cap B^c)^c$$
$$= A \cap (A^c \cup B)$$
$$= (A \cap A^c) \cup (A \cap B)$$
$$= \emptyset \cup (A \cap B) = A \cap B$$

$$(A \cap B) \cup (A \cap B^c)$$
$$= A \cap (B \cup B^c) =$$
$$= A \cap U = A$$

$$(A \setminus C) \cup (B \setminus C) =$$
$$= (A \cap C^c) \cup (B \cap C^c) =$$
$$= (A \cup B) \cap C^c =$$
$$= (A \cup B) \setminus C$$

5. $U = \mathbb{R}$
$f, g : \mathbb{R} \to \mathbb{R}$

$A = \{x \in \mathbb{R} ; f(x) = 0\}$ - vse ničle enačbe
$B = \{x \in \mathbb{R} ; g(x) = 0\}$ vse ničle enačbe

$C = \{x ; f(x) \cdot g(x) = 0\}$
$\Downarrow$
$f(x) = 0$ ali $g(x) = 0$

$D = \{x, f^2(x) + g^2(x) = 0\}$
oba morata biti nič

$C = A \cup B$

$D = A \cap B$

$E = \{x ; g^2(x) \cdot f^2(x) + g^2(x) = 0\}$
$E = \cancel{(A \cup B) \cap B}$ $\quad g^2(x) \cdot (f^2(x) + 1) = 0$
$E = \cancel{(A \cap B) \cup} (B \cap B)$
$E = B$

ENAČBE IN NEENAČBE

1. $2x^2 + 7x - 15 = 0$

$$ax^2 + bx + c = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x_{1,2} = \frac{-7 \pm \sqrt{49 - 120}}{4} = \frac{-7 \pm \sqrt{169}}{4}$$
$$x_1 = \frac{-7 + 13}{4} = \frac{6}{4}$$
$$x_2 = \frac{-7 - 13}{4} = -5$$

2. $\sqrt{5x+1} - \sqrt{2x+3} = \sqrt{7x-20}$ $\qquad \sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$

$\cancel{5x}+1 - 2\sqrt{(5x+1)(2x+3)} + \cancel{2x}+3 = \cancel{7x}-20$

$-2\sqrt{(5x+1)(2x+3)} = -24$

$\sqrt{(5x+1)(2x+3)} = 12 \quad /^2$

$144 = 10x^2 + 17x + 3$

$0 = 10x^2 + 17x - 141$

$x_{1,2} = \dfrac{-17 \pm \sqrt{289 + 5640}}{20}$

$x_1 = \dfrac{-17+77}{20} = 3 \qquad x_2 = \dfrac{-17-77}{20} = \cancel{-}$

17·17
17
119
289

3. $x + |x+1| = 3$ $\qquad |x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

$x+1 \geq 0$

$x + (x+1) = 3$

$2x = 2$

$x = 1$

$x+1 < 0$

$x - (x+1) = 3$

$x - x - 1 = 3$

$-1 = 3$

ni rešitev

$\underbrace{|x+1|}_{0} + \underbrace{|x-1|}_{0} = 2$

ko $x = -1$ $\qquad$ ko $x = 1$

$--\quad +-\quad ++$

-1 $\quad$ 0 $\quad$ 1

I $\quad x < -1$

$-(x+1) + (-x+1) = 2$

$-x - 1 - x + 1 = 2$

$-2x = 2$

$x = -1$

II $\quad -1 \leq x < 1$

$x+1 - x + 1 = 2$

$2 = 2$

$x \in [-1, 1)$

III. $\quad 1 \leq x$

$x+1+x-1 = 2$

$2x = 2$

$x = 1$

ZAPRTI INTERVAL

$x \in [-1, 1] =$

4. $2x < x+1 < 2x-1$

$2x < x+1$

$x < 1$

$x+1 < 2x-1$

$2 < x$

$x \in (-\infty, 1) \cup (2, \infty)$ vendar ni nobenega x ni

$\mathbb{R} = \emptyset$

5. $|x-5| < 2x-3 \leq x+2$

$x-5 < 2x-3$

$-2 < x$

$2x-3 \leq x+2$

$x \leq 5$

$x \in (-2, 5]$

-2 $\qquad$ 5

6.

6. $\sqrt{x} + \sqrt{x+1} > 3 \quad /^2$

$x + 2\sqrt{x(x+1)} + x + 1 > 9$

$2\sqrt{x(x+1)} > -2x + 8 \quad /^2$

$x(x+1) > x^2 + 16 - 32x$

$x^2 + x - x^2 + 16 + 8x > 0$

$9x > 16$

$x > 16/9$

R: $x \in (16/9, \infty)$

7. $\dfrac{2x-3}{x-2} \leq 3 \quad /(x-2)^2,\ x \neq 2$, ce ne bi morali razdelit.

$(2x-3)(x-2) \leq 3 \cdot (x-2)^2$

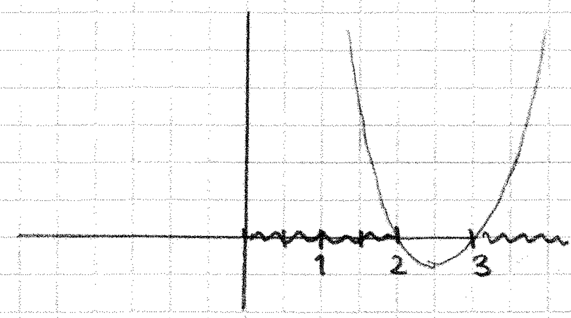
$2x^2 - 4x - 3x + 6 \leq 3 \cdot (x^2 - 4x + 4)$

$2x^2 - 7x + 6 \leq 3x^2 - 12x + 12$

$0 \leq x^2 - 5x + 6$

$0 \leq (x-2)(x-3)$

$x_1 = 2 \qquad x_2 = 3$



$x \in (\infty, 2) \cup [3, \infty)$

8. $|x^2 + 3x - 1| < 3$ $\qquad |x| < a \Leftrightarrow -a < x < a$

$-3 < x^2 + 3x - 1 < 3$

$-3 < x^2 + 3x - 1$

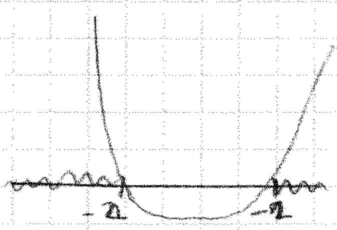
$0 < x^2 + 3x + 2$

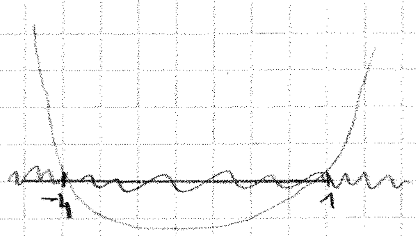
$0 < (x+1)(x+2)$

$x^2 + 3x - 1 < 3$

$x^2 + 3x - 4 < 0$

$(x+4)(x-1)$





$x_1 \in (-\infty, -2) \cup (-1, \infty)$ $\qquad x \in (-4, 1)$

R: $x \in (-4, -2) \cup (-1, 1)$

9. $|2|x|-4|<2$

I. $x \geq 0$

$|2x-4|<2$

$-2 < 2x-4 < 2$

$-2 < 2x-4$

$0 < 2x-2$

$x \geq 1$

$2x-4<2$

$2x-6<0$

$2x<6$

$x<3$

R: $x \in (1,3)$

II. $x<0$

$-2 < -2x+4$

$-2 < -2x-4$

$-2x > 2$

$x < -1$

$-2x+4<2$

$-2x<6$

$x<-3$

R: $x \in (-3,-1)$

$x \in (-3,-1) \cup (1,3)$

Nariši:

$\{(x,y);\ 1 \leq x^2+y^2 < 9\}$

KROŽNICA $(x-p)^2+(y-q)^2=r^2$

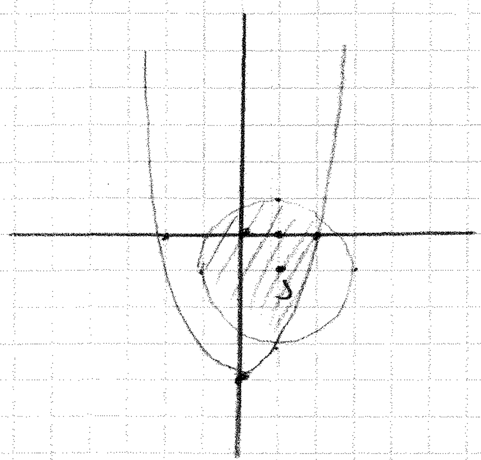
$S(p,q)$

$\{(x,y);\ \frac{x^2}{4}+\frac{y^2}{9} \leq 1,\ y \geq \frac{1}{|x|}\}$

elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

hiperbola $y=\frac{1}{x}$

$\{(x,y),\ y \geq x^2-4 \ \widehat{\text{in}}\ (x-1)^2+(y+1)^2 \leq 4$



5

MAT I-V.
16.10.2007

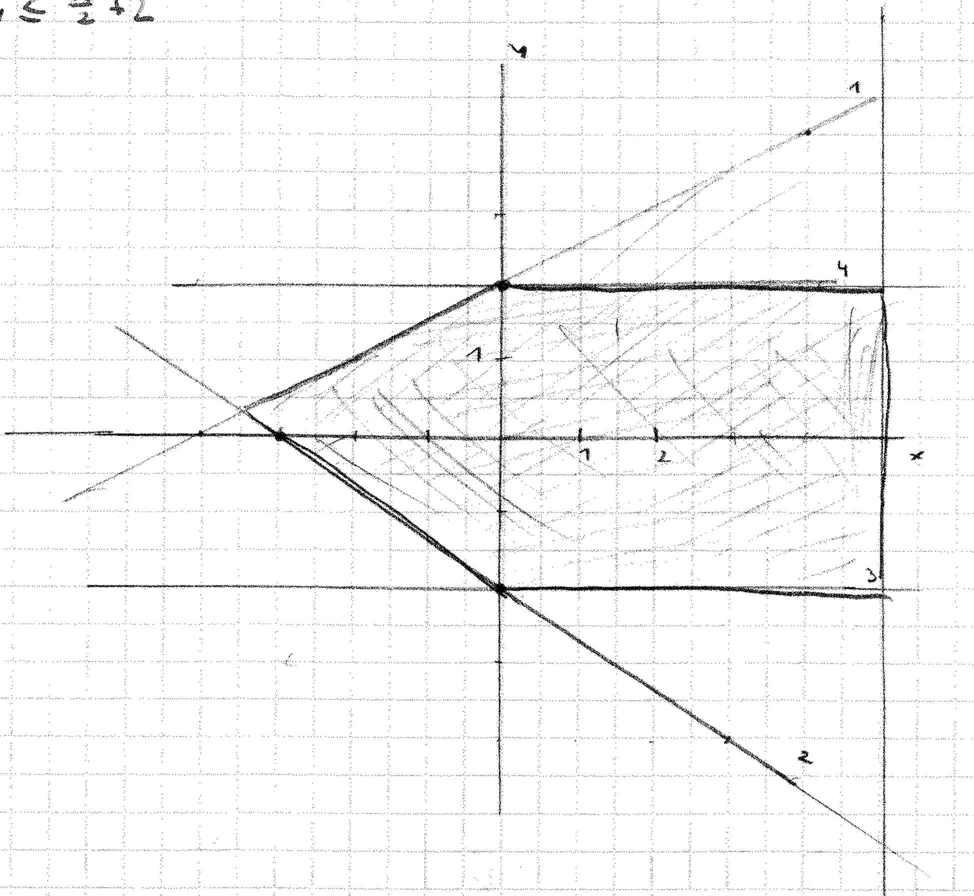
2

1 Nariši množico točk:

$\{(x,y);\ x-2y+4\geq 0;\ 2x+3y+6\geq 0,\ y+2\geq 0,\ y-2\leq 0,\ x-5\leq 0\}$

1 $2y \leq x+4$ 2 $y \geq -\frac{2x}{3} - 2$ 3 $y \geq -2$ 4 $y \leq 2$ 5 $x \leq 5$

$y \leq \frac{x}{2}+2$



2. $\{(x,y);\ |x|+|y|\leq 1$

$x\leq 0$, $y\leq 0$

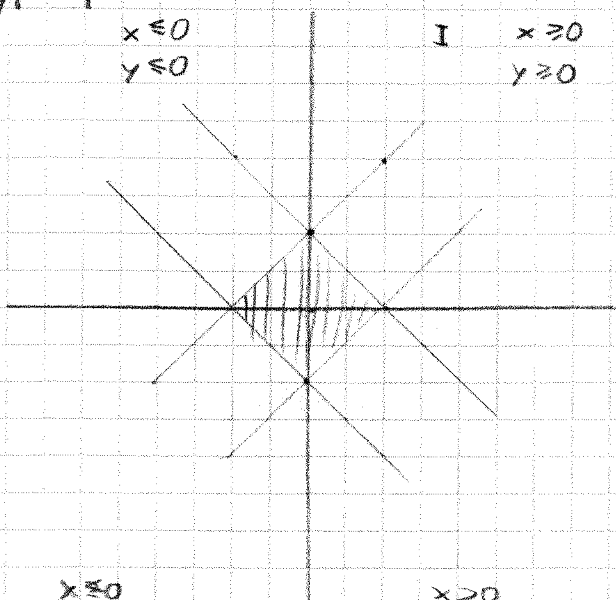
I $x\geq 0$, $y\geq 0$

1 $x+y\leq 1$

$y\leq -x+1$

$-x+y\leq 1$

$y\leq x+1$



$-x-y\leq 1$

$-x-1\leq y$

$x-y\leq 1$

$x-1\leq y$

$x\leq 0$, $y\leq 0$

$x\geq 0$, $y\leq 0$

6

MATEMATIČNA INDUKCIJA

če velja za $n=1$ - baza indukcije
če velja princip indukcije - indukcijski korak

$$1\cdot 2 + 2\cdot 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

$n=1$

$$1(1+1) = \frac{1(1+1)(1+2)}{3}$$
$$1\cdot 2 = 1\cdot 2$$

$n \to n+1$

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + \ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$$

$$\underline{1\cdot 2 + 2\cdot 3 + \ldots + n(n+1)} + (n+1)(n+1+1) = \frac{(n+1)(n+1+1)(n+1+2)}{3}$$

$$\frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = (n+1)(n+2)\left(\frac{n}{3}+1\right) = \frac{(n+1)(n+2)(n+3)}{3}$$

■

1. $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$

$n=1$

$$2^{1-1} = 2^1 - 1$$
$$2^0 = 2 - 1$$
$$1 = 1$$

$n \to n+1$

$$1 + 2 + 2^2 + \ldots 2^{n-1} = 2^n - 1$$

$$1 + 2 + \ldots 2^{n-1} + 2^{n+1-1} = 2^{n+1} - 1$$

$$2^n - 1 + 2^n = 2\cdot 2^n - 1 = 2^{n+1} - 1$$

2. $1^2 + 2^2 + \ldots + n^2 = \frac{n\cdot(n+1)\cdot(2n+1)}{6}$

$n=1$

$$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$
$$1 = \frac{1\cdot 2\cdot 3}{6}$$
$$1 = 1$$

$n \to n+1$

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$ - IND. PREPOST.

$$1^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right) = (n+1)\left(\frac{2n^2+n+6n+6}{6}\right) = \frac{(n+1)(n+2)(2n+3)}{6}$$

3. $9 \mid n^3 + (n+1)^3 + (n+2)^3$

$n=1$

$$1 + 2^3 + 3^3 = 1+8+27 = 36 = 9 \cdot 4$$

$n \to n+1$

I.P. $n^3 + (n+1)^3 + (n+2)^3 = 9k$

$(n+1)^3 + (n+1+1)^3 + (n+2+1)^3 = 9\tilde{c}$ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\cancel{n^3 + 3n^2 + 3n + 1 + n^3 + 6n^2 + 2}$

$$n^3 + 9n^2 + 3n + 9 = \cancel{9k}$$

$$= 9k + \cancel{n^3} + 9n^2 + 9n + 27$$

$$= 9k + 9(\cancel{k'}) = 9(k')$$

4. $133 \mid 11^{n+1} + 12^{2n-1}$

$n=1$

$133 \mid 11^2 + 12^{2-1}$

$133 \mid 121 + 12 = 133$

I.P. $133 \mid 11^{n+1} + 12^{2n-1} = 133k'$

$n+1$ $\quad 11^{n+2} + 12^{2n+1} \;\; | \; 133 \cdot k'$

$$11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} = 11 \cdot (11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{n+1}$$

$$= 11 \cdot 133k + 133 \cdot \tilde{c}$$

$$= 133(11k + \tilde{c})$$

KOMPLEKSNA ŠTEVILA

$\mathbb{C} = \{ z = x + iy \; ; \; x, y \in \mathbb{R}, \quad i^2 = -1 \quad i = \sqrt{-1}$

$x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

$w = u + iv$

$$z \pm w = (x \pm u) + i(y \pm v)$$

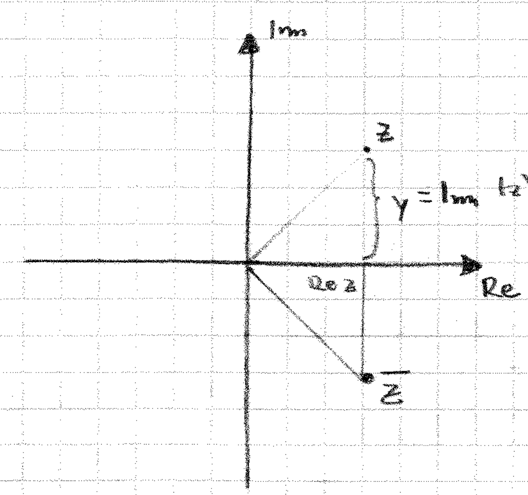
$$z \cdot w = (xu - yv) + i(xv + y \cdot u)$$

$$\overline{z} = x - iy$$

$$|z| = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$$

$$z^{-1} = \frac{1}{z} = \frac{1 \cdot \overline{z}}{z \overline{z}} = \frac{\overline{z}}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$$



8

1. $(1+2i)^3 = 1^3 + 3\cdot 2i + 3\cdot(2i)^2 + 8i^3$
$= 1 + 6i - 12 - 8i$
$= -11 - 2i$

$i =$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$\frac{5}{(-3+4i)} = \frac{5\,(-3-4i)}{(-3+4i)\,(-3-4i)} = \frac{-15-20i}{9+16} = \frac{-15-20i}{25} = \frac{-3-4i}{5}$

2. $\frac{2-3i}{3-i} - \frac{4+i}{3+i} = \frac{(2-3i)(3+i) - (4+i)(3-i)}{9+1} =$
$= \frac{6+3+i(2-9) - (12+1)+(-4+3)i}{10} =$
$w = \frac{9-7i-13+i}{10} = \frac{-4-6i}{10} = \frac{-(2+3i)}{5}$

w : $Re_{(w)} = -2/5$ $\quad \bar{w} = \frac{-2+3i}{5}$
$Im_{(w)} = -3/5$
$|w| = \sqrt{\frac{4}{25}+\frac{9}{25}} = \frac{\sqrt{13}}{5}$

3. $w = \frac{(3+i)\cdot(1+i)}{(2-i)}$
$w = \frac{(3-1)+(3+1)i\cdot(2+i)}{2+1} = \frac{(2+4i)\cdot(2+i)}{3} =$
$= \frac{(4-4)+(2+8)i}{5} = \frac{10i}{5} = 2i$ $\quad Re_{(w)}=0$, $Im_{(w)} = 10/5 = 2$

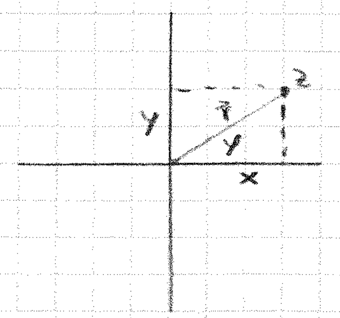
$\bar{w} = -2i$

$|w| = \sqrt{4} = 2$

4. $w = \frac{z-\bar{z}}{2} = \frac{x+iy-(x-iy)}{2} = \frac{2y\cdot i}{2} = yi$ $\quad Re = 0$, $Im = y$ $\quad \bar{w} = -yi$, $|w| = |y|$

$w = z\cdot\bar{z}^2 = (x+iy)(x^2-2xyi-y^2) = x^3-2x^2yi-xy^2+x^2iy+2y^2-y^3i$
$= x^3-x^2iy+yi-x^2+2y^2$
$\ldots = (x^3+xy^2) - i(x^2y+y^3)$

POLARNI ZAPIS:



$z = x+iy = r(\cos\varphi + i\sin\varphi)$
$= r\cdot e^{i\varphi}$

$x = r\cdot\cos\varphi$ $\quad r = \sqrt{x^2+y^2}$
$y = r\cdot\sin\varphi$
$\varphi = \mathrm{arctg}\,\frac{y}{x}$!!!
↓
perioda π

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin\varphi$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\varphi$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\mathrm{tg}\,\varphi$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0

$z = |z|(\cos\varphi$

$z = |z| (\cos \gamma + i \sin \gamma)$
$w = |w| (\cos \alpha + i \sin \alpha)$

$z \cdot w = |z| \cdot |w| [\cos(\gamma + \alpha) + i \cdot \sin(\gamma + \alpha))$
$z^2 = |z|^2 (\cos 2\varphi + i \cdot \sin 2\varphi$

$z^n = |z|^n (\cos(n\varphi) + i \cdot \sin(n\varphi))$ - De Moirre - ova formula

1. $z = 1+i \quad = \sqrt{2} \left(\cos \frac{\pi}{4} + i \cdot \sin \frac{\pi}{4}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z = -1 - i\sqrt{3} \quad = 2 \left(\cos \frac{4\pi}{3} + i \cdot \sin \frac{4\pi}{3}\right)$

2. $z = -\sqrt{2} + \sqrt{2} i \quad = 2 \left(\cos \frac{3\pi}{4} + i \cdot \sin \frac{3\pi}{4}\right)$

3. $(-\sqrt{3} + 3i)^7 = (2\sqrt{3})^7 \left(\cos\left(7 \cdot \frac{2\pi}{3}\right) + i \cdot \sin\left(7 \cdot \frac{2\pi}{3}\right)\right)$
$= 2^7 \cdot 27\sqrt{3} \left(\cos \frac{2\pi}{3} + i \cdot \sin \frac{2\pi}{3}\right)$
$= 128 \cdot 27\sqrt{3} \cdot \left(\left(-\frac{1}{2}\right) + i \frac{\sqrt{3}}{2}\right)$

4. $z^3 = 1$ - ciklometrična enačba $\qquad z = |z|(\cos\varphi + i \sin\varphi)$
$z = 1 \qquad z^3 = |z|^3 (\cos 3\varphi + i \cdot \sin 3\varphi)$

$|z^3| (\cos 3\varphi + \sin 3\varphi) = 1 \cdot (\cos 0 + i \cdot \sin 0)\ 1 = 1 \cdot (\cos 0 + i \sin 0)$

$|z|^3 = 1 \qquad \cos 0 = \cos 3\varphi$
$|z| = 1 \qquad 3\varphi = 0 + k 2\pi$
$\varphi = \frac{2k\pi}{3} \qquad k = 0, 1, 2, 3$
$\varphi_1 = \frac{2\pi}{3}, 0, \frac{4\pi}{3}$

$w_k = |z| (\cos \varphi_k + i \sin \varphi_k) \qquad k = 0, 1, 2$

$w_0 = 1$
$w_1 = 1 \cdot \left(\cos \frac{2\pi}{3} + i \cdot \sin \frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}$
$w_2 = 1 \cdot \left(\cos \frac{4\pi}{3} + i \cdot \sin \frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}$

5. $z^3 = i \qquad i = 1 \cdot (\cos$
$|z|^3 \cdot (\cos 3 \cdot \varphi + i \sin 3\varphi) = 1 \cdot \left(\cos \frac{\pi}{2} + i \cdot \sin \frac{\pi}{2}\right)$

$|z|^3 = 1 \qquad \cos \frac{\pi}{2} = \cos 3\varphi$
$|z| = 1$
$3\varphi = \frac{\pi}{2} + 2k\pi$
$\varphi = \frac{\pi}{6} + \frac{2k\pi}{3} \qquad k \in \mathbb{Z}$
$\varphi = \frac{\pi + 4k\pi}{6} \qquad k = 0, 1, 2$

$\varphi_1 = \frac{\pi}{6} \qquad \varphi_2 = \frac{5\pi}{6} \qquad \varphi_3 = \frac{9\pi}{6} = \frac{3\pi}{2}$

$w_0 = 1 \cdot \left(\cos \frac{\pi}{6} + i \cdot \sin \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2} i$
$w_1 = 1 \cdot \left(\cos \frac{5\pi}{6} + i \cdot \sin \frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i$
$w_2 = 1 \cdot \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right) = -i$

10

MAT.I.V
23.10.2007

1. $\sqrt[4]{-2+2i\sqrt{3}} = \quad z^4 = -2+2i\sqrt{3}$

$r = \sqrt{a^2+b^2} = \sqrt{(-2)^2+(2\sqrt{3})^2}$

$r = \sqrt{4+12} = \sqrt{16} = 4$

$\varphi = \mathrm{arctg}\,\frac{y}{x} = \frac{2\sqrt{3}}{2} = \sqrt{3} = -\frac{\pi}{3}+\pi = \frac{2\pi}{3}$

$z = 4\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)$

$= \sqrt[4]{4}\left(\cos\frac{\pi}{6} + i\cdot\sin\frac{\pi}{6}\right)$

$k_0 = \sqrt{2}\cdot\frac{\sqrt{3}}{2} + \sqrt{2}\cdot\frac{1}{2}$

$k_1 = \sqrt{2}\cdot\left(\cos\frac{\pi}{6}+2\pi + i\cdot\sin 2\pi+\frac{\pi}{6}\right) = \sqrt{2}\cdot\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt{2}\cdot -\frac{\sqrt{3}}{2}+(i\cdot$

$k_2 = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\cdot\left(-\frac{1}{2}\right)\right) \quad \frac{5\pi}{6}$

$k_3 = \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\cdot\left(-\frac{1}{2}\right)\right)$

2. $z^4 = -\sqrt{3}+i$

$r^4(\cos 4\cdot\varphi + i\cdot\sin 4\cdot\varphi) = 2\cdot\left(\cos\frac{5\pi}{6} + i\cdot\sin\frac{5\pi}{6}\right)$

$r^4 = 2$

$r = \sqrt[4]{2}$

$\cos 4\varphi = \cos\frac{5\pi}{6}$

$4\varphi = \frac{5\pi}{6} + 2k\pi$

$\varphi = \frac{5\pi}{24} + \frac{k\pi}{2} = \frac{5\pi + 12k\pi}{24}$

$\varphi_0 = \frac{5\pi}{24} \rightarrow z_0 = \sqrt[4]{2}\left(\cos\frac{5\pi}{24} + i\cdot\sin\frac{5}{24}\right)$

$\varphi_1 = \frac{\pi}{2} \quad z_1 = \sqrt[4]{2}\left(\cos\frac{17\pi}{24} + i\sin\frac{17\pi}{24}\right.$

$\varphi_2 = \frac{29\pi}{24} \quad z_2 = \sqrt[4]{2}\left(\cos\frac{29\pi}{24} + i\sin\frac{29\pi}{24}\right)$

$\varphi_3 = \frac{41\pi}{24} \quad z_3 = \sqrt[4]{2}\left(\cos\frac{41\pi}{2} + i\sin\frac{41\pi}{24}\right)$

3. $|z| + z = 2+i$

$\sqrt{a^2+b^2} + a + ib = 2+i$

$bi = 1i$

$\sqrt{a^2+b^2} + a = 2$

$\sqrt{a^2+1} + a = 2 \quad /^2$

$a^2+1 = 4-4a+a^2$

$a = 3/4$

$z = \frac{3}{4}+i$

11

4. $|z-2|=3$

$|z+1|=3$

$|a+ib-2|=3$ $\quad$ $|a+ib+1|=3$

$\sqrt{(a-2)^2+b^2}=3$ $\quad$ $\sqrt{(a+1)^2+b^2}=3$

$(a-2)^2+b^2=9$ $\quad - \quad$ $(a+1)^2+b^2=9$ $\quad = a^2-4a+4-a^2-2a-1$

$= -6a+3$

$6a=3$

$a=1/2$

$-b^2=(\frac{3}{2})^2 \cancel{+b} -9$

$z= \quad -b^2=\frac{9}{4}-9$

$-b^2=-\frac{27}{4}$

$b=\sqrt{\frac{27}{4}}=\frac{\sqrt{27}}{2}=\pm\frac{3\sqrt{3}}{2}$

$z_1=\frac{1}{2}-\frac{3\sqrt{3}}{2}i$

$z_2=\frac{1}{2}+\frac{3\sqrt{3}}{2}i$

5. $\left|\frac{z}{z+1}\right|=1$

$\frac{z}{\bar{z}}=i$

$\frac{a+bi}{a-bi}=i$

$a+ib=(a-ib)i$

$a+ib=ai+b$

$a=b$

$\frac{|a+bi|}{|a+bi+1|}$

$|a+bi|=|a+bi+1|$

$\sqrt{a^2+b^2}=\sqrt{(a+1)^2+b^2}$

$a^2+b^2=(a+1)^2+b^2$

$a^2=a^2+2a+1$

$2a=-1$

$a=-1/2$

$z=-\frac{1}{2}-\frac{1}{2}i$

6. $z\cdot\bar{z}+(2+i)\cdot z+(2-i)\bar{z}\cancel{+4}=0$

$-(1-i)z+(1+i)\bar{z}\cancel{+4}=0$ $\quad = z\bar{z}+(1+2i)z+(1-2i)\bar{z}=0$

$a+bi=z$

$a^2-b^2+(2+i)(a+bi)+(2-i)(a+bi)+4=0$

$(1-i)(a+bi)+(1+i)(a+bi)+4=0$ $\quad b=-a-2$

$a^2+b^2+\cancel{2a}+2bi+\cancel{ai+b}+\cancel{2a}+2bi=\cancel{ai}+b+4$

$a^2+b^2+4a+b=-4$

$\cancel{a^2}+a^2+4a-4+4a+a-2+4=0$

$2a^2+10a+12=0$

$a^2+5a+6=0$

$(a+2)(a+3)$

$a_1=-2 \quad b_1=0$

$a_2=-3 \quad b_2=\cancel{4}$

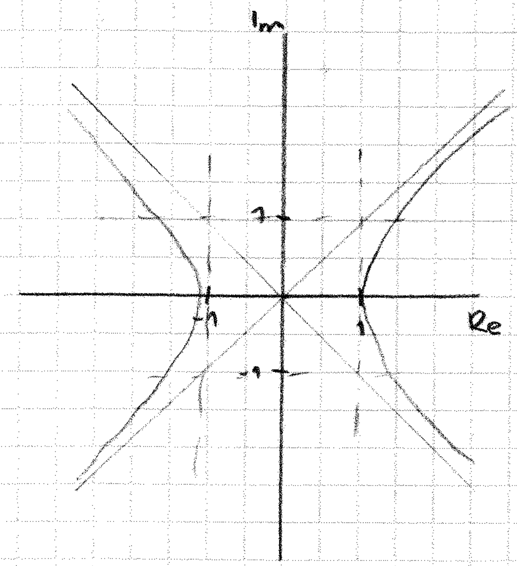
12

7. Nariši množico kompleksnih števil, kjer velja:

$$Re\ (\bar{z}^2)=4$$

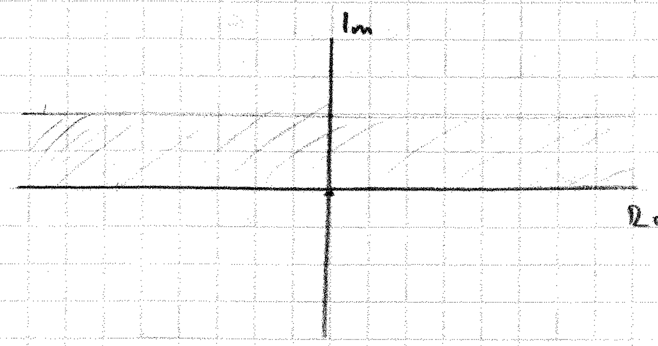
$z=a+bi$
$\bar{z}=a-bi$
$\bar{z}^2=a^2-2abi-b^2$

$\frac{a^2}{4}-\frac{b^2}{4}=1$ -elipsa , če bi bilo -1 bi bila zgoraj

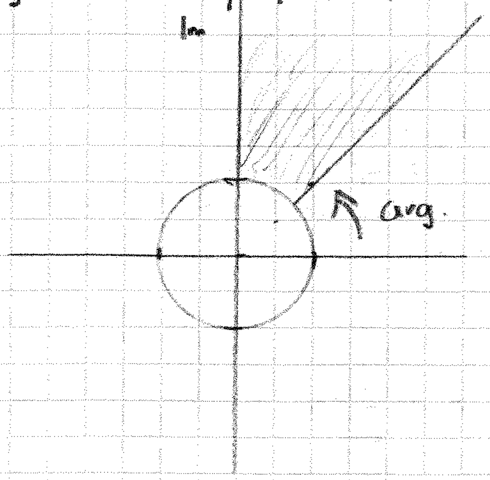


Nariši množico točk za katere velja:

$0 \leq Im(z) < 1$
$0 \leq a+bi < 1$
$0 \leq bi < 1$



$\frac{\pi}{4} < arg.(z) < \frac{\pi}{2}$, $|z|>1$



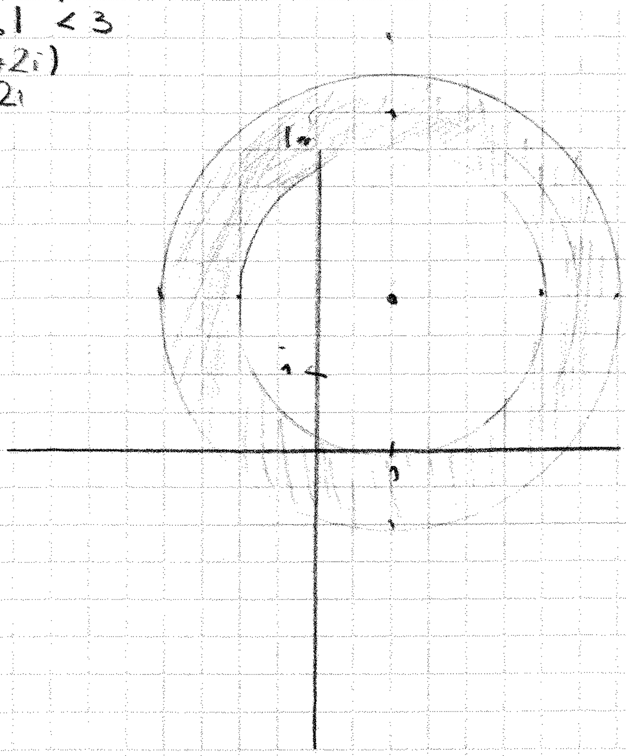
Nariši množico točk:

$2 \le |z-1-2i| < 3$

$|z-z_0| < 3$

$(1+2i)$

$z = 1+2i$



ZAPOREDJA

Predpis, ki naravnemu številu priredi realno število

$\mathbb{N} \to \mathbb{R}$

$n \to a_n$

$1, 2, 3, 4, \ldots$

a_n je naraščajoče če $a_n \le a_{n+1}$ $\forall n$

a_n je padajoče če $a_n \ge a_{n+1}$ $\forall n$

1 Pogledamo razliko členov $a_{n+1} - a_n \ge 0$ naraščajoče

Pogledamo kvocient členov $\frac{a_{n+1}}{a_n} \ge 1$ naraščajoče

1. $a_n = \frac{1}{n+1}$

$a_1 = \frac{1}{2}$, $a_2 = 1/3$, $a_3 = 1/4$, $\vdots$

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \frac{n+1}{n+2} < 1$ $\Rightarrow$ strogo padajoče

2. $a_n = n^2$

$a_{n+1} - a_n = (n+1)^2 - n^2$
$= n^2 + 2n + 1 - n^2$
$= 2n+1$ strogo naraščajoče

3. $a_n = \frac{3^n}{n!}$

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$(n+1)! = n!\,(n+1)$

$a_1 = 3$, $a_2 = 9/2$, $a_3 = 9/2$

$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3 \cdot \cancel{3^n}}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{\cancel{3^n}} = \frac{3}{n+1} \le$ za $n \ge 2$

n

$a_n = \sin(n\pi)$
- konstantno zaporedje,
  vsi členi so enaki

M je zg. meja zap. a_n, če je $a_n \leq M$ za $\forall n$
supremum je natančna zg meja ali najbolj natančna
m je spodnja meja od a_n
infimum je največja spodnja meja

4. $a_n = \frac{n}{n+1}$

$\inf_n a = \frac{1}{2}$

$\sup_n a = 1$

$\max_n a$

$\min_n a \quad \frac{1}{2}$

5. $a_n = \frac{2^n}{n!} =$

MAT 1. – V
6.11.2007

1 $a_n = (1 - \frac{1}{n})(-1)^n$

$\rightarrow a_1 = 0$
$* \; a_2 = 1/2$ alternirajoče zaporedje
$\rightarrow a_3 = -2/3$
$* \; a_4 = 3/4$
$\rightarrow a_5 = -4/5$ $\quad \rightarrow 0, -\frac{2}{3}, -4/5 \quad \rightarrow -1$
$* \; a_6 = 5/6$

$* \; 1/2, \; 3/4, \; 5/6,$

$\inf_n a_n = -1$ min in maks ne obstajata
$\sup_n a_n = 1$

2\. a) 2, 3, 4, 5, 6
a_1 a_2 a_3 a_4

$a_n = \cancel{1+a_n}$ $n+1$

b) 1, -4, 9, -16, 25

$a_n = n^2 \cdot (-1)^{n+1}$

a je stekališče zaporedja a_n, če je v vsaki okolici a neskončno členov zaporedja.

a je limita zaporedja a_n, če je v vsaki okolici a neskončno členov zaporedja, zunaj pa končno mnogo.

Zaporedje je konvergentno, če ima limito, sicer je divergentno.

Če je zaporedje padajoče i naraščajoče in omejeno potem ima limito.

1\. a

$a_n = 1 - \frac{1}{n}$ $a_1 = 0$ $a_2 = \frac{1}{2}$
$a_3 = 2/3$ $a_4 = 3/4$

$\lim a_n = 1$

b) $a_n = \sin \frac{n\pi}{3}$
$a_1 = \sqrt{3}/2$
$a_2 = \sqrt{3}/2$
$a_3 = 0$
$a_4 = -\sqrt{3}/2$
a_5 $-\sqrt{3}/2$
$a_6 = 0$

je divergentno

2\. a $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}$

je divergentno, ima 2 stekališči (0, 1)

b) $\lim_{n\to\infty} \frac{8n^2+9n-6}{2n^3+3n+1} = 0$ deliš z največjo potenco

3\. a) $\lim_{n\to\infty} \frac{7n^2+2}{3n^2+11n-2} = \frac{7}{3}$

b) $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{2n+1} = \frac{1}{2}$

4\. $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{1+1} = \frac{1}{\infty} 0 = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}\cdot(\sqrt{n+1}-\sqrt{n})} = \frac{\cancel{1}\sqrt{n+1} + \sqrt{n}}{\sqrt{n}\cdot(\cancel{n+1}-\cancel{n})} = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}} = 2$

16

$$\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n} \quad \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^{n+1}+1^{n+1}}{\frac{2^n}{3^{n+1}}+\left(\frac{3}{3}\right)^{n+1} \cdot 1/3} = 3$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{-n} = e$$

1. $$\lim_{n\to\infty} \left(\frac{n+5}{n+3}\right)^n = \left(1+\frac{2}{n+3}\right)^n$$

$$\frac{2}{n+3} = \frac{1}{m}$$
$$2m = n+3$$
$$2m+3 = n$$

$$\lim_{m\to\infty} = \left(1+\frac{1}{m}\right)^{2m-3}$$

li

$$= \left(\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m\right)^2 \cdot \left(1+\frac{1}{m}\right)^{-3} = e^2 \cdot 1$$

2. $$\lim_{n\to\infty} (n+3)(\ln(n+1) - \ln n)$$

$\ln a + \ln b = \ln ab$
$\ln a - \ln b = \ln \frac{a}{b}$
$r \cdot \ln a = \ln a^r$

$$\lim_{n\to\infty} (n+3)\left(\ln \frac{n+1}{n}\right)$$

$$\lim_{n\to\infty} \left(\ln \left(\frac{n+1}{n}\right)^{n+3}\right) = \left(\ln \left(1+\frac{1}{n}\right)^{n+3}\right) = \ln \cdot \lim \left(1+\frac{1}{n}\right)^{n+3}$$

bez $\frac{1}{!}$ ævina

$$= \ln e^3 = .1$$

3. $$a_n = \frac{n^2+n}{n^2+1} \qquad \varepsilon = 10^{-1} \qquad n > 9$$

$$|a_n - \lim a_n| < 10^{-1}$$

$$\left|\frac{n^2+n}{n^2+1} - 1\right| < 10^{-1} \quad = \left|\frac{n^2+n-n^2-1}{n^2+1}\right| < 10^{-1}$$

$$= \frac{n-1}{n^2+1} < 10^{-1}$$

$$n-1 < \frac{n^2}{10} + \frac{1}{10}$$

$$n^2 - 10n + 11 > 0$$

$$(n - \quad)(- \quad)$$

$$\frac{10 \pm \sqrt{56}}{2} \qquad n_1 = 1{,}26 \quad n_2 = 8{,}74$$

4. $$a_n = \frac{5^n - 1}{5^n}$$

$$|a - a_n| < 25^{-25} \qquad \frac{1}{5^n} < \frac{1}{25^{25}}$$

$$\left|1 - \frac{5^n-1}{5^n}\right| < 25^{-25} \qquad 25^{25} < 5^n$$

$$\left|\frac{5^n - 5^n + 1}{5^n}\right| < 25^{-25} \qquad 5^{50} < 5^n \quad 50 < n$$

17

5. $a_1 = 0$ Pokaži, da je konvergentno:

$a_{n+1} = \frac{a_n}{3} - 2$

$a_2 = \frac{a_1}{3} - 2 = -2$

$a_3 = -\frac{8}{3}$

$a_4 = \frac{-26}{9}$

je navzdol omejeno

$a_n \geq -3$

$n = 1$

$a_1 = 0$

za vsak n

$a_n \geq 3$

potem za vsak $n+1$

$a_{n+1} \geq 3$

$a_{n+1} = \frac{a_n}{3} - 2 \geq -3 \quad a_n \geq 3 \; /:3$

$\frac{a_n}{3} \geq \frac{3}{3} - 1 \; / \; (-2)$

$\frac{a_n}{3} - 2 \geq -3$

2. da je zaporedje padajoče:

$a_{n+1} - a_n < 0$

$= \frac{a_n}{3} - 2 - a_n = -\frac{2a_n}{3} - 2 < 0$ vemo: $a_n \geq -3 \; / \cdot (-\frac{2}{3})$

$-\frac{2}{3} a_n \leq +2 \; / -2$

$-\frac{2}{3} a_n - 2 \leq 0$

Zaporedje je padajoče in omejeno $\Rightarrow$ konvergentno

$\lim\limits_{n \to \infty} a_n = a$ $\qquad \lim a_{n+1} = \left(\frac{a_n}{3} - 2\right) = a$

$\Downarrow$

$\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to} a_{n+1}$

$a = \lim \frac{a_n}{3} - \lim -2$

$a = \frac{a}{3} - 2$

$\frac{2}{3} a = -2$

$a = -3$

0. $a_1 = 1$

$a_{n+1} = \frac{a_n}{2} + 1$

$a_1 = 1$
$a_2 = 3/2$
$a_3 = 7/4$
$a_4 = 15/8$
$a_5 = 31/16$

$\lim_{n\to\infty} a_n = a$

$\lim_{n\to\infty} \left(\frac{a_n}{2} + 1\right) = \frac{a}{2} + 1$

$\frac{a}{2} + 1 = a$

$\frac{a}{2} = a - 1$

$-a/2 = -1$

$a = 2$

i) navzgor omejeno $a_n \leq 2$

$n = 1$

$1 \leq 2$

$n \to n+1 \quad a_{n+1} \leq 2$

$a_{n+1} = \frac{a_n}{2} + 1$

$a_n \leq 2 \;/:2$

$\frac{a_n}{2} \leq 1$

$\frac{a_n}{2} + 1 \leq 2$

da je zaporedje naraščajoče

$a_{n+1} - a_n > 0$

$\frac{a_n}{2} + 1 - a_n > 0$

$-\frac{a_n}{2} + 1 > 0$

vem

$a_n \leq 2 \;/:2$

$-\frac{a_n}{2} \geq -1$

$-\frac{a_n}{2} + 1 \geq 0$

7. $a_1 = 1$

$a_{n+1} = \frac{1}{5} a_n^2 + 1$

$a_1 = 1$
$a_2 = 6/5$
$a_3 = \frac{161}{125}$
$a_4 =$

i) navzgor omejeno

$a_n \leq 2$

$a_1 < 1$

$n \to n+1 \quad a_{n+1} = \frac{1}{5} a_n^2 + 1$

$a_n \leq 2$
$a_{n+1} < 2$

$a_n \leq 2 \;/^2$

$a_n^2 < 4 \;/\cdot\frac{1}{5}$

$\frac{1}{5} a_n^2 < 4/5 \;(+1)$

$1/5\, a_n^2 + 1 < 9/5 < 2$

$a_{n+1} - a_n > 0$

vem $a_n \leq 2 \Rightarrow -a_n \geq -2 \rightarrow$

$a_n^2 < 4$

$a_{n+1} - a_n = 1/5\, a_n^2 + 1 - \frac{1}{5} a_{n-1}^2 - 1$

$= 1/5\,(a_n^2 - a_{n-1}^2) =$

$= 1/5\,(a_n - a_{n-1})(a_n + a_{n-1})$

>0

>0

iz predpostavke

↓

Pokažemo z indukcijo

$$\lim_{n} a_{n+1} = \lim \left(\frac{1}{5} a_n^2 + 1\right)$$

$$a = \frac{1}{5}a^2 + 1$$
$$a^2 - 5a + 5 = 0$$

$$\frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2} = a_{1,2}$$

$$\lim a_n = \frac{5-\sqrt{5}}{2}$$

GEOMETRIJSKA VRSTA

$$\sum_{n=1}^{\infty} a \cdot q^{n-1} = \frac{a}{1-q} \qquad |q| < 1$$

1. $$\sum_{n=1}^{\infty} \frac{3}{4^{n-1}} = \sum_{n=1}^{\infty} 3 \cdot \left(\frac{1}{4}\right)^{n-1} \qquad S = \frac{a}{1-k} = \frac{3}{1-\frac{1}{4}} = \frac{3}{\frac{3}{4}} = 4$$

2. $$\sum_{n=1}^{\infty} \frac{37}{100^n} = \sum 37 \cdot \frac{1}{100}^{n-1} \cdot \frac{1}{100} \qquad a = \frac{37}{100} \quad q = \frac{1}{100} \qquad S = \frac{a_1}{1-q} = \frac{\frac{37}{100}}{\frac{99}{100}} = \frac{37}{99}$$

MAT. I V
13. 11. 2007

ŠTEVILSKE VRSTE

je oblike

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \dots + a_N$$

N-ta delna vsota

$$S = \lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n$$

1. $$\sum_{n=1}^{\infty} \frac{1 = 0 \cdot n + 1}{n \cdot (n+1)} = 1 = \text{PARCIALNI ULOMKI} = \frac{B}{n+1} + \frac{A}{n} = \frac{A(n+1) - Bn}{n(n+1)}$$

$$\frac{1}{n(n+1)} = a_n$$

$$= \frac{An + A - Bn}{n(n+1)} = \frac{n(A-B) + A}{n(n+1)}$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$A = 1$$
$$B = -1$$

$$S_n = a_1 + a_2 + \dots + a_n$$
$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \cancel{\frac{1}{N}} + \frac{1}{N+1}$$

$$= 1 - \frac{1}{N+1}$$

$$s = \lim_{n\to\infty} s_n = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

$$a_n = \frac{1}{4n^2-1} = \frac{1}{(2n+1)(2n-1)} = \frac{A}{(2n+1)} + \frac{B}{(2n-1)} = \frac{(2n-1)\cdot A + B(2n+1)}{(2n+1)(2n-1)}$$

$$= \frac{A2n - A + B2n + B}{4n^2-1}$$

$$= \frac{n(A2A+2B) + (B-A)}{4n^2-1}$$

$B - A = 1$ $\quad A + B = 1$

$B = 1/2$

$2A + 2B = 0$

$A + B = 0$ $\quad A = -\frac{1}{2}$

$A = -B$

$$a_n = \frac{1}{2(2n-1)} - \frac{1}{(2(2n+1))}$$

$$s_n = \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \cancel{\frac{1}{20}}\right) + \ldots \left(\cancel{\frac{1}{2(2N-1)}} - \frac{1}{2(2N+1)}\right)$$

$$\frac{An+B}{n^2+1}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2(2N+1)} = \frac{1}{2}$$

3. $0,23232323 = 23 \cdot \frac{1}{100} + 23 \cdot \frac{1}{10000} + \ldots$

$$\sum_{n=1}^{\infty} = 23 \cdot \frac{1}{100^n} = 23/100 \cdot \left(\frac{1}{100}\right)^{n-1}$$

$$\frac{\frac{23}{100}}{1 - \frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}} = \frac{23}{99}$$

KONVERGENCA VRSTE

s poz. členi

$$\sum_{n=1}^{\infty} a_n \qquad a_n > 0$$

1. kvocientni

$$\sum_{n=1}^{\infty} a_n \qquad q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

< 1 konv

> 1 diver

$= 1$ kriterij odpove

3. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$q = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{n!(n+1)} = \frac{1}{n+1} = 0 < 1$$

4. $\sum_{n=1}^{\infty} \frac{2n}{n+1}$

$$q = \lim_{n\to\infty} \frac{\frac{2n+2}{n+2}}{\frac{2n}{n+1}} = \lim_{n\to\infty} \frac{(2n+2)(n+1)}{2n(n+2)} = \frac{2(n+1)(n+1)}{2n(n+2)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+2n} = 1$$

kriterij odpove

2. KORENSKI KRITERIJ

$\sum_{n=1}^{\infty} a_n$ $\qquad q < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ konv

$q = \lim_{n\to\infty} \sqrt[n]{a_n}$ $\qquad q = 1$? $\qquad q > 1$ divergira

1. $\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n+1)}$

$$q = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{n-1}{n+1}\right)^{\cancel{n}(n+1)}} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right)^{n+1} = e$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{u}\right)^{2u} = e^{-2} < 1 \text{ vrsta}$$

$\frac{2}{n+1} = \frac{1}{u}$

$2u = n+1$

$u = \frac{n+1}{2}$

2. $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n \cdot 5^n}$

$$q = \lim_{n\to\infty} \sqrt[n]{\frac{3^{2n+1}}{n 5^n}} = \lim_{n\to\infty} \frac{9 \sqrt[n]{3}}{\sqrt[n]{n} \cdot 5} = \frac{9}{5} \lim_{n\to\infty} \frac{\sqrt[n]{3}}{\sqrt[n]{n}} \overset{= 1}{} \cdot 1 = \frac{9}{5}$$

DIVERGIRA

3. PRIMERJALNI KRITERIJ

$$0 < a_n \leq b_n$$

a) $\sum_{n=1}^{\infty} a_n$ DIV ; $\sum_{n=1}^{\infty} b_n$ DIV

b) $\sum_{n=1}^{\infty} b_n$ KONV $\Rightarrow \sum_{n=1}^{\infty} a_n$ KONV

1. $\sum_{n=1}^{\infty} \frac{1}{\log n}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n}$ - divergentna harmonična vrsta

$$\log n < n$$
$$\frac{1}{\log} > n$$

$$\sum_{n=1}^{\infty} \frac{1}{\log n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

divergira $\qquad$ divergira

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ $\alpha > 1$ konvergira, $\alpha \leq 1$ divergira

$$\Downarrow$$

$$\sum \qquad n^3+n > n$$
$$\frac{1}{\sqrt{n^3+n}} < \frac{1}{\sqrt{n^3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+n}} < \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

konv. $\qquad$ konv.

ALTERNIRAJOČE VRSTE

OBLIKE

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

JE KONVERGENTNA ČE JE ZAPOREDJE a_n PADAJOČE IN IMA $\lim_{n\to\infty} = 0$

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ $\qquad a_n = \frac{1}{n(n+1)}$

~~$a_{n+1} \leq a_n$~~

$$\frac{a_{n+1}}{a_n} < 1$$

$$\frac{\cancel{(n+1)}\cancel{(n+2)}\, n}{n\cancel{(n+1)}(n+2)} < 1 \quad \text{padajoče}$$

$$\lim \frac{1}{n(n+1)} = \frac{0}{1} = 0$$

2 $\sum_{n=1}^{\infty} \frac{(-1)^3}{2^n}$

$a_n = \frac{1}{2^n}$ $\qquad \frac{a_{n+1}}{a_n} \overset{?}{\leq} 1$

$\frac{1 \cdot 2^{n+1}}{2^{n+1}} = \frac{1}{2} <$ je padajoče

$\lim_{n\to\infty} \frac{1}{2^n} = \frac{\frac{1}{2^n}}{(\frac{2}{2})^n} = 0$

KOLOKVIJ

a) $z^5 + \frac{\sqrt{3}}{2} z + \frac{\sqrt{3}}{2} i \cdot z = 0$

$z(z^4 + \sqrt{3}_2 + i\frac{\sqrt{3}}{2}) = 0$

$z = 0$

$z^4 = -\frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{2}$

$|z|^4(\cos 4\varphi + i \cdot \sin\varphi) = \sqrt{3}^4 \left(\cos\frac{5\pi}{4} + i \cdot \sin\frac{5\pi}{4}\right)$

$|z|^4 = \sqrt{3}$

$z = \sqrt[8]{3}$

$4\varphi = 5\pi/4 + 2k\pi$

$\varphi = \frac{\frac{5\pi}{4} + 2k\pi}{4}$

$k_0 = \sqrt[8]{3}\left(\cos\frac{5\pi}{16} + i \cdot \sin\frac{5\pi}{16}\right)$

$k_1 = \sqrt[8]{3}\left(\cos\frac{13\pi}{16} + i \cdot \sin\frac{13\pi}{16}\right)$

$k_2 = \sqrt[8]{3}\left(\cos\frac{21\pi}{16} + i \cdot \sin\frac{21\pi}{16}\right)$

$k_3 = \sqrt[8]{3}\left(\cos\frac{29\pi}{16} + i \cdot \sin\frac{29\pi}{16}\right)$

2005

$|z|^2 + \frac{1}{z} - (\mathrm{Re}(z))^2 = 2$ $\qquad z = a + ib$

$\cancel{a^2} + b^2 + \frac{1}{a-ib} \cdot \frac{a+ib}{a+ib} - \cancel{a^2} = 2$

$b^2 + \frac{a+ib}{a^2+b^2} = 2$

$b = 0$ $\qquad z = \frac{1}{2}$

$\frac{a}{a^2} = 2$

$a = 2a^2$

$a = 1/2$

$x = 1$

$|x-1| > |x| + x$

$x < 0$ $\quad 0 \leq x < 1$ $\quad x > 1$

$-x+1 > \cancel{-x+x}$

$1 > x$ //

$x-1 > x+x$

$-1 > x$ //

$-x+1 > x+x$

$1 > 2x$

$\frac{1}{2} > x$

$x\ [0; \infty)$

$x \in [0, \frac{1}{2})$

FUNKCIJE

MAT. I V
20. 11. 2007

Definicijsko območje D_f

$f: D \to Z$

Zaloga vrednosti

$Z_f = \{ f(x);\ x \in D \}$

1. $f(x) = \sqrt{4-x^2} + \frac{1}{x}$ $\quad x > 0$

$4 - x^2 \geq 0$

$x^2 < 4$

$|x| < |2|$

$D_f \{-2, 2\} \setminus \{0\}$

$x \in [-2, 2] - \{0\}$

2. $f(x) = x + \ln \frac{x-1}{x+1}$

da se ohrani predznak

$\frac{\cancel{x-1}}{x+1} \geq 0 \quad (x+1)^2 \quad x \neq 1$

$\cancel{(x-1)(x+1)} \geq 0$

$x^2 - 1 \geq 0$

$x^2 \geq 1$

$|x| > 1$

$x \in (-\infty, -1) \cup (1, \infty)$

3. $f(x) = 1 - x^2$

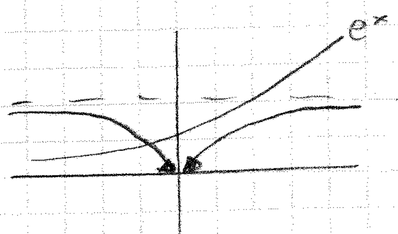
$D_f: \mathbb{R}$

$Z_f: [-\infty, 1]$

4. $f(x) = e^{-\frac{1}{x^2}}$

$D_f: \mathbb{R} \setminus \{0\}$

$Z_f: (0, 1)$



$x \to 0 \quad -\frac{1}{x^2} = -\infty \to e^{-\frac{1}{x^2}} = 0$

$x \to \pm\infty \quad \frac{1}{x^2} \to 0 \to e^{\frac{1}{x^2}} = 0$

f: SODA, če je $f(-x)=f(x)$ $\{x^2, \cos x\}$
f: LIHA, če je $f(-x)=-f(x)$ $\{x^3, \sin x, \ldots\}$

1. $f(x) = \frac{x}{1+x^2}$ $f(-x) = \frac{-x}{1+(-x)^2} = \frac{-x}{1+x^2} = -\frac{x}{1+x^2} = -f(x)$ - liha

$g(x) = \frac{\sin x}{x^3}$; $g(-x) = \frac{-\sin x}{-x^3} = \frac{\sin x}{x^3}$ => soda

$f: D \to Z$ je injektivna
če za vsak x_1, x_2 iz D $x_1 \neq x_2$ velja
$f(x_1) \neq f(x_2)$

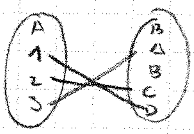
$(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

Z_f = je množica rešitev

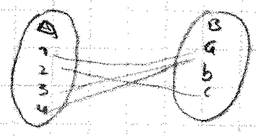
Z - je predpis kam slike
velja kot Z_f

DOMENA KODOMENA
$f: D \to Z$ je surjektivna
če za vsak y iz Z obstaja $x \in D: y = f(x)$
$(Z = Z_f)$

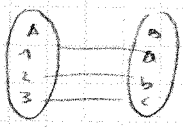
f je bijektivna, če je surjektivna in injektivna



je injektivna

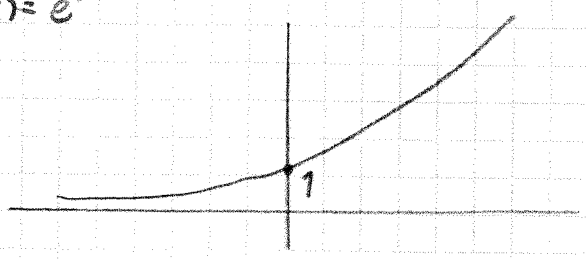


ni injektivna
je surjektivna



je bijektivna

1. $f(x) = e^x$



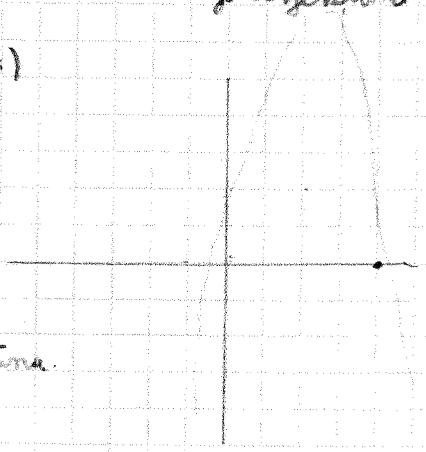
$D_f: \mathbb{R}$

$Z_f: \mathbb{R}_+$

a) $f: \mathbb{R} \to \mathbb{R}$
je injektivna
ni surjektivna

b) $\mathbb{R} \to \mathbb{R}_+$
je injektivna
je surjektivna
je bijektivna

2. $f(x) = 8x - 2x^2 = 2x(4-x)$
$D_f: \mathbb{R}$
$Z_f: \cancel{(8, -\infty)}$ $(-\infty, 8]$

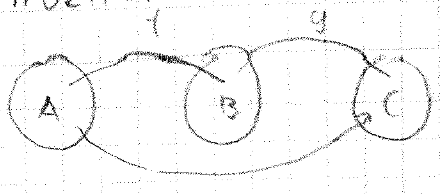


a) $f: \mathbb{R} \to \mathbb{R}$
ni injektivna in surjektivna

$f: [2, \infty) \to \mathbb{R}$
je injektivna, ni surjektivna

$f: [2, \infty) \to (-\infty, 8]$
je bijektivna

Kompozitum



$$g\circ f = g(f(x))$$
$$(g\circ f)(x) = g(f(x))$$

1. $f(x) = e^x$
   $g(x) = x^2$

   $(g\circ f)(x) = g(f(x)) = (e^x)^2 = e^{2x}$
   $(f\circ g)(x) = f(g(x)) = f(x^2) = e^{x^2}$

   $f\circ g \neq g\circ f$

2. $f(x) = \frac{1}{1+x}$
   $g(x) = 1+x^2$

   $(g\circ f)(x) = g\left(\frac{1}{1+x}\right) = 1+\left(\frac{1}{1+x}\right)^2 = 1+\frac{1}{1+2x+x^2} = \frac{2+2x+x^2}{1+2x+x^2}$
   $(f\circ g)(x) = f(1+x^2) = \frac{1}{2+x^2}$

3. $f(x) = -1+2x$
   $g(x) = 2-3x$

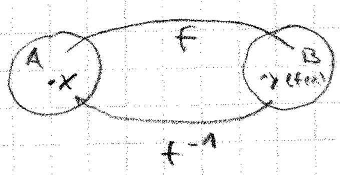
   $(g\circ f)x = 2+3-6x = 5-6x$

   $(f\circ g)x = -1+4-6x = 3-6x$

   $(f\circ f)x = -1-2+4x = -3+4x$

   $(g\circ g)x = 2-6+9x = -4+9x$

Inverznost funkcije



$f^{-1}(y) = x$ $\qquad$ $f^{-1}\circ f = id$
$f^{-1}(f(x)) = x$

1. $f(x) = 1+\operatorname{arctg} 3x$ $\qquad$ $y = 1+\operatorname{arctg} 3x$
   $f^{-1}(x) =$

   $x = 1+\operatorname{arctg} 3y$
   $x-1 = \operatorname{arctg} 3y$
   $\frac{\operatorname{tg}(x-1)}{3} = y$ $\qquad$ $f^{-1}(x) = \frac{\operatorname{tg}(x-1)}{3}$

2. $f(x) = \frac{2x+3}{x-2}$

   $x = \frac{2y+3}{y-2}$

   $2y+3 = xy-2x$
   $2y-xy = -3\cdot$
   $y(2-x)$

   $y = \frac{2x+3}{x-2}$
   $f^{-1}(x) = \frac{2x+3}{x-2}$

LIMITE FUNKCIJ

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

1. $$\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{3x\to 0} \frac{3\sin 3x}{3x} = 3$$

$$\lim_{x\to 0} \frac{\sin ax}{x} = a$$

2. $$\lim_{x\to 0} \frac{x\sin ax}{x\sin bx} = \lim_{x\to 0} \frac{a\sin ax\cdot bx}{ax\sin bx} = \lim_{x\to 0} = \frac{a}{b}$$

3. $$\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{(\frac{\pi}{2}-x)^2} = \frac{1-\sin(\frac{\pi}{2}-y)}{y^2} = \frac{1-\cos y}{y^2}$$

$y = \frac{\pi}{2} - x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$= \lim_{y\to 0} \frac{\sin^2\frac{y}{2} + \cos^2\frac{y}{2} - \cos^2\frac{y}{2} + \sin^2\frac{y}{2}}{y^2}$$

$$= \lim_{y\to 0} \frac{2\cdot \sin^2\frac{y}{2}}{y^2} = 2\lim\left(\frac{\sin\frac{y}{2}}{y}\right)^2$$

$$= 2\left(\lim_{y\to 0} \frac{\frac{1}{2}\sin\frac{y}{2}}{\frac{y}{2}}\right)^2 = 1/2$$

4. $$\lim_{x\to\infty} \left(\frac{x^2+1}{x^2-1}\right)^{x^2} = \left(1+\frac{2}{x^2-1}\right)^{x^2} = \left(\left(1+\frac{1}{\frac{x^2-1}{2}}\right)^{\frac{x^2-1}{2}}\right)^{\frac{x^2}{\frac{x^2-1}{2}}} = e^2$$

$$\lim_{x\to\infty} P(x)\, e^{-x} = 0 \qquad P(x) \text{ polinom}$$

$$\lim_{x\to 0} x\cdot \ln x = \lim_{y\to\infty} \frac{1}{y} \ln\frac{1}{y} = \lim_{z\to -\infty} \frac{1}{e^{-z}}\cdot \ln e^{-z}$$

$$= \lim_{z\to\infty} \frac{1}{e^{-z}}\cdot \ln e^{-z} = 0$$

$\ln\frac{1}{y} = -\ln y$

$z = -\ln y$

$e^{-z} = y$

1. $\lim_{x\to4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}\cdot\frac{(\sqrt{x}+2)(\sqrt{1+2x}+3)}{(\sqrt{x}+2)(\sqrt{1+2x}+3)}$

$= \lim_{x\to4}\frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} =$

$= \lim_{x\to4}\frac{(2x-8)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \frac{2(x-4)\cdot}{(x-4)} = \frac{2\cdot4}{6} = \frac{4}{3}$

2. $\lim_{x\to1}\frac{x^m-1}{x^n-1} = \lim_{x\to1}\frac{(x-1)(\overbrace{x^{m-1}+\dots}^{m})}{(x-1)(\underbrace{x^{n-1}+\dots}_{n})}$ $\quad x^n-1=(x^n-1)(x^n+x^{n-1}+\dots+x+1)$

$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+ab^{n-2}+b^{n-1})$

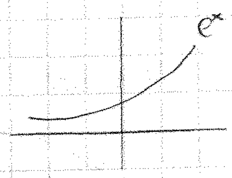
$=\lim_{x\to1} = \frac{m}{n}$

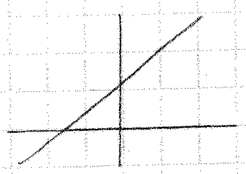
3. $\lim_{x\to1}\frac{x-\sqrt{x}}{\sqrt{x}-1} = \frac{x^2-x(\sqrt{x}+1)}{(x+1)(x+\sqrt{x})} = \frac{x(x-1)(\sqrt{x}+1)}{(x-1)(x+\sqrt{x})} = \frac{2}{2} = 1$

4. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}} = \frac{\sqrt{1}-0}{\sqrt{1}-0} = 1$

ZVEZNOST FUNKCIJ

$f(x)=\begin{cases} e^x; & x<0 \\ x+a; & x\geq 0\end{cases}$





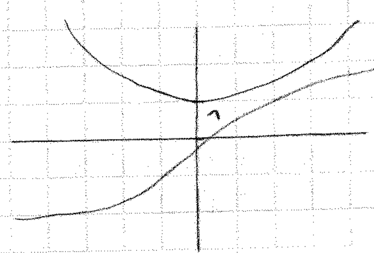
f je zvezna v b, če je $\lim_{x\uparrow b} f(x) = \lim_{x\downarrow b}$

L. $\lim_{x\uparrow0} f(x) = \lim_{x\uparrow0} e^x = 1$

$\lim_{x\downarrow0} f(x) = \lim_{x\downarrow0} x+a = a$ $\qquad a=1$

$\text{ch}\,x = \frac{e^x+e^{-x}}{2}$

$\text{sh}\,x = \frac{e^x-e^{-x}}{2}$



$\text{ch}^2x-\text{sh}^2x=1$

$\text{ch}\,2x=\text{ch}^2x+\text{sh}^2x$

$\text{sh}\,2x = 2\,\text{sh}\,x\,\text{ch}\,x$

$\text{sh}'x=\text{ch}\,x$

$\text{ch}'x=\text{sh}\,x$

$$f(x) = \begin{cases} x \cdot \sin\frac{1}{x} & x \neq 0 \\ a & x = 0 \end{cases}$$

$$a = \lim_{x\to 0}\left(x \cdot \sin\frac{1}{x}\right) = \lim_{y\to\infty} \frac{1}{y} \cdot \sin y = \frac{\sin y}{y} = 0$$

$y = \frac{1}{x}$

$y \to \infty$

4. $$f(x) = \begin{cases} 2\cdot\sqrt{x} & ; 0 \leq x < 1 \\ 4-2x & ; \frac{5}{2} > x \geq 1 \\ 2x-7 & \frac{5}{2} \leq x \leq 4 \end{cases}$$

točka 1

$\lim_{x\uparrow 1} 2\sqrt{x} = 2$ $\quad \lim_{x\downarrow 1} f(x) = \lim_{x\downarrow 1} 4-2x = 2$ $\quad \vee$ $x=1$ je zvezna

$\lim_{x\uparrow \frac{5}{2}} 4f(x) = \lim_{x\uparrow\frac{5}{2}} 4-5 = -1$ $\quad \lim_{x\downarrow \frac{5}{2}} 2x-7 = -2$ $\quad$ ni zvezna

27. 11. 2007

ODVODI

Tabela elementarnih odvodov:

$(C') = 0$
$(e^x)' = e^x$
$(\ln x)' = \frac{1}{x}$
$(\sin x) = \cos x$
$(\cos x) = -\sin x$
$(\tan x) = \frac{1}{\cos^2 x}$
$(\operatorname{ctg} x) = -1/\sin^2 x$

$(x^n)' = n x^{n-1}$
$(\sqrt{x})' = 1/2\sqrt{x}$
$(\frac{1}{x})' = -\frac{1}{x^2}$
$(\arcsin x)' = 1/\sqrt{1-x^2}$
$(\arccos x)' = -1/\sqrt{1-x^2}$
$(\arctan x)' = 1/1+x^2$
$(\operatorname{arcctg} x)' = -1/1+x^2$

Pravila za odvajanje

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$
$(C \cdot f(x))' = f'(x) \cdot C$

$(f(x) \cdot g(x))' = f'(x) g(x) + f(x) g'(x)$
$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) g'(x)}{g^2 x}$

$f(g(x)) = f'(g(x)) \cdot g'(x)$

1. $f(x) = (x \cdot \sqrt{1+x^2})'$

$= (1 \cdot \sqrt{1+x^2} + x \frac{1}{2\sqrt{1+x^2}} \cdot 2x)$

$= \frac{1+x^2-x^2}{\sqrt{1+x^2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$

2.

$f(x) = 3\sin x - 5\cdot\cos x$

$f'(x) = 3\cdot\cos x + 5\cdot\sin x$

3. $f(x) = x \cdot \operatorname{arctg} x$

$f'(x) = 1 \cdot \operatorname{arctg} x + x \cdot \frac{1}{1+x^2}$

4. $f(x) = \ln^3(x^2)$

$x \xrightarrow{f} x^2 \xrightarrow{h} \ln(x^2) \xrightarrow{()^3}$

$= 3\ln^2 x^2 \frac{1}{x\cancel{x}} \cdot 2\cancel{x}$

$= \frac{6\ln^2(x^2)}{x}$

5. $f(x) = e^x \cdot \cos x$

$f'(x) = e^x \cdot \cos x - e^x \cdot \sin x$

$= e^x(\cancel{\sin}\cos x - \sin x)$

6. $f(x) = \frac{\ln(\sin(x))}{\ln(\cos x)}$ $f'(x) = \frac{\frac{1}{\sin x} \cdot \cos x \cdot \ln(\cos x) - \ln(\sin x) \cdot \frac{1}{\cos x} \cdot (-\sin x)}{\ln^2(\cos x)}$

$f'(x) = \frac{\operatorname{ctg} x \cdot \ln(\cos x) + \operatorname{tg} x \cdot \ln(\sin x)}{\ln^2(\cos x)}$

7. $f(x) = \ln(\ln^2(\ln^3 x)))$

$f'(x) = \frac{1}{\ln^2(\ln^3 x))} \cdot 2 \cdot \ln \cdot (\ln^3 x)) \cdot 3 \cdot \ln^2 x \cdot \frac{1}{x} \cdot \frac{1}{\ln^3(x)}$

$= \frac{6}{\ln(\ln^3(x) \cdot \ln(x) \cdot x}$

8. $f(x) = x^{\frac{1}{x}}$

$f'(x) = e^{\ln x \cdot \frac{1}{x}} \Rightarrow f'(x) = e^{\ln x \cdot \frac{1}{x}} \cdot (\frac{1}{x} \cdot \frac{1}{x} + \ln x)(-\frac{1}{x^2})$

$x = e^{\ln x}$ $F'(x) = \frac{1}{x^2}\, e^{\ln x \cdot \frac{1}{x}}(1 - \ln x)$

9. $f(x) = \frac{1}{x+2} + \frac{3}{x^2+1}$ $f'(0)$ in $f'(-1) = 2$

$f'(x) = (x+2)^{-1} + 3\cdot(x^2+1)^{-1}$

$= -(x+2)^{-2} + 3(x^2+1)^{-2} \cdot 2x$ $f'(0) = -(-2)^{-2} + 3\cdot(1)^{-2}$

$f'(0) = -\frac{1}{4} + 0 = -\cancel{3}\frac{1}{4}$

$f(-1) = -1 + \frac{+6}{4} = \frac{1}{2}$

3.1

10. $\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$

$\operatorname{sh}(x) = \frac{e^x - e^{-x}}{2}$

$(\operatorname{ch} x)' = \frac{e^x - e^x}{2} = \operatorname{sh} x$

$(\operatorname{sh} x)' = \frac{e^x + e^x}{2} = \operatorname{ch} x$

11. $x + y = x^2 + x^3$

$1 + y' = 2x + 3x^2$

$y' = 3x^2 + 2x - 1$

12. $\ln(x^2+y^2) = \operatorname{arctg}\left(\frac{y}{x}\right)$

$\frac{1}{x^2+y^2}(2x+2yy') = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{y'x - y \cdot 1}{x^2}$

$\frac{2x+2yy'}{x^2+y^2} = \frac{1y'x - y}{\cancel{1+(\frac{y}{x})^2}\, x^2+y^2} = .$

$2x + 2yy' = y'x - y$

$2yy' - y'x = -y - 2x$

$y' \cancel{(2y - x)} = \frac{-y-2x}{2y-x}$

L'HOPITALOVO PRAVILO:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

$\Rightarrow \frac{0}{0}$ ali $\frac{\infty}{\infty}$ - ce vstavimo

13 $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$

14. $\lim_{x \to 0} \frac{2 \cdot \arcsin x}{3x} = \frac{2}{3} \lim_{x \to 0} \frac{\frac{1}{\sqrt{1+x^2}}}{1} = \frac{2}{3} \cdot 1$

15. $f(x) = e^{\sin x} \cos x$

$f'(x) = e^{\sin x} \cdot (\cos^2 x - e^{\sin x} \cdot \sin x = e^{\sin x}(\cos^2 x - \sin x)$

$f''(x) = e^{\sin x} \cdot \cos x(\cos^2 x - \sin x) + e^{\sin x}(-2\sin x \cos x \cdot \sin x - \cos x)$

$f''(x) = e^{\sin x} \cos x (\cos^2 x - \sin x - 2 \sin x - 1)$

$e^{\sin x} \cos x (\cos^2 x - 3 \sin x - 1)$

16 $f(x) = x(\sin(\ln x) + \cos(\ln x))$

$f'(x) = 1(\sin(\ln x) + \cos(\ln x)) + x(\cos(\ln x)\frac{1}{x} + \sin \ln x \cdot \frac{1}{x})$

$= (\sin(\ln x) + \cos(\ln x)) + (\cos \ln(x) - \sin(\ln x))$

$= 2 \cdot \cos \ln(x)$

$f''(x) = -2\sin(\ln x)\frac{1}{x}$

17. $x = \ln(1+y)$

$y'' = ?$

$1 = \frac{1}{1+y} \cdot y'$

$y' = \cancel{1 - \frac{1}{1+y}} = 1+y$ $\quad y'' = y' = 1+y$

$y' = \cancel{\frac{y}{1+y}}$

$y'' = \cancel{\frac{y'(1+y) - y \cdot y'}{(1+y)^2}}$

$y'' = y' = 1+y$

18. $f(x) = e^{-3x}$

$f'(x) = e^{-3x} \cdot (-3)$ $\quad y^{(n)} = (-3)^n \cdot e^{-3x}$

$f''(x) = -3e^{-3x} \cdot (-3) = 9e^{-3x}$

$f'''(x) = -3e^{-3x}(-3)^2 = 27e^{-3x}$

19. $f(x) = \frac{1}{x^2 - a^2} =$

$= -\frac{\frac{1}{2a}}{x+a} + \frac{\frac{1}{2a}}{x-a} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{(x+a)}\right)$

$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{(x+a)} + \frac{B}{(x-a)}$

$= \frac{(x-a)A + B(x+a)}{x^2-a^2}$

$= \frac{Ax - aA + Bx + Ba}{}$

$A+B = 0 \quad -aA + Ba = 1$

$A = -B$

$-A + B = 0 \; 1$

$+B + B \cdot a = 1$

$2b \cdot a = 1$

$a = \frac{1}{2B}$

$B = \frac{1}{2a}$

$A = -\frac{1}{2a}$

$f'(x) = \frac{1}{2a}(-(x-a)^{-2} + (x+a)^{-2})$

$f''(x) = \frac{1}{2a}(2(x-a)^{-3} - 2(x+a)^{-3})$

$f'''(x) = \frac{1}{2a}(-6(x-a)^{-4} + 6(x+a)^{-4})$

$f^{(n)}(x) = \frac{1}{2a}((-1)^n \cdot n!(x-a)^{-n-1} + (-1)^{n-1} n!(x+a)^{-n-1})$

$= \frac{1}{2a}(-1)^n \cdot n!((x-a)^{-n-1} - (x+a)^{-n-1})$

19. $f(x) = \begin{cases} e^x + x^2 + 1 & x < 0 \\ ax + b & x \geq 0 \end{cases}$ f je zvezna, če je

$$f'(x) \begin{cases} e^x + 2x & x < 0 \\ a & x \geq 0 \end{cases}$$

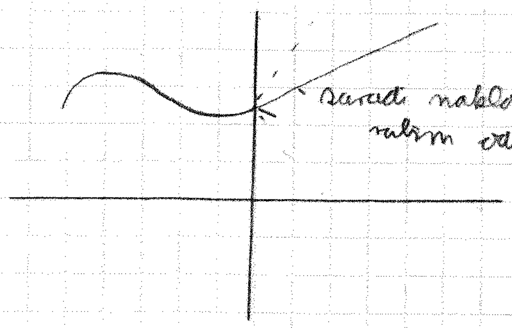
$$\lim_{x \uparrow 0} f(x) = \lim_{x \downarrow 0} f(x)$$

$$\lim_{x \uparrow 0} f(x) = \lim_{x \uparrow 0} (e^x + x^2 + 1) = 2$$

$$\lim_{x \downarrow 0} f(x) = \lim_{x \downarrow 0} (ax + b) = b \qquad b = 2$$

$$\lim_{x \uparrow 0} f'(x) = \lim_{x \uparrow 0} e^x + 2x = 1$$

$$\lim_{x \downarrow 0} f'(x) = \lim_{x \downarrow 0} a = a \qquad a = 1$$



20. $f(x) = \frac{2x}{1+x^2}$ f narašča / pada če je $f' \gtrless 0$

$$f'(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$2 - 2x^2 > 0$$
$$-2x^2 > -2$$
$$x^2 < 1$$
$$|x| < 1$$
$$-1 < x < 1$$

pada $(-\infty, -1] \cup [1, \infty)$

EKSTREMI

a je stacionarna točka, če je $f'(a) = 0$

$f''(a) < 0$ maksimum
$f''(a) > 0$ minimum
$f''(a) = 0$ sedlo

21. $f(x) = 8x - 2x^2$

$f'(x) = 8 - 4x$ $\qquad$ $f''(x) = -4$

$8 - 4x = 0$

$8 = 4x$ $\qquad$ $F''(2) = -4 < 0$ maksimum

$x = 2$

22. $f(x) = x^2 \cdot e^{-x}$

$f'(x) = 2x \cdot e^{-x} - x^2 \cdot e^{-x}$

$f'(x) = x(2e^{-x} - xe^{-x})$

$f'(x) = x \cdot e^{-x}(2 - x)$

$x = 0$ $\qquad$ $x = 2$

$\cancel{f'(x)}$

$f'(-1) = -3e^{1}$ – manjše od 0 pada

$f(0) =$ minimum

$f'(1) = e^{-1} > 0$ narašča

$f(2)$ maksimum

$f'(3) = -3e^{-3} < 0$ pada

23. $f(x) = 2\,tg(x) - tg^2(x)$ $\qquad$ $I = [0, \frac{\pi}{2})$

$f'(x) = 2 \cdot \frac{1}{\cos^2 x} - 2\,tg(x) \cdot \frac{1}{\cos^2 x}$

$f'(x) = \frac{2 - 2tg(x)}{\cos^2 x}$

$2 - 2tg(x) = 0$ $\qquad$ $f'(\frac{\pi}{6}) = \frac{2 - 2\frac{\sqrt{3}}{3}}{\frac{3}{4}} > 0$

$1 - tg(x) = 0$

$1 = tg(x)$

$x = \frac{\pi}{4} + k\pi$ $\qquad$ $f(\frac{\pi}{3}) = \frac{2 - 2\sqrt{3}}{\frac{1}{4}} < 0$

24. DIFERENCIAL

$f(x_0) \simeq f(a) + (x_0 - a) f'(a)$

$f(x) = \sqrt{3x^2 + 3x + 4}$ $\qquad$ $f(0) = 2$

$x_0 = \frac{8}{100}$

$a = 0$ $\qquad$ $f'(x) = \frac{1}{2\sqrt{3x^2+3x+4}} \cdot 6x + 3$

$f'(a) = \frac{3}{4}$

$f(\frac{8}{100}) = f(0) + (\frac{8}{100} - 0) \cdot f'(a) =$

$= 2 + \frac{2}{100} \cdot \frac{3}{\cancel{4}} = 2{,}06$

MAT IV
4. 12. 2007

$f(x) = x^4 - x^3 - x^2$
$f(x) = x^2(x^2 - x - 1)$
0

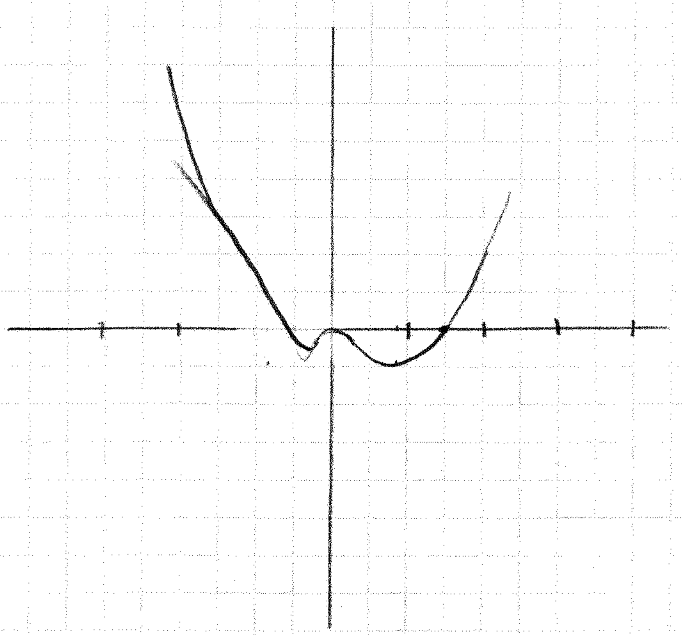
$\frac{1 \pm \sqrt{5}}{2}$ $x_2 = \frac{1-\sqrt{5}}{2}$ $x_3 = \frac{1+\sqrt{5}}{2}$

$f'(x) = 4x^3 - 3x^2 - 2x = 0$
$x(4x^2 - 3x - 2) = 0$
0

$\frac{3 \pm \sqrt{9+32}}{12}$ $x_1 = \frac{3+\sqrt{41}}{8} \doteq 1{,}1$

$x_2 = \frac{3-\sqrt{41}}{8} \doteq 0\bar{5}$

$f''(x) = 12x^2 - 6x - 2$



$f(x) = \frac{2x}{1-x^2}$

ničle : 0

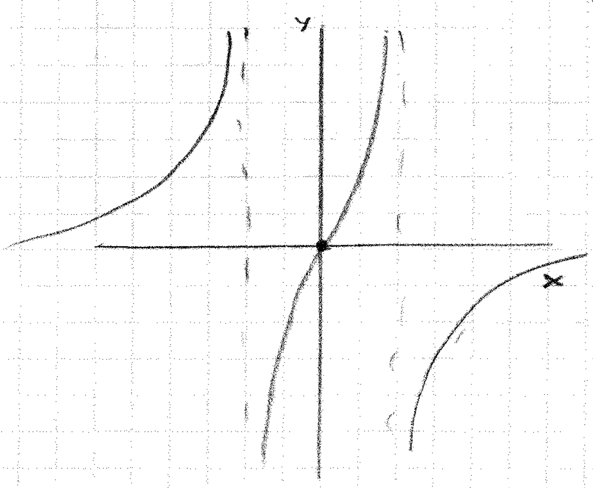
poli : $1 - x^2$
$(1-x)(1+x)$
$x = -1, \quad x = 1$

asimptota
$y = 0$

$f'(x) = \frac{2\cdot(1-x^2) - 2x(-2x)}{(1-x^2)^2}$

$f'(x) = 2 - 2x^2 + 4x^2$
$2 + 2x^2 \neq 0$
ni

če je pol lihe stopnje, nadaljujemo na drugi strani, kjer smo končali



$$f(x) = \frac{2x}{1+x^2}$$

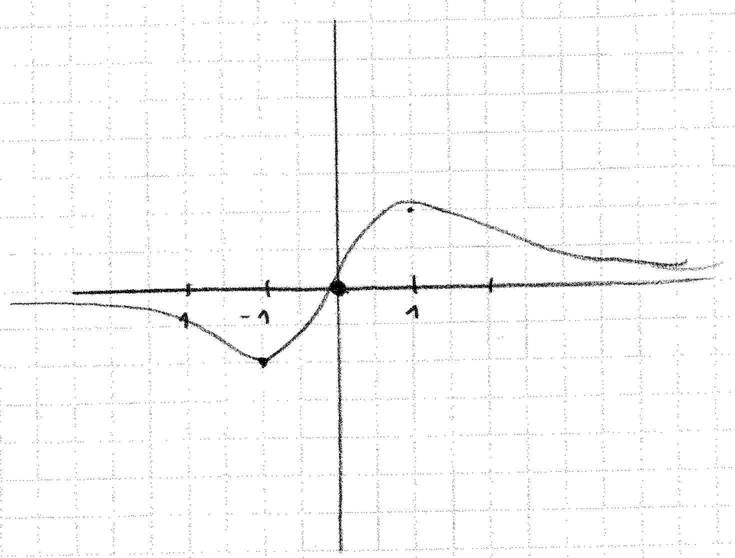
ničle : 0

$$f'(x) = \frac{2+2x^2-4x^2}{(1+x^2)^2}$$

$$2-2x^2=0$$
$$x^2=1$$
$$x_1=1$$
$$x_2=-1$$

1 $-4x$ <0 maks

-1 >0 min



$$f(x) = \frac{x^2+x}{x+2}$$

pol $x=-2$

asimptota $\frac{x^2+x}{x+2} = x-1$

ničle : $x=0$
$x=-1$

če je ostanek pri deljenju ni nič in da se deli z asimp $\Rightarrow$ 0 - seka tam graf asimp.

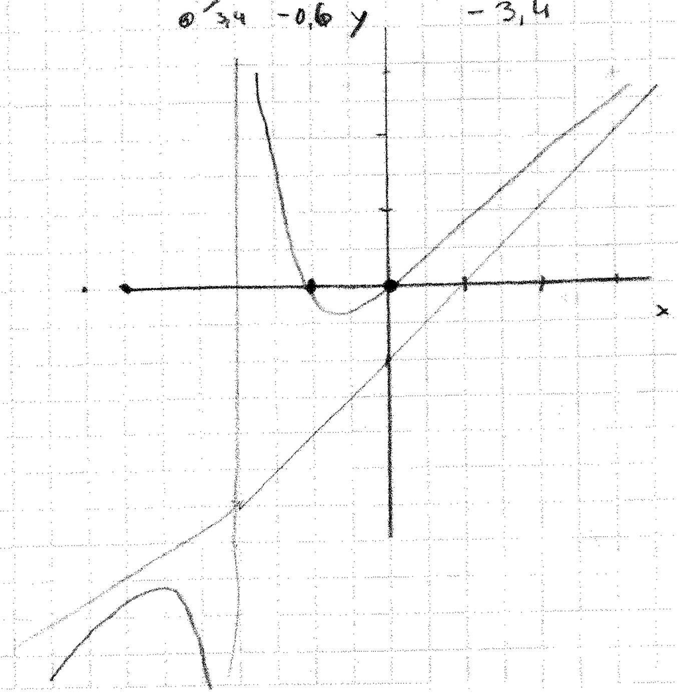
$$f'(x) = \frac{(2x+1)(x+2)-(x^2+x)\cdot 1}{(x+2)^2}$$

$$2x^2+x+4x+2$$
$$2x^2+5x+2$$

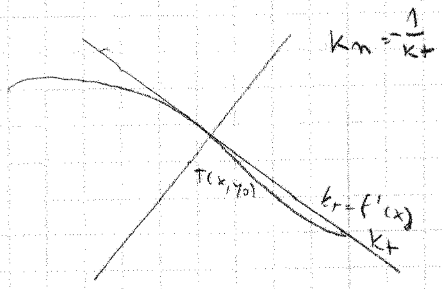
poglej vmesne točke

$x_1 = -2+\sqrt{2}$ $x_2 = -2-\sqrt{2}$

$-0,6$ $-3,4$



RACUNANJE TANGENT in NORMAL



$y = x^3 + 2x^2 - 4x - 3$ $\quad$ $T(-2, 5)$

$y' = 3x^2 + 4x - 4$

$k_t = 3 \cdot (-2)^2 + 4 \cdot (-2) - 4$
$= 12 - 8 - 4 = 0$

$y - y_0 = k(x - x_0)$ $\quad$ $x = -2$

$y - 5 = 0$

$y = 5$

$y = x^{\cos x} = e^{\ln x \cdot \cos x}$ $\quad$ $x_0 = \pi$

$y' = e^{\ln x \cdot \cos x} \cdot (\cos x \frac{1}{x} - \ln x \cdot \sin x)$

$y_0 = y(x_0)$

$y_0 = \pi^{-1}$

$k_t = y'(\pi)$

$(e^{\ln \pi})^{\cos \pi} \cdot (-1) \frac{1}{\pi} - \ln \pi \cdot 0)$

$= \frac{1}{\pi} (-1) \cdot \frac{1}{\pi} = -\frac{1}{\pi^2}$

$y - \frac{1}{\pi} = -\frac{1}{\pi^2}(x - \pi)$

$y = -\frac{1}{\pi^2} x + \frac{2}{\pi}$

$k_n = \pi^2$

$y = \pi^2 x - \pi^3 + \frac{1}{\pi}$

$y^2 + y - 6x = 0$ $\quad$ $T(1, -3)$

$2yy' + y' - 6 = 0$

$y'(2y + 1) = 6$

$k_t = y' = \frac{6}{2y+1}$

$y' = \frac{6}{-5} = -\frac{6}{5}$

$y + 3 = -\frac{6}{5}(x - 1)$

$y = -\frac{6}{5}x - \frac{9}{5}$

$k_n = -\frac{1}{-\frac{6}{5}} = \frac{5}{6}$

$y + 3 = \frac{5}{6}x - \frac{5}{6}$

$y = 5/6\, x - \frac{23}{6}$

$f(x) = x^2$ $\quad$ $[0, 3]$ $\quad$ $A(0, f(0))$, $B(3, f(3))$

$y = \frac{y_B - y_A}{x_B - x_A}(x - x_A)$ $\quad$ $k = 3$

$k = \frac{9}{3}$ $\quad$ $3 = 2x$

$x = 3/2$

$y = x \cdot \ln x$

$2x - 2y + 3 = 0$

$y = x + \frac{3}{2}$

$k_n = 1$

$k_r = -1$

$1 \cdot \ln x + x \cdot \frac{1}{x}$

$\ln x + 1 = -1$

$\ln x = -2$

$x = e^{-2}$

$y_0 = e^{-2} \cdot (-2)$

$y_0 = -2e^{-2}$

$y = x - e^{-2} - 2e^{-2}$

$y = x - 3e^{-2}$

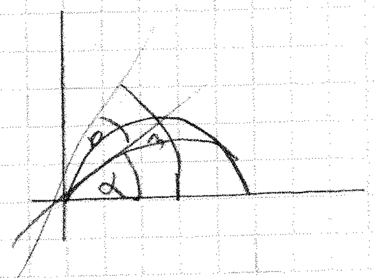
1. $y = \sin x$
2. $y = \sin 2x$

$y' = \cos x = k_1$

$y' = 2 \cdot \cos 2x = k_2$

$k_1 = 1$

$k_2 = 2$



$\tg \alpha = k_1$

$\tg \beta = k_2$

$\varphi = \beta - \alpha$

$\tg = \tg(\beta - \alpha) = \frac{\tg\beta - \tg\alpha}{1 + \tg\alpha \cdot \tg\beta} = \frac{k_2 - k_1}{1 + k_1 \cdot k_2}$

$\tg \varphi = \frac{2-1}{1+2} = \frac{1}{3}$, $\tg \varphi = \frac{1}{3}$ $\varphi = 18°26'$

EKSTREMALNI PROBLEMI

Imamo število 36

$36 = x \cdot y$ $y = \frac{36}{x}$

$\Rightarrow x^2 + y^2$ je min.

$\left(x^2 + \left(\frac{36}{x}\right)^2\right)' = 0$

$x^2 + \frac{36^2}{x^2}$

$x^2 + 36 \cdot x^{-2}$

$+2x - 72x^{-3} = 0$

$\frac{2x^4 - 72}{x^3} = 0$

$\frac{2x^4 - 36}{} = 0$

$(x^2 + 36)(x^2 - 36)$

$x_1 = \sqrt{6}$ $x_2 =$ $\pm\sqrt{6}$

$(x-6)(x+6)$

6 -6

$x = 6$

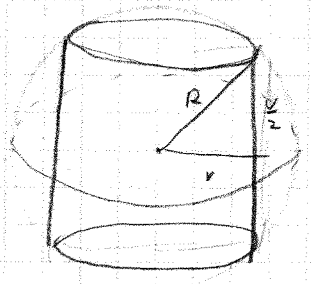
$y = 6$

$x = -6$

$y = -6$

Imamo kroglo z radijem R, vrtamo valj z maks. volumnom

$h=?\quad r=?$



$$4\left(\frac{v}{2}\right)^2 + r^2 = R^2$$

$$V = \pi r^2 \cdot v$$

$$V = \pi \left(R^2 - \frac{v^2}{4}\right) \cdot v$$

$$V_{(v)} = \pi R^2 \cdot v - \frac{v^3}{4}\pi$$

$$\frac{dV}{dv} = \pi R^2 - \frac{3\pi}{4} v^2 = 0$$

$$4\pi R^2 = 3\pi v^2$$

$$v = \sqrt{4/3 R^2}$$

$$v = \frac{2}{\sqrt{3}} R = \frac{2\sqrt{3}R}{3}$$

$$r^2 = R^2 - \frac{\frac{\cancel{4}\cdot 3R^2}{9}}{4}$$

$$r^2 = \frac{3R^2 - R^2}{3} = \frac{2R^2}{3}$$

$$r = \sqrt{\frac{2}{3}} R = R \cdot \sqrt{6}$$

Stožec z volumnom $V = \frac{\sqrt{2\pi}}{3}$

$r=?$

$P = \cancel{\pi r^2} + \pi \cdot s \cdot r$

nima dna

$$V = \frac{\pi r^2 \cdot h}{3} = \frac{\sqrt{2\pi}}{3} \qquad s^2 = r^2 + h^2$$

$$\pi r^2 \cdot h = \sqrt{2\pi}$$

$$P = \pi r \sqrt{r^2 + v^2} = \pi r \sqrt{r^2 + \left(\frac{\sqrt{2\pi}}{\pi r^2}\right)^2}$$

$$= \pi r \sqrt{r^2 + \frac{2\pi}{\pi^2 r^4}}$$

$$= \pi r \sqrt{\frac{\pi^2 r^6 + 2\pi}{\pi^2 r^4}}$$

$$= \cancel{\pi r}\frac{\sqrt{\pi^2 r^6 + 2\pi}}{\cancel{\pi} r^{\cancel{2}}} = \frac{\sqrt{\pi^2 r^4 + 2\pi}}{r} =$$

$$= \frac{\frac{3\pi^2 \cancel{6} r^5}{2\sqrt{\pi^2 r^6 + 2\pi}}\cdot r - \sqrt{\pi^2 r^4 + 2\pi}\cdot 1}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 2\pi}{r^2 \sqrt{\pi^2 r^4 + 2\pi}}$$

$$= \frac{2\pi^2 r^6 - 2\pi}{r^2 \sqrt{\pi^2 r^4 + 2\pi}}$$

$$2\pi^2 r^6 - 2\pi = 0$$

$$\pi r^6 = 1$$

$$r^6 = \frac{1}{\pi}$$

INTEGRAL

Tabela elementarnih integralov

$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq 1$

$\int \frac{dx}{x} = \ln|x| + C$

$\int e^x = e^x + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \cos x \, dx = \sin x + C$

$\int \sin x \, dx = -\cos x + C$

$\int sh\, x\, dx = ch\, x + C$

$\int ch\, x\, dx = sh\, x + C$

$\int \frac{dx}{\cos^2 x} = tg\, x + C$

$\int \frac{dx}{\sin^2 x} = -ctg\, x + C$

$\int \frac{dx}{x+a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \arcsin \frac{x}{a} + C$

$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln x + \sqrt{x^2 \pm a} + C$

Pravila za integriranje

$\int (f(x) \pm g(x)) dx = \int f(x) dx = \int g(x) dx$

$\int c \cdot f(x) dx = C \cdot \int f(x) dx$

perpartes:

$\int u dv = u \cdot v - \int v du$

uvedba nove spremenljivke

$\int f(x) dx = \int f(g(t)) \cdot g'(t) dt$

$x = g(t)$

$dx = g'(t) dt$

1. $\int (1-x^2)(1-x) dx = \int 1 - x - x^2 + x^3 dx$
$= x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + C$

2. $\int tg^2 x \, dx = \int \left(\frac{\sin x}{\cos x}\right)^2 dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx$

$= \int \left(\frac{1}{\cos^2 x} - 1\right) dx = tg\,x - x + C$

3. $\int \frac{x^4}{1+x^2} dx = \int x^2 - 1 + \frac{1}{1+x^2} dx = \frac{x^3}{3} + x + arctg\, x + C$

če je rg. večja najprej delimo

$x^4 : x^2+1 = x^2-1$

$x^4 + x^2$

$-x^2$

$+1$

4. $\int tg x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{\cancel{\sin x} \cdot dt}{t} = -\ln|t| = -\ln|\cos x| + C$

vzameš tisto, ki je spodaj (ponavadi)

$t = \cos x$

$dt = -\sin x \, dx$

5. $\int \frac{x}{3x+2} dx = \int \frac{\frac{t-2}{3\cancel{x}} \cdot dt}{t \cdot 3} = \int \frac{(t-2)\, dt}{9t} = \frac{1}{9} \int 1 - \frac{2}{t}\, dt$

$= \frac{1}{9} t - 2 \ln t$

$= \frac{1}{9}(3x+2) - 2 \cdot \ln|3x+2|_{/9} + C$

$t = 3x+2$

$dt = 3 \cdot dx$

$dx = \frac{dt}{3}$

6. $\int \sqrt[3]{1-3x}\, dx = \int \sqrt[3]{t} \; \frac{dt}{3} = -\frac{1}{3} \frac{t^{4/3}}{4/3} + C = -\frac{t^{4/3}}{4} + C$

$= \frac{(1-3x)^{4/3}}{4} + C$

$1-3x = t$

$dt = 3 dx$

9. $\int x^{3} \sqrt{1-x^2}\,dx = \int (1-t^2)t(-t\,dt) =$

$= \int (t^4-t^2)\,dt = \frac{t^5}{5} - \frac{t^3}{3} + C$

$= \frac{(\sqrt{1-x^2})^5}{5} - \frac{(\sqrt{1-x^2})^3}{5} + C$

$\sqrt{1-x^2} = t$

$dt = \frac{2x}{2\sqrt{1-x^2}}dx$

$1-x^2 = t^2$

$2x = 2t\,dt$ $\quad x = \sin t$

$2x\,dx$

10. $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2} =$

$= \frac{1}{2}\int 1-\cos 2x\,dx$

$= \frac{1}{4}\int 1-\cos t\,dt$

$= \frac{2x}{4} - \sin 2x + C$

$2x = t$

$dt = 2dx$

$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$

$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

11. $\int \frac{\ln^2 x}{x}\,dx = \int t^2\,dt = \frac{t^3}{3} = \frac{\ln^3 x}{3}$

$\ln x = t$

$dt = \frac{1}{x}dx$

12. $\int e^{\sin x}\cdot \cos x\,dx = \int e^t\,dt = e^t + c = e^{\sin x} + C$

$t = \sin x$

$dt = \cos x\,dx$

13. $\int \frac{dx}{x\ln x \cdot \ln(\ln x)} = \int \frac{dt}{t\cdot \ln(t)} = \int \frac{du}{u} = \ln u = \ln(\ln(\ln x)) + c$

$t = \ln x$

$dt = \frac{1}{x}dx$

$u = \ln t$

$du = \frac{1}{t}dt$

14. Per partes $\quad \int u\,dv = u\cdot v - \int v\,du$

$\int x^3 e^x\,dx = x^3\cdot e^x - 3\int x^2\cdot e^x\,dx = x^3\cdot e^x - 3(x^2\cdot e^x - 2\int x\cdot e^x\,dx$

$u = x^3$ $\quad du = 3x^2$

$dv = e^x dx$

$v = e^x$

$u = x^2$ $\quad dv = e^x dx$

$du = 2x$ $\quad v = e^x$

bolje, koji je važniji za int.

e^x

$\sin x, \cos x$

x^n

$\ln x$

$x^3\cdot e^x - 3x^2\cdot e^x + 6\int x\cdot e^x\,dx = x^3\cdot e^x - 3x^2 e^x + 6(xe^x - \int e^x\,dx)$

$= x^3\cdot e^x - 3x^2 e^x + 6xe^x - 6e^x + C$

$u = x$ $\quad dv = e^x dx$

$du = 1\,dx$ $\quad v = e^x$

15 $\int x \cdot \ln(x-1)\,dx = \frac{x^2}{2} \cdot \ln(x-1) - \int \cancel{\frac{x^2}{2}} \cdot \cancel{\frac{1}{(x-1)}}\,dx =$

$dv = x\,dx \quad v = \frac{x^2}{2}$ $\quad u = \ln(x-1) \quad du = \frac{1}{(x-1)}\,dx$

$\frac{x^2}{2}\ln|x-1| - \frac{1}{2}\int x - 1 + \frac{1}{x-1}$

$= \frac{x^2}{2}\ln(x-1) - \frac{1x^2}{4} - \frac{x}{2} + \frac{1}{2}\int \frac{dt}{t}$

$= \frac{x^2}{2}\ln(x-1) - \frac{x^2}{4} - \frac{x}{2} + \frac{1}{2}\ln(x-1)$

ZAPOMNI SI:

$\int \frac{dx}{x+a} = \ln(x+a)$

16 $\int \arctan x\,dx = x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2}\,dx$

$u = \arctan x \quad du = \frac{1}{x^2+1}\,dx$ $\quad dv = 1 \cdot dx \quad v = x$ $\quad 1+x^2 = t \quad dt = 2x\,dx$

$= x \cdot \arctan x - \frac{1}{2}\int \frac{dt}{t} =$

$= x \cdot \arctan x - \frac{1}{2}\ln(1+x^2) + C$

17 $\int \overbrace{e^{ax}\cos bx\,dx}^{I}$

$u = \cos bx \quad du = -b\sin x\,dx$ $\quad dv = e^{ax} \quad v = \frac{1}{a}e^{ax}$

$\cancel{= \frac{b}{a}e^{ax}\sin x} - \frac{1}{a}\int$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\cdot \sin bx\,dx$

$u = \sin bx \quad du = b\cos x\,dx$ $\quad dv = e^{ax} \quad v = \frac{1}{a}e^{ax}$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\cdot\sin bx - \frac{b}{a}\int e^{ax}\cos x\,dx\right)$

$I = \frac{1}{a}e^{ax}\cdot\cos bx + \frac{b}{a^2}e^{ax}\cdot\sin bx - \frac{b^2}{a^2}\int \overbrace{e^{ax}\cos bx\,dx}^{I}$

$I + \frac{b^2}{a^2}\cdot I = \frac{1}{a}e^{ax}\cdot\cos bx + \frac{b}{a^2}e^{ax}\cdot\sin bx$

$I\left(1+\frac{b^2}{a^2}\right) = \frac{1}{a}e^{ax}\cdot\cos bx + \frac{b}{a^2}e^{ax}\cdot\sin bx$

$I = \frac{\frac{1}{a}e^{ax}\left(\cos bx + \frac{b}{a}\sin bx\right)}{1+\frac{b^2}{a^2}} = \frac{1/a\, e^{ax}\left(\cos bx + b/a\,\sin bx\right)}{a^2+b^2}$

18 $\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x+1)(x-2)} = \int -\frac{1}{3}\frac{1}{(x+1)} + \frac{1}{3(x-2)}\,dx) =$

$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$

$\frac{Ax-2A+Bx+B}{(x+1)(x-2)}$

$= \frac{1}{3}\int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx =$

$= \frac{1}{3}\ln|x-2| - \ln|x+1|$

$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|$

$Ax + Bx = 0 \quad A = -B$

$B - 2A = 1$

$B + 2B = 1/3$

$B = 1/3 \quad A = -1/3$

nastavek $\quad A\ln|x+1| + B\ln|x-2| + C$, odvajamo

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{-x^2 - \frac{b}{a}x - \frac{c}{a}}} = \int \frac{dx}{\sqrt{-x^2+px+q}}$$

$$= \int \frac{dx}{\sqrt{k^2-(x-p/2)^2}} = \int \frac{k\,dt}{\sqrt{k^2-k^2t^2}} = \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \arcsin t + C =$$

$$= \arcsin \frac{x-p/2}{\frac{\sqrt{p^2+4q}}{2}} = \arcsin \frac{2x-p}{\sqrt{p^2+4q}} + C$$

$$-x^2+px+q = -\left[x^2-px-q\right] =$$
$$= -\left[(x-p/2)^2 - \frac{p^2}{4} - q\right]$$
$$= \frac{p^2+4q}{4} - \left(x-\frac{p}{2}\right)^2$$

$x - p/2 = kt$

$dx = k \cdot dt$

če imamo izraz

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + d\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

najprej odvajamo

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = Q_{n-1}'(x)\sqrt{ax^2+bx+c} + Q_{n-1}(x)\frac{2ax+b}{2\sqrt{ax^2+bx+c}} + \frac{d}{\sqrt{ax^2+bx+c}} \Big/ \sqrt{ax^2+bx+c}$$

$$\Rightarrow P_n(x) = Q'_{n-1}(x)(ax^2+bx+c) + Q_{n-1}(x)\cdot(ax+b/2) + d$$

45

$$\int \frac{dx}{(x+1)(x+2)(x+3)} = A \cdot \ln|x+1| + B \ln|x+2| + C \ln|x+3|$$

odvajamo

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)} = \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+2)(x+1)}{(x+1)(x+2)(x+3)}$$

$$x = Ax^2 + 5Ax + 6A + Bx^2 + 4Bx + 3B + Cx^2 + 3C + 2C$$

$$0 = A + B + C \qquad C = A + B \qquad A = -\frac{1}{2}$$

$$4A + 5B = 2 \begin{cases} 1 = 5A + 4B + 3C \quad :2 \\ 0 = 6A + 3B + 6C \quad \Rightarrow \end{cases} \qquad B = 2$$

$$C = -\frac{3}{2}$$

$$\int \frac{dx}{x^3 + x^2 + 2x + 2} = \int \frac{dx}{x^2(x+1) + 2(x+1)} = \int \frac{dx}{(x+1)(x^2+2)}$$

$$\frac{A}{(x+1)} + \frac{Bx+C}{x^2+2} = \frac{Ax^2 + 2A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+2)}$$

$$\underbrace{(A+B)}_{=0} x^2 + x \underbrace{(C+B)}_{=0} + \underbrace{2A+C}_{=1} \qquad \int \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+2}\, dx$$

$$A = -B$$

$$\frac{1}{3} \ln|x+1| +$$

$$C + B = 0$$
$$A + B = 0 \qquad B = -\frac{1}{3}$$
$$2A + C = 1 \qquad C = \frac{1}{3}$$

$$2A - B = 1$$
$$A + B = 0 \qquad -\frac{1}{3} \int \frac{x}{x^2+2}\, dx + \frac{1}{3} \int \frac{dx}{x^2+2} = -\frac{1}{6} \int \frac{dt}{t} + \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{x}{\sqrt{2}}$$

$$3A = 1$$
$$A = \frac{1}{3}$$

$$t = x^2 + 2 \qquad \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$$
$$dt = 2x\,dx$$

$$= \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln|x^2+2| + \frac{1}{3\sqrt{2}} \operatorname{arctg} \frac{x}{\sqrt{2}}$$

$$=$$

splošno

$$A \cdot \ln|x+1| + B \ln|x^2+2| + C' \operatorname{arctg} \frac{(x^2+2)'}{\sqrt{-D}}$$

v splošnem

$$\int \frac{P(x)\,dx}{(ax+b)(cx^2+dx+e)} = A \ln|ax+b| + B \ln|cx^2+dx+e| + C \operatorname{arctg} \frac{2cx+d}{\sqrt{4ac-b^2}}$$

MAT I-V.
18. 12. 2007

1. $\int \frac{x^2+1}{(x+1)^2(x-1)} =$ nastavek

$\frac{A}{x+1} + B \cdot \ln|x+1| + C\ln|x-1| + D$

odvajamo:

$$\frac{x^2+1}{(x+1)^2(x-1)} = -A\cdot (x+1)^{-2} + B\cdot\frac{1}{x+1} + C\,\frac{1}{x-1}$$

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{-A(x-1)+B(x+1)(x-1)+C(x+1)^2}{(x+1)^2(x-1)}$$

$$x^2+1 = -Ax + A + Bx^2 + \cancel{2Bx} - B + Cx^2 + 2x + C$$

$$B + C = 1$$
$$-A + 2C = 0$$
$$A - B + C = 1$$

$$A + 2C = 2$$
$$+ -A + 2C = 0$$
$$4C = 2$$
$$C = \tfrac{1}{2}$$

$$A = 1$$
$$B = \tfrac{1}{2}$$

$$= \frac{1}{x+1} + \frac{1}{2}\ln(x+1) + \frac{1}{2}|x-1| + D$$

DOLOČENI INTEGRAL

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{Leibnitzova formula}$$

$F(x) = \int f(x)\,dx$

1. $\int_1^2 \left(x^2 + \frac{1}{x^4}\right)dx = \frac{x^3}{3} + \frac{x^{-3}}{3}\Big|_1^2$

$$= \left(\frac{2^3}{3} - \frac{2^{-3}}{3}\right) - \left(\frac{1}{3} - \frac{1}{3}\right)$$

$$= \left(\frac{8}{3} - \frac{1}{24}\right) = \frac{63}{24}$$

2. $\int_0^{\pi/4} \sin 4x\,dx = \int_0^{\pi} \sin t\,dt = -\frac{1}{4}\cos t\Big|_0^{\pi} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

$t = 4x$ $x=0 \Rightarrow t=0$
$dt = 4dx$ $x=\frac{\pi}{4} \Rightarrow t=\pi$
$dx = \frac{dt}{4}$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3. $\int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x}\,dx = \frac{2\cancel{\sin x}\cdot\cancel{\cos x}}{\cos^{4\,3} x}\,dx = -2\int_1^{\frac{\sqrt{2}}{2}} \frac{dt}{t^3}$

$t = \cos x$
$dt = -\sin x\,dx$

$$= 2\int_{\frac{\sqrt{2}}{2}}^{1} \frac{dt}{t^3} = -1\,\frac{t^{-2}}{\cancel{2}}\Big|_{\frac{\sqrt{2}}{2}}^{1}$$

$$-\left(1^{-2} - \left(\tfrac{\sqrt{2}}{2}\right)^{-2}\right) = -(1-2) = 1$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

4. $\int_{-2}^{2} x^2 \sin x\, dx$

$u = x^2 \qquad dv = \sin x\, dx$

$du = 2x\,dx \qquad v = -\cos dx$

$$= -x^2\cos x\Big|_{-2}^{2} - \int_{-2}^{2} -\cos x \cdot 2x\, dx = -x^2\cos x\Big|_{-2}^{2} + 2\int_{-2}^{2} x\cdot\cos x\, dx$$

$$= -4\cdot\cancel{\cos 2} + 4\cancel{\cos(-2)} + 2\cdot\int_{-2}^{2} x\cdot\cos x\, dx$$

$u = x \qquad dv = \cos x\, dx$

$du = 1dx \qquad v = \sin x$

$$2\cdot\left(x\cdot\sin x\Big|_{-2}^{2} - \int_{-2}^{2} dx\cdot\sin x\right.$$

$$= 2\left(\cancel{2\cdot\sin 2} - \cancel{(-2\sin(-2))} - (-\cos x)\Big|_{-2}^{2}\right.$$

$$= 2\cdot(\cancel{-\cos(-2)} + \cancel{\cos 2}) = 0$$

integral lihe funkcije na simetričnem intervalu je enak 0

5. $\int_0^{\ln 5} \frac{\sqrt{e^x-1}}{1+3e^{-x}}\, dx = \int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\, dx$

$t = \sqrt{e^x-1}$

$t^2 = e^x - 1$

$2t\,dt = e^x\, dx$

$e^0 = 0$

$e^{\ln 5} = 5 - 1 = \sqrt{4} = 2$

$$= \int_0^2 \frac{2t^2\,dt}{t^2+4} = \int_0^2 2 - \frac{8}{t^2+4}\, t$$

$2t^2 : t^2+4 = 2$

$-2t^2 + 8$

-8

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$$

$$= 2t - 8\left(\frac{1}{2}\operatorname{arctg}\frac{x}{2}\right)\Big|_0^2$$

$$= (4-0) - (4\cdot\operatorname{arctg} 1 - 4\cdot\operatorname{arctg} 0)$$

$$= 4 - 4\cdot\frac{\pi}{4} = 4 - \pi$$

6. $\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\, dx = \left(-\frac{\sqrt{1+x^2}}{x}\right)\Big|_1^2 = -\frac{\sqrt{5}}{2} + \sqrt{2}$

$R(x, \sqrt{1-x^2})$

$x = \sin t$ ali

$x = \cos t$

$\int \frac{1}{x^2\sqrt{1+x^2}}\, dx$

$tg^2\sqrt{1+tg^2}$

$\frac{1}{|\cos t|}$

$x = tg\, t$

$dx = \frac{1}{\cos^2 t}\, dt$

$x = \frac{\sin t}{\cos t} \Rightarrow$

$\sin t = x\cos t$

$\frac{1}{\sqrt{1+x^2}}$

$R(x, \sqrt{1+x^2})$

$x = tg\, t$

$\cancel{x} = sh\, t$

$R(x, \sqrt{x^2-1})$

$x = 1/\cos t$

$$\int \frac{1}{\tan^2 t\ \frac{1}{\cos t}}\ \frac{dt}{\cos^2 t} = \int \frac{\cos t}{\sin^2 t}\, dt = \int \frac{dv}{v^2} = -\frac{1}{v}$$

$\frac{\sin^2 t}{\cos^2 t}$

$$= -\frac{1}{\sin t} = -\frac{\sqrt{1+x^2}}{x}$$

$v = \sin t$

$dv = \cos t\, dt$

$$\sin t = tg\, t\cdot\cos t = x\cdot\frac{1}{\sqrt{1+x^2}}$$

7. $\int_1^{\infty} \frac{1}{(x^2+1)(x^2+3)}\,dx$ nepravi integral

odvod od x^2+1

$$\int \frac{1}{(x^2+1)(x^2+3)} = A \ln|x^2+1| + B \cdot \operatorname{arctg} \frac{2x}{\cancel{2}} + C \cdot \ln|x^2+3| + D \cdot \operatorname{arctg} \frac{2x}{2\sqrt{3}} + E$$

$\sqrt{-D}$ $\sqrt{4}$

odvajamo:

$$\frac{1}{(x^2+1)(x^2+3)} = \frac{A}{x^2+1}2x + \frac{B}{1+x^2} + \frac{C\,2x}{x^2+3} + \frac{D\cdot\frac{\sqrt{3}}{3}}{1+\left(\frac{x\sqrt{3}}{3}\right)^2}\ \frac{x^2}{3}$$

$$\frac{2Ax}{x^2+1} + \frac{B}{1+x^2} + \frac{2Cx}{x^2+3} + \frac{D\sqrt{3}}{x^2+3}$$

$$\frac{2Ax(x^2+3)+B(x^2+3)+2Cx(x^2+1)+D\sqrt{3}(x^2+1)}{(x^2+3)(x^2+1)}$$

$$= \frac{2Ax^3+6Ax+Bx^2+3B+2Cx^3+2Cx+\sqrt{3}Dx^2+D\sqrt{3}}{}$$

x^3: $2A+2C=0$
x^2: $B+\sqrt{3}D=0$
x: $6A+2C=0$
$3B+D\sqrt{3}=1$

odšt = $4A=0$
$A=0$ $C=0$

$2B=1$
$B=\frac{1}{2}$

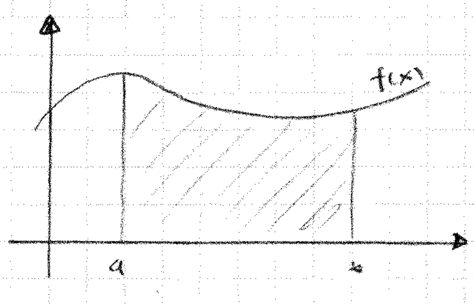
$A=0$
$B=\frac{1}{2}$
$C=0$
$D=-\frac{1}{2\sqrt{3}}$

$$=\left(\frac{1}{2}\operatorname{arctg} x - \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\right)\Big|_1^{\infty}$$

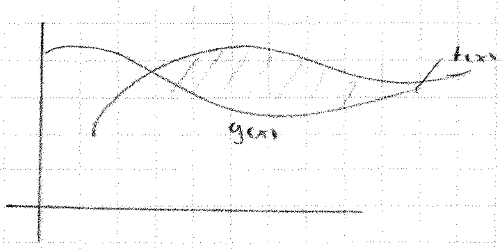
$$=\left(\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2\sqrt{3}}\cdot\frac{\pi}{6}\right) + \left(\frac{1}{2}\underbrace{\operatorname{arctg}\infty}_{\frac{\pi}{2}} - \frac{\sqrt{3}}{6}\underbrace{\operatorname{arctg}\frac{\infty}{\sqrt{3}}}_{\frac{\pi}{2}}\right)$$

$$=\left(\frac{\pi}{4} - \frac{\pi\sqrt{3}}{12}\right) - \left(\frac{\pi}{8} - \frac{\sqrt{3}\pi}{30}\right)$$

$$=\frac{\pi}{8} - \frac{2\pi\sqrt{3}}{18}$$



$$S=\int_a^b f(x)\,dx$$



$$S=\int_a^b (f(x)-g(x))\,dx$$

Primer:

$$\int \cos^3 x \sin^3 x \, dx = \int \cos^2 x \cdot \sin^3 x \cdot \cos x \, dx$$

$$= \int (1-\sin^2 x)^2 \cdot \sin^3 x \cos x = \int (1-t^2) t^3 \, dt =$$

$\sin x = t$

$dt = \cos x \, dx$

$$= \int t^3 - t^5 \, dt$$

$$= \frac{t^4}{4} - \frac{t^6}{6} = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$$

8. $$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \cos^m x \, dx = \frac{\cos^{m-1} x \cdot \sin x}{m} + \frac{m-1}{m} \int \cos^{m-2} x \, dx$$

9. $\int \sin ax \cos bx \, dx$; $\int \sin ax \cdot \sin bx \, dx$; $\int \cos ax \cos bx \, dx$

$$\int \sin ax \cdot \cos bx = \frac{1}{2} \int [\sin (a-b)x + \sin (a+b) x] \, dx$$

$$\int \sin ax \cdot \cos bx = \frac{1}{2} \int [\cos (a-b) x - \cos (a+b) x] \, dx$$

$$\int \cos ax \cdot \cos bx = \frac{1}{2} \int [\cos (a-b) x + \cos (a+b) x] \, dx$$

Primer:

$$\int \sin 5x \cdot \cos x \, dx = \frac{1}{2} \int [\sin 4x + \sin 6x] \, dx =$$

$$= \frac{1}{2} [-\cos 4x/4 - \cos 6x/6] + C$$

10. $$I = \int f(e^{ax}) \, dx = \int \frac{f(t)}{at} \, dt$$

$e^{ax} = t$

$dt = a e^{ax} dx = at \, dt$

$dx = \frac{dt}{at}$

Primer:

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{e^{-x} dx}{\sqrt{1-e^{-2x}}} = -\int \frac{dt}{\sqrt{1-t^2}}$$

$e^{-x} = t$

$dt = -e^{-x} dx = -t \, dt$

$dx = -\frac{dt}{t}$

$$= \arcsin t + C =$$

$$= \arcsin e^{-x} + C$$

11. $\int G(x) e^{ax} dx = \frac{G(x) e^{ax}}{a} - \frac{1}{a} \int G'(x) e^{ax} dx$

$G(x) = u$ $\quad e^{ax} dx = dv$
$du = G'(x) dx$ $\quad v = 1/a\, e^{ax}$

isti prijem dokler ne pridemo do polinoma 0 stopnje.

Če je negativna potenca, moramo zamenjati odvajanje in integriranje (kaj zamenjamo)

$\int \frac{e^x}{x^2} dx \qquad \int \frac{e^x}{x}$

Teh integralov ne moremo definirati z elementarno funkcijo, zato smo vpeljali novo funkcijo

12. $\int R(\ln x) dx = \int R(u) e^u du$

$\ln x = u$
$du = \frac{dx}{x}$

$x = e^u$

$\int f(x) \ln(x) dx$ podobno

Primer:

1. $\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n dx = \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} x^{n+1} + C$

$\ln x = u$; $x^n dx = dv$
$du = \frac{dx}{x}$, $v = \frac{x^{n+1}}{n+1}$

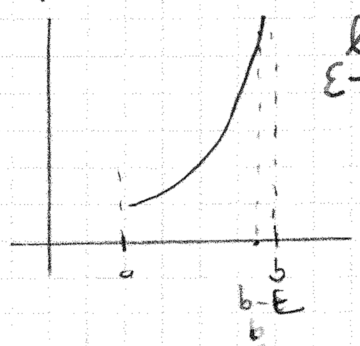
2. $\int \frac{dx}{x \cdot \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

$u = \ln x$
$du = \frac{dx}{x}$

NEPRAVI ali POSPLOŠENI INTEGRALI

$\int_a^b f(x) dx$

Taki so integrali, ki nimajo limite

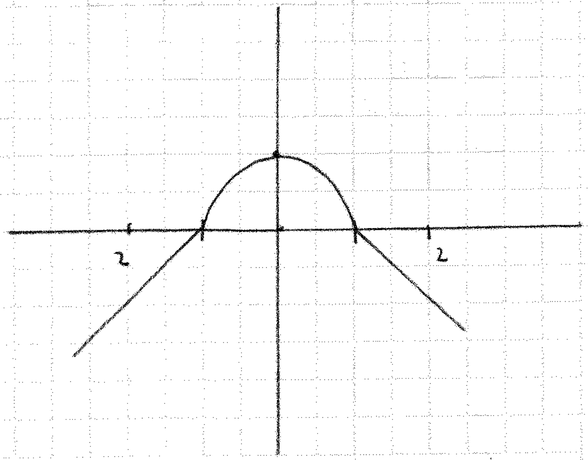


$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x) dx$ se da izračunati, ker je končna funkcija

Če limita ne obstaja je integral divergenten. Če pa ima limito je divergenten ali posplošen integral

$\int_a^b f(x) dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x) dx$

1. $\int_{-2}^{2} f(x)\,dx$ $\qquad f(x) \begin{cases} 1-x^2 ; & |x| \leq 1 \\ 1-|x| ; & |x| > 1 \end{cases}$



$2\cdot\int_{-2}^{-1} (1+x)\,dx + \int_{-1}^{1}(1-x^2)\,dx$

$2\cdot\left(x+\frac{x^2}{2}\right)\Big|_{-2}^{-1} + \left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 2\cdot\left((-1+\frac{1}{2})-(-2+2)\right) + \left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)$

$= 2\left(\frac{1}{2}-1\right) - 0 + \left(1-\frac{1}{3}+1-\frac{1}{3}\right)$

$= -1 + \frac{4}{3} = \frac{1}{3}$

3. $f(x) = \sin x + \cos x + x$

$x=0$

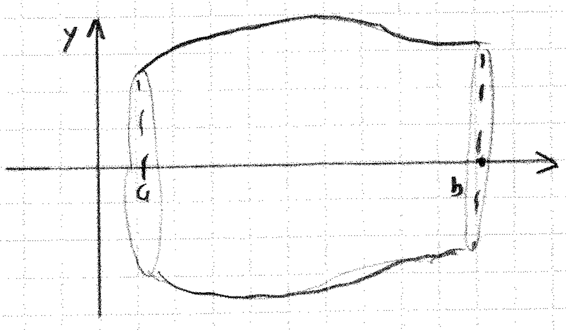
$x=\pi$

$S = \int_0^{\pi} (\sin x + \cos x + x)\,dx = (-\cos x + \sin x + \frac{x^2}{2})\Big|_0^{\pi}$

$= (1+0+\frac{\pi^2}{2}) + 1 - 0 + 0$

$= 2 + \frac{\pi^2}{2}$

VOLUMEN ROTACIJSKEGA TELESA



$V = \pi \int_a^b f^2(x)\,dx$

$y = 3\sqrt{x(1-x)^3}$ $\qquad 0 < x < 1$

$V = \pi \int_0^1 \left(3\sqrt{x(1-x)^3}\right)^2 dx = 9\pi \int_0^1 x\cdot(1-x)^3\,dx = 9\pi \int_0^1 x\,(1-3x+3x^2+x^3)$

$= 9\pi \left(\frac{x^2}{2} - \frac{3x^3}{3} + \frac{3x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1$

$= 9\pi\left(\frac{1}{2} - 1 + \frac{3}{4} + \frac{1}{5}\right)$

$= \frac{9}{20}\pi$

52

4 $y = e^{-x}$, $0 < x < 3$

$$V = \pi \int_0^3 (e^{-x})^2 dx = \pi \cdot -\frac{1}{2} e^{-2x} \Big|_0^3$$
$$= \pi(-\frac{1}{2})e^{-6} - (-\frac{1}{2}\pi \cdot e^0)$$
$$= -\frac{1}{2}\pi e^{-6} + \frac{1}{2}\pi$$
$$= 1/2\pi (1 - e^{-6})$$

WWW. STROMAR, SI