MATEMATIKA I

zapiski z avditornih vaj

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Izvajalec	Melita Hajdinjak
Avtor dokumenta	Blaž Potočnik
Sodelavci	Blaž Potočnik, Aljoša Praznik

UREJANJE DOKUMENTA

VERZIJA	01	**REVIZIJA**	01
DATUM	24. 2. 2009		

ZADNJI POPRAVLJAL	/
PREGLEDAL	Blaž Potočnik, Aljoša Praznik

OPOMBE

POPRAVKI

8.10.07

Množice

I

A ... množice, $x \in A$, $y \notin A$
U ... univerzalna mn.
$\emptyset$ ... prazna mn, $\{\}$

$A \subseteq B$ za vsak $(\forall x \in A \Rightarrow x \in B)$ x iz A sledi, da je tudi v B

$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

$A \cap B = \{x; x \in A \wedge x \in B\}$

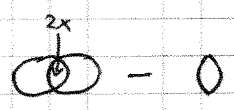
$A \cup B = \{x; x \in A \vee x \in B\}$
$A - B = \{x; x \in A \wedge x \notin B\}$

$A^c = \{x; x \notin A\}$

$A \cap B = \emptyset$ množici sta disjunktni

$A - B = A \cap B^c$

moč $|A| = m(A)$
$m(A \cup B) = m(A) + m(B) - m(A \cap B)$



lastnosti
- komutativnost: ni važno v katerem vrstnem redu naredimo presek ali unijo

$$A \cup B = B \cup A$$

- asociativnost: $(A \cap B) \cap C = A \cap (B \cap C)$
- distributivnost: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

- $A \cap A^c = \emptyset$
- $A \cup A^c = U$
- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $A \cap U = A$
- $A \cup U = U$
- $(A^c)^c = A$

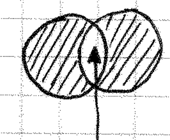
De Morganova zakona

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

Simetrična razlika množic

- Vennov diagram

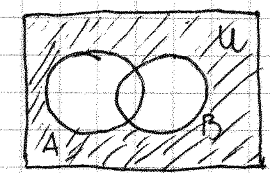


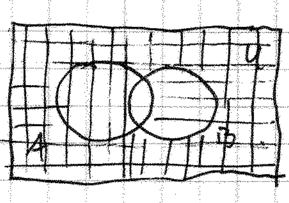
tega ni

$A \Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

1. grafično (Vennov diagram) dokaži de Morganova zakona

$(A \cup B)^c = A^c \cap B^c$





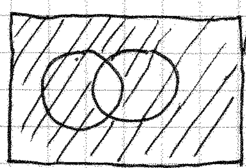
≡ vse, kar ni v A

|||| vse, kar ni v B

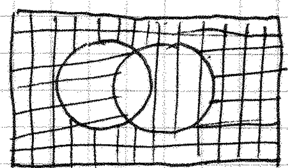
\# to je dokaz

$(A \cup B)^c$ $\qquad$ $A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$



$(A \cap B)^c$



$A^c \cup B^c$

2. Dokaži DM zakona (teorija množic)

$(A \cap B)^c = A^c \cup B^c$

$x \in (A \cap B)^c \Leftrightarrow x \notin (A \cap B) \Leftrightarrow (x \notin A) \underset{\text{ali}}{\vee} (x \notin B)$
$\Leftrightarrow (x \in A^c) \vee (x \in B^c) \Leftrightarrow x \in (A^c \cup B^c)$

$(A \cup B)^c = A^c \cap B^c$

$x \in (A \cup B)^c \Leftrightarrow x \notin (A \cup B) \Leftrightarrow x \notin A \wedge x \notin B \Leftrightarrow x \in A^c \wedge$
$x \in B^c \Leftrightarrow x \in (A^c \cap B^c)$

3. Poenostavi izraze

- $A - (A - B) =$

  $A - B = A \cap B^c$

  $= A - A \cap B^c = A \cap (A \cap B^c)^c = A \cap (A^c \dot\cup B) =$
  $= \emptyset \cup A \cap B = A \cap B$ ✓

- $(A \cap B) \dot\cup (A \cap B^c) = (A \cap B) \cup (A \cap (A - B)) =$
  $= ((A \cap B) \cup A) \cap ((A \cap B) \cup B^c) =$
  $= A \cap ((A \cup B^c) \cap (B \cup B^c)) = A \cap (A \cup B^c) \cap \mathcal{U} = A \cap (A \cup B^c) = A$

- $(A - C) \cup (B - C)$ R: $(A \cup B) - C$

Potenčna množica

množica vseh podmnožic

$P(A) = \{x; x \subseteq A\}$

$|A| \Rightarrow |P(A)| = 2^n$!

4. Določi potenčne množice

- $P(A) = P(\{a, b, c\}) = P(\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\} \{a,c\}, \{b,c\}, \{a,b,c\})$
- $P(P(\emptyset)) = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$ $\{\emptyset\} \neq \emptyset$ .... moč mn.
- $P(\{a\}) = \{\emptyset, \{a\}\}$
- $P(\{1, \{1,2\}, \{3\}\})$ 3 je elementi $= \{\emptyset, \{1\}, \{1, \{1,2\}\}, \{1, \{3\}\}, \{\{1,2\}, \{3\}\},$
  $\{1, \{1,2\}, \{3\}\}, \{\{1,2\}\}, \{\{3\}\}\}$

Kartezični produkt

$A \times B = \{(a,b);\ a \in A,\ b \in B\}$

6. Določi kart. p. množic $A = \{1,2\}$ $B = \{A, B, C\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

- $U = \mathbb{R}$

  $f, g : \mathbb{R} \to \mathbb{R}$

  $A = \{x \in \mathbb{R},\ f(x) = 0\}$
  $B = \{x \in \mathbb{R},\ g(x) = 0$

  Izrazi z množicama A in B rešitve:

- $f(x) \cdot g(x) = 0$

  $C = \{x;\ f(x) g(x) = 0\} = \{x;\ f(x) = 0 \vee g(x) = 0\} =$
  $= \{x;\ f(x) = 0\} \cup \{x;\ g(x) = 0\} = A \cup B$

  $\vee$ ... unija

- $f^2(x) + g^2(x) = 0$ R: $A \cap B$

  $C = \{x;\ f^2(x) + g^2(x) = 0\} = \{x;\ f(x) = 0 \wedge g(x) = 0\} =$
  $= \{x;\ f(x) = 0\} \wedge \{x;\ g(x) = 0\} = A \cap B$

-

7. Naj bo $U = [0, 3\pi)$ univerzalna

$A = \{x,\ \sin x < 0\}$
$B = \{x,\ \cos x < 0\}$
$C = \{x,\ \operatorname{tg} x > 0\}$

Zapiši kot intervale ali unije intervalov množice

$A, B, C,\ A \cap B,\ A^c \cap C,\ B^c \cup C$

R: $A = \{\pi, 2\pi\}$
$B = \{\frac{\pi}{2}, 3\pi/2\} \cup \{5\pi/2, 3\pi\}$
$C = (0, \pi/2) \cup (\pi, 3\pi/2) \cup (2\pi, 5\pi/2)$
$A \cap B = (\pi, 3\pi/2)$
$A^c \cap C = (0, \pi/2) \cup (2\pi, 5\pi/2)$
$B^c \cup C = (0, \pi/2] \cup (\pi, 5\pi/2]$

Realna števila

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

- koeficienti z več:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$(a+b)^4 = 1 \cdot a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Potence

$x^{a+b} = x^a \cdot x^b$
$(xy)^a = x^a y^a$
$x^{ab} = (x^a)^b$
$x^{a-b} = x^a / x^b$
$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
$x^{-a} = \frac{1}{x^a}$

Koreni

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$

Abs. vrednost

$|x| = \begin{cases} x; x \geq 0 \\ -x; x < 0 \end{cases}$

$|1|$
$|-1| = 1 = -(-1)$

$\sqrt{x^2} \neq x$ ..... $\sqrt{x^2} = |x|$ sodi koreni

$1 = \sqrt{1^2} = \sqrt{(-1)^2} \neq -1$

$1^2 = (-1)^2$

$\sqrt[3]{x^3} = x$

$|x| = |-x|$ $\quad$ $|x| = 0 \Leftrightarrow x = 0$

$x \leq |x|$ $\quad$ $|x \cdot y| = |x| \cdot |y|$

$|x| \geq 0$ $\quad$ $|x + y| \leq |x| + |y|$

$|x - y| \geq |x| - |y|$

Kvadratna enačba

$ax^2 + bx + c = 0$

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}} \qquad \boxed{D = b^2 - 4ac}$$

8. Reši kv. enačbe

√ • $\frac{1}{2}x^2 - 2x + 3 = 0$ $\qquad$ $x^2 - 4x + 6 = 0$

$x_{1,2} = 2 \pm \sqrt{-2}$ $\qquad$ $x_{1,2} = 2 \pm \sqrt{2}\, i$

√ • $\sqrt{3x^2 - 7x + 3} = 1 - x \quad /^2$

$3x^2 - 7x + 3 = 1 - 2x + x^2$

$3x^2 - x^2 - 7x + 2x + 3 - 1 = 0$

$2x^2 - 5x + 2 = 0$

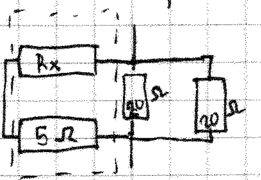
$x_{1,2} = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$

~~$x_1 = 2$~~

$x_2 = 1/2$ ✓

Preveri rešitve! – kvadriranje

9. Poišči vrednosti upora R_x, pri katerih je nadomestna upornost vezja manjša kot 8Ω in večja ali enaka 4Ω



R_z

$$R_z = R_x + 5$$

$$\frac{1}{R_{nad}} = \frac{1}{R_z} + \frac{1}{20} + \frac{1}{20} = \frac{1}{R_z} + \frac{1}{10}$$

$$R_{nad} = \left(\frac{1}{R_z} + \frac{1}{10}\right)^{-1} = \left(\frac{10 + R_z}{10 R_z}\right)^{-1} = \frac{10 R_z}{10 + R_z}$$

$$R_{nad} = \frac{10(R_x + 5)}{R_x + 15}$$

$$R_z = R_x + 5$$

- R_{nad}

$$4 \leq R_{nad} < 8$$

$$4 \leq \frac{10(R_x + 5)}{R_x + 15} < 8$$

$$4 \leq \frac{10(R_x + 5)}{R_x + 15} \quad / \cdot (R_x + 15)$$

$$4(R_x + 15) \leq 10(R_x + 5)$$
$$4R_x + 60 \leq 10 R_x + 50$$
$$10 \leq 6 R_x$$
$$R_x \geq 10/6$$
$$R_x \geq 5/3\ \Omega$$

$$\frac{10(R_x + 5)}{R_x + 15} < 8 \quad / \cdot (R_x + 15)$$

$$10 R_x + 50 < 8 R_x + 120$$
$$2 R_x < 70$$
$$R_x < 35\ \Omega$$

$$5/3 \leq R_x < 35$$

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10. Poišči množico rešitev naslednjih neenačb

√ • $2x + 3 \leqslant 3x + 4$ R: $[-1, \infty)$

$2x + 3 \leqslant 3x + 4$
$-x \leqslant 1$
$x \geqslant -1$

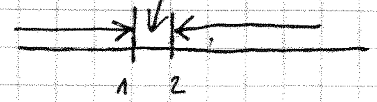
$[-1, \infty)$

√ • $2x < x + 1 < 2x - 1$

$2x < x + 1$
$x < 1$

$x + 1 < 2x - 1$
$x - 2x < -2$
$-x < -2$
$x > 2$

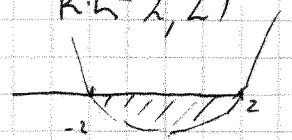
Rešitev je presek.



R: $x < 1 \wedge x > 2$
$(-\infty, 1) \cap (2, \infty) = \emptyset$

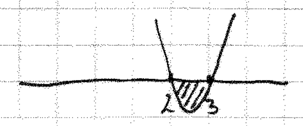
in → $\cap$
ali → $\cup$

√ • $x^2 < 4$ R: $[-2, 2)$
$x^2 - 4 < 0$
$(x + 2)(x - 2) < 0$
$[-2, 2)$!



• $x \cdot (x - 5) < -6$
$x^2 - 5x < -6$
$x^2 - 5x + 6 < 0$
$(x - 2)(x - 3) < 0$

$x_1 = 2$
$x_2 = 3$



R: $x \in \{2, 3\}$

8

15. 10. 07

Reši enačbe in neenačbe

$$\sqrt{5x+1} - \sqrt{2x+3} = \sqrt{7x-20}$$

DeF: $5x+1 \geq 0 \rightarrow x \geq -1/5$ vse mora obstajati

$2x+3 \geq 0 \rightarrow x \geq -3/2$

$7x-20 \geq 0 \rightarrow x \geq 20/7$

$\boxed{x \geq 20/7}$

$/^2$

$$(5x+1) - 2\cdot\sqrt{5x+1}\sqrt{2x+3} + (2x+3) = 7x - 20$$

$$\cancel{7x} + 4 - \cancel{7x} + 20 = 2\sqrt{5x+1}\,\sqrt{2x+3} \qquad /^2$$

$$144\ \cancel{400} = \cancel{4}(5x+1)(2x+3) \qquad D = b^2 - 4ac$$

$$144\ \cancel{400} = \cancel{4}(10x^2 + 15x + 2x + 3)$$

$$144\ \cancel{400} = \cancel{4(10x^2+17x+3)}\ 10x^2 + 17x + 3$$

$$0 = 10x^2 + 17x + 3 - 144$$

$$0 = 10x^2 + 17x - 141$$

$$x_{1,2} = \frac{-17 \pm 77}{20}$$

$x_1 = 3$ $\qquad \cancel{x_2 = -47/10}$

$\downarrow$ pr

$\sqrt{16} - \sqrt{9} = \sqrt{1}$

$4 - 3 = 1$ ✓ $\qquad \boxed{x = 3}$

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Akta®

✓ $\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=\sqrt{2}$ $\qquad x \geq 1/2$

✓ $\sqrt{x+1}+\sqrt{x} > 3$

$-2 < 5 \rightarrow 4 < 25$

$-6 < 5 \rightarrow 36 > 25$

$p_1 < p_2 \rightarrow p_1^2 < p_2^2$

$n_1 < p_1 \rightarrow$ • $|n_1| < p_1 \quad n_1^2 < p_2^2$

• $|n_1| > p_2 \quad n_1^2 > p_2^2$

obrne se neenačaj !

Def: $x \geq 0$

$x+1 \geq 0 \rightarrow x \geq -1$

$x \geq 0$

$x + 2\sqrt{x}\sqrt{x+1} + x + 1 > 9$

$2\sqrt{x}\sqrt{x+1} > 8 - 2x \qquad /:2$

$\underbrace{\sqrt{x}\sqrt{x+1}}_{\text{pozitivno}} > 4 \underbrace{-x}_{?}$

1. $4 - x \geq 0$

$x \leq 4$

$x(x+1) > 16 - 8x + x^2$

$9x > 16 \rightarrow x > 16/9$

presek: $\left.\begin{array}{l} x \leq 4 \\ x > 16/9 \end{array}\right\} R_1$

$R_1: x \in \left(\frac{16}{9}, 4\right]$

2. $4 - x < 0$

$\boxed{x > 4}$

$\sqrt{x}\sqrt{x+1} > |4-x| \quad /^2$

$x^2 + x > 16 - 8x + x^2$

$9x > 16$

$x > 16/9$

neenačaj se ne obrne

$x(x+1) > 16 - 8x + x^2$

$R_2: x \in (4, \infty)$

$\sqrt{x}\sqrt{x+1} < |4-x| \quad /^2$

$x(x+1) < 16 - 8x + x^2$

$9x < 16$

$x < 16/9$

$R_3: \emptyset$

$R = R_1 \cup R_2 \cup R_3 = \boxed{(16/9, \infty)}$

10

$$\sqrt{19-x} - \sqrt{x+1} > 2$$

Def: $x \leq 19$ $\quad x \geq -1$

$x \in (-1, 19)$

• $\sqrt{19-x} - \sqrt{x+1} > 0$

$19 - x > x + 1$

$18 > 2x$

$\boxed{x < 9}$ je leva str pozitivna

• $\sqrt{19-x} - \sqrt{x+1} < 0$

ni rešitve (ker mora biti leva str > 2)

$19 - \cancel{x} - 2\sqrt{19-x}\sqrt{x+1} + \cancel{x} + 1 > 4$

$-2\sqrt{19-x}\sqrt{x+1} > 4 - 16$

$\sqrt{19-x}\sqrt{x+1} < 8$

$(19-x)(x+1) < 64$

$19x - x - x^2 + 19 < 64$

$18x - x^2 - 45 < 0$

$x^2 - 18x + 45 > 0$

$(x-3)(x-15) > 0$

3 15

$x \in ((-\infty, 3) \cup (-15, \infty)) \cap (-\infty, 9]$

R: $x \in (-\infty, 3)$

Dn: $\dfrac{2x-3}{x-2} \leq 3$, $x \neq 2$

R: $(-\infty, 2) \cup [3, \infty)$

M

$$\sqrt{\frac{1+x^2}{1-x^2}} \leq 1 \qquad \text{Def: } x^2 \neq 1,\quad x \neq 1,\quad x \neq -1$$

$$\frac{1+x^2}{1-x^2} - 1 \leq 0$$

$$\frac{1+x^2-1+x^2}{1-x^2} \leq 0$$

$$2x^2/1-x^2 \leq 0 \qquad /\cdot (1-x^2)^2$$

– da se znebimo neg. množenja

$$2x^2(1-x^2) \leq 0$$
$$2x^2-2x^4 \leq 0$$
$$2x^2(1-x^2) \leq 0$$

$$\frac{1+x^2}{1-x^2}(1-x^2)^2 \leq (1-x^2)^2$$

$$(1+x^2)(1-x^2) \leq (1-x^2)^2$$
$$(1+x^2)(1-x^2)-(1-x^2)^2 \leq 0$$
$$(1-x^2)(1+x^2-1+x^2) \leq 0$$
$$(1-x^2)(2x^2) \leq 0$$

• $(1-x^2) \geq 0 \qquad x^2 \leq 0$

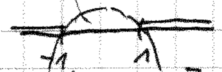
$$\left.\begin{array}{l} -x^2 \geq -1 \\ x^2 \leq 1 \\ x^2 \leq 0 \end{array}\right\} \rightarrow x = 0 \text{ edina rešitev}$$

• $(1-x^2) \leq 0 \qquad x^2 \geq 0$

$x^2 \geq 0 \rightarrow$ 1

$1 - x^2 < 0$

$(1-x)(1+x) \leq 0$



$x \in (-\infty, 1] \cup [1, \infty)$

$x \in (-\infty, -1) \cup (1, \infty)$

$$R: R_1 \cup R_2 = \boxed{(-\infty,-1) \cup (1,\infty) \cup \{0\}}$$

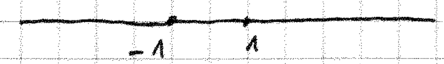
Dn: $\sqrt{-x} < \sqrt{\sqrt{x+2}+2}$ $\qquad$ R: $(-2,-1)$

ABSOLUTNA VREDNOST

$|x+1| + |x-1| = 2$

$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

$x+1 \geq 0$	$x+1 \geq 0$	$x+1 < 0$	$x+1 < 0$
$x-1 \geq 0$	$x-1 < 0$	$x-1 \geq 0$	$x-1 < 0$
$x \geq -1$	$x \geq -1$	$x < -1$	$x < -1$
$x \geq 1$	$x < 1$	$x \geq 1$	$x < 1$
$x \geq 1$	$-1 \leq x < 1$	ni rešitve	$x < -1$



① $x + 1 + x - 1 = 2$

$2x = 2$

$x = 1$

$R_1: \{1\}$

② $x + 1 - x + 1 = 2$

$2 = 2$

$x \in \mathbb{R}$

$R_2:$ $[-1, 1)$

③ /

④ $-x - 1 - x + 1 = 2$

$-2x = 2$

$x = -1$

$R_3 = \{\}$

$R = R_1 \cup R_2 \cup R_3 = [-1, 1]$

$|-x^2+2x-3|=1$

• $(-x^2+2x-3) \geq 0$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{-2 \pm \sqrt{4-12}}{-2}$

$D<0$ ni $\mathbb{R}$ rešitev

ni večje ali enako 0.
ni $\mathbb{R}$ rešitev

$\boxed{R: \emptyset}$

• $(-x^2+2x-3)<0$

Vsa $\mathbb{R}$ za ta pogoj

$x^2-2x+3=1$ 1 · 2

$x^2-2x+2=0$

$\frac{2 \pm \sqrt{4-8}}{2}$

$D<0$ ni $\mathbb{R}$ rešitev

← ni rešitev $\mathbb{R}$!

Reši neenačbe

$|x^2+3x-1|<3$

$x^2+3x-1 \geq 0$ $x^2+3x-1<0$

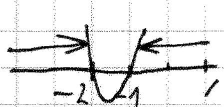
$-3<(x^2+3x-1)<3$

① $-3<x^2+3x-1$

$x^2+3x+2>0$

$(x+2)(x+1)>0$

$x_1=-1,\ x_2=-2$



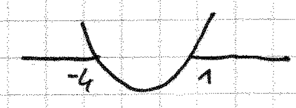
$R_1: x \in (-\infty,-2) \cup (-1,\infty)$

② $x^2+3x-1<3$

$x^2+3x-4<0$

$(x-1)(x+4)<0$

$x_1=1,\ x_2=-4$



$R_2: x \in (-4,1)$

$R: R_1 \cap R_2 = (-4,-2) \cup (-1,1)$

$$\left|\frac{x}{x+4}\right| < 1$$

$-1 < \frac{x}{x+4} < 1 \rightarrow x \neq -4$

$-1 < \frac{x}{x+4} \quad / (x+4)^2$

$-(x+4)^2 < x(x+4)$

$-(x+4)^2 - x(x+4) < 0$

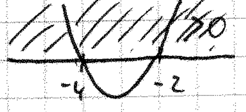
$(x+4)^2 + x(x+4) > 0$

$(x+4)(x+4+x) > 0$

$0 < 2(x+4)(x+2)$

$\cancel{x_1 = -4} \quad x = -2$

R:



$R_1: (-\infty, -4) \cup (-2, \infty)$

$R: R_1 \cap R_2 = (-2, \infty)$

$\frac{x}{x+4} < 1 \quad / (x+4)^2$

$x(x+4) < (x+4)^2$

$(x+4)(x-x-4) < 0$

$(x+4) - 4 < 0$

$x + 4 > 0$

$x > -4$

$R_2: x \in \{-4, 0\}$ ×

- $Dn: |2x+3| \leq |4x-3|$ $\qquad R: (-\infty, 0] \cup [3, \infty)$

$|\,|2|x| - 4| < 2$

$x \geq 0$

$x < 0$

$\boxed{x \geq 0}$

$|2x - 4| < 2$

$2x - 4 \geq 0$

$\boxed{x \geq 2}$

$2x - 4 < 2$

$\boxed{x < 3}$

$R_1 = [2, 3)$

$2x - 4 < 0$

$\boxed{x < 2}$

$-2x + 4 < 2$

$2x > 2$

$\boxed{x > 1}$

$R_2 (1, 2)$

$\boxed{x < 0}$

$|-2x - 4| < 2$

$-2x - 4 \geq 0$

$\boxed{x \leq -2}$

$2x - 4 < 2$

$2x > -6$

$\boxed{x > -3}$

$R_3: (-3, -2)$

$-2x - 4 < 0$

$\boxed{x > -2}$

$-(-2x - 4) < 2$

$2x < -2$

$\boxed{x < -1}$

$R_4: (-2, -1)$

$R: R_1 \cup R_2 \cup R_3 \cup R_4 = (1, 3) \cup (-3, -1)$

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Akta®

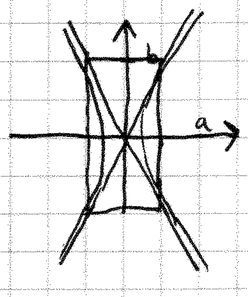
Krivulje 2. reda

Krožnica: $(x-a)^2+(y-b)^2=r^2$

$S(a,b)$, polmer r

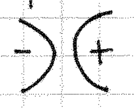
$>$ zunanjost + krožnica
$<$ notranjost

Elipsa: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Hiperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1$



a, b - polosi

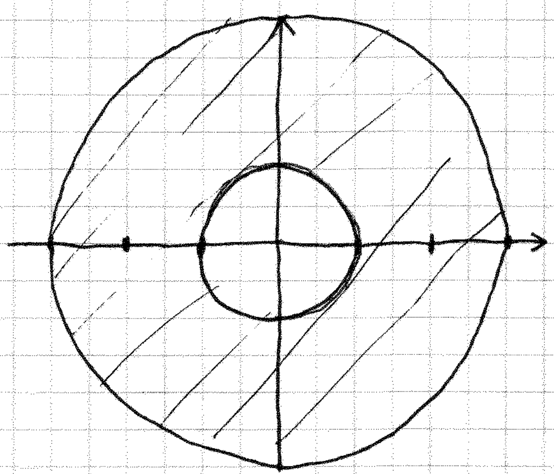
Parabola: $y^2=2px$



$p/2$ .... razdalja med G in temenom

Določi podmnožico realne ravnine

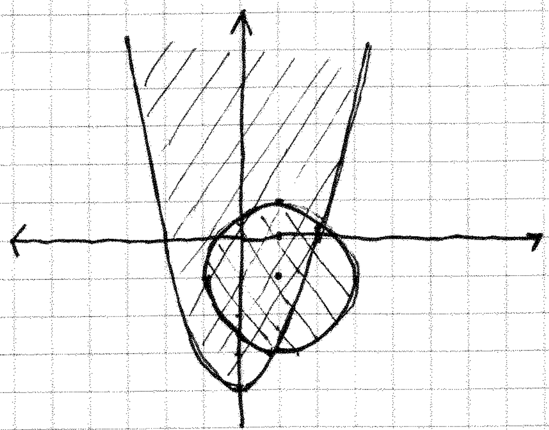
a) $\{(x,y); 1 \leq x^2+y^2 < 9\}$



b) $\{(x,y);\ y \geq x^2-4 \wedge (x-1)^2+(y+1)^2 \leq 4\}$

$y \geq x^2-4$
$y \geq (x-2)(x+2)$

$4 \geq (x+1)^2+(y+1)^2$
$S(1,-1),\ r=2$



c) $\{(x,y);\ 5xy+5x+y^2+y \geq 0\}$

$5xy+5x+y^2+y \geq 0$
$5x(1+y)+y(1+y) \geq 0$
$(y+1)(5x+y) \geq 0$

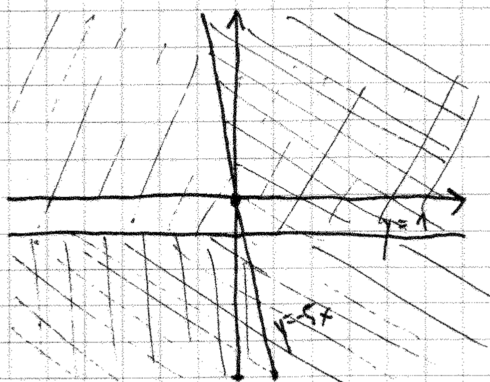
1. oba pozitiv

$y \geq -1$
$y \geq -5x$

$y \geq -1 \wedge y \geq -5x$

2. oba negativ

$y \leq -1$
$y \leq -5x$



d) $D_n: \{(x,y);\ x^2+2xy+y^2 \leq 4\}$

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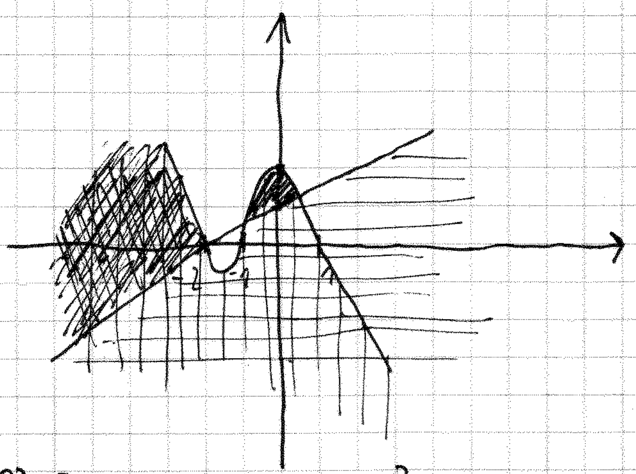
e) $\left\{(x,y);\ 2+x-2x^2-x^3 \geqslant y,\ y \geqslant \frac{x}{2}+1\right\}$

$2+x-2x^2-x^3=y$
$2(1-x^2)+x(1-x^2)=y$
$(1-x^2)(2+x)=y$
$(1-x)(1+x)(2+x)=y$
$x_1=1,\quad x_2=-1,\quad x_3=-2$

$y=x/2+1$



f) $\{(x,y);\ |x|+|y|\leqslant 1\}$

$|x|+|y| \leqslant 1$

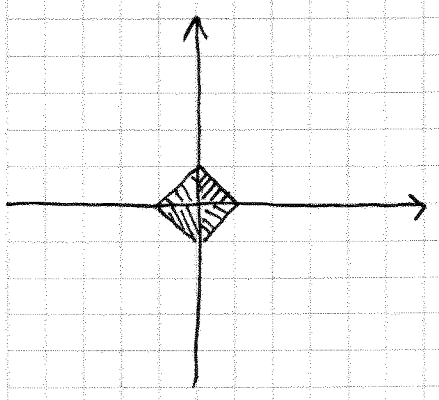
$x+y\leqslant 1 \qquad -x+y\leqslant 1 \qquad x-y\leqslant 1 \qquad -x-y\leqslant 1$

① $y,x \geqslant 0$ (prvi kvadrant)
$y \leqslant 1-x$

② $x\geqslant 0$ (IV. k)
$y<0$
$x-y\leqslant 1$
$y\geqslant x-1$

④ $y<0$
$x<0$
$-x-y\leqslant 1$
$-y\leqslant 1-x$
$y\geqslant x-1$

③ $x<0$ (II kv)
$y>0$
$y-x\leqslant 1$
$y\leqslant x-1$



22.10.2007

1. kolokvij: 19. 11. 2007 19.00
2. kolokvij: 7. 1. 2008 19.00

Matematična indukcija

Vsaka podmnožica naravnih števil, ki vsebuje število 1 in skupaj s številom n tudi njegovega naslednika $(n+1)$, vsebuje vsa $\mathbb{N}$ števila

1. Z indukcijo dokaži naslednje enakosti:

a) $1+2+2^2+\ldots+2^{n-1}=2^n-1$

$\boxed{n=1}\rightarrow$ BAZA INDUKCIJE

$1\overset{?}{=}2^1-1$ ✓

$\boxed{n\rightarrow n+1}$ indukcijski korak

indukcijska predpostavka:

$1+2+2^2+\ldots+2^{n-1}=2^n-1$

Ali velja tudi $n+1$

$1+2+2^2+\ldots 2^n=\boxed{2^{n+1}-1}$

$1+2+2^2+2^n=$

$\underbrace{1+2+2^2+\ldots+2^{n-1}}_{2^n-1}+2n=2^n-1+2^n=2\cdot 2^n-1=\boxed{2^{n+1}-1}$ indukcijska predpostavka

enakost velja

b) $1\cdot 2+2\cdot 3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

$$1^2+2^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$\boxed{n=1}$

$1^2 \stackrel{?}{=} 6/6$

$1=1$ ✓

i.p. $n \to n+1$

$$1^2+2^2+\dots+n^2=\frac{n(n+1)(2n+2)}{6}$$

ali velja za $n \to n+1$

$$1^2+2^2+\dots+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$1^2+2^2+\dots+n^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)^2+\frac{n(n+1)(2n+1)}{6} \stackrel{?}{=} \frac{(n+1)(n+2)(2n+3)}{6}$$

$$6(n+1)^2+n(n+1)(2n+1) \stackrel{?}{=} (n+1)(n+2)(2n+3)$$

$$(n+1)(6n+6+2n^2+n) \stackrel{?}{=} (n+1)(2n^2+7n+6)$$

$$\frac{(7n+2n^2+6)(n+1)}{6} = \frac{(7n+2n^2+6)(n+1)}{6}$$

$$=\frac{(n+1)(n+2)(2n+3)}{6}$$ smo dokazali

- $(\cos x + i\sin x)^n = \cos(nx) + i(\sin(nx))$ adicijski izrek

- $n < 2^n$

$\boxed{n=1}$

$1 \overset{?}{<} 2^1$ ✓

$\boxed{n \to n+1}$ i.p.

$n < 2^n$

$n+1 \leq 2^{n+1}$

$\boxed{n+1} < 2^n + 1 < 2^n + 2 < 2^n + 2^2 \ldots < 2^n + 2^n = 2 \cdot 2^n = \boxed{2^{n+1}}$ ✓

$n < 2^n$

- DN $n^2 < 2^n$, če je $n \geq 5$

- $133 \mid (11^{n+1} + 12^{2n-1}) \leftrightarrow 11^{n+1} + 12^{2n-1} = 133 \cdot k$

$\boxed{n=1}$

$133 \mid 11^2 + 12^1$

$133 \mid 133$

$\boxed{n \to n+1}$ i.p.

$11^{n+1} + 12^{2n-1} = 133 \cdot k, \quad k \in \mathbb{N}$

$11^{n+2} + 12^{2n+1} = 133 \cdot t, \quad t \in \mathbb{N}$

$11^{n+2} + 12^{2n+1} = 11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} =$

$144 = 11 + 133$

$= 11 \cdot (11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{2n-1} = 11 \cdot 133 \cdot k + 12^{2n-1} \cdot 133 =$

$= \boxed{133 \underbrace{(11k + 12^{2n-1})}_{t}}$

✓ dokazano

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- $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$

KOMPLEKSNA ŠTEVILA

$z = x + iy \qquad x, y \in \mathbb{R}$

$i = \sqrt{-1}$ oz $i^2 = -1$ imaginarna enota

$x = Re(z)$ realna komponenta
$y = Im(z)$ imaginarna komponenta

- konjugirana vrednost

$z = x + iy \rightarrow \bar{z} = x - iy$

- $\overline{(\bar{z})} = z$

$\overline{(z \mp w)} = \bar{z} \pm \bar{w}$

$\overline{(z \cdot w)} = \bar{z} \cdot \bar{w}$

$\overline{(z^{-1})} = (\bar{z})^{-1}$

- absolutna vrednost

$|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$

$|z \cdot w| = |z| \cdot |w|$

$|z + w| \leq |z| + |w|$

Poenostavi in določi $Re(z)$, $Im(z)$ ter $|z|$

- $\left(\frac{1-i}{3-2i}\right)^2 = \left(\frac{(1-i)(3+2i)}{(3-2i)(3+2i)}\right)^2 = \left(\frac{3+2i-3i+2}{9+4}\right)^2 = \left(\frac{5-i}{13}\right)^2 = \frac{25-10i-1}{13^2} =$

$= \frac{24-10i}{13^2} = \boxed{\frac{24}{169} - \frac{10i}{169}} \qquad |z| = \sqrt{\left(\frac{24}{169}\right)^2 + \left(-\frac{10}{169}\right)^2}$

$Re = 24/169 \qquad Im = -10/169$

• DN $\frac{2-3i}{3-i} - \frac{4+i}{3+i}$

• $\frac{y+ix}{x+iy} + \frac{x+iy}{x-iy} = \frac{(y+ix)(x-iy)+(x+iy)(x+iy)}{(x+iy)(x-iy)} =$

$= \frac{xy + x^2 i - y^2 i + xy + x^2 + 2xyi - y^2}{x^2+y^2} =$

$= \frac{(x^2+2xy-y^2)(1+i)}{x^2+y^2}$ Re: $\frac{x^2+2xy-y^2}{x^2+y^2}$

Im: $\frac{1(x^2+2xy-y^2)}{x^2+y^2}$

$|z| = \sqrt{\frac{2(x-y)^2}{x^2+y^2}} = \sqrt{2}\cdot\left|\frac{x^2-\cancel{2x}y-y^2}{x^2+y^2}\right|$ $\sqrt{x^2} = |x|$!

• DN: $\frac{z-\bar{z}}{z}$ $z = x+iy$ Re $= 0$ Im $= y$ $|w| = |y|$

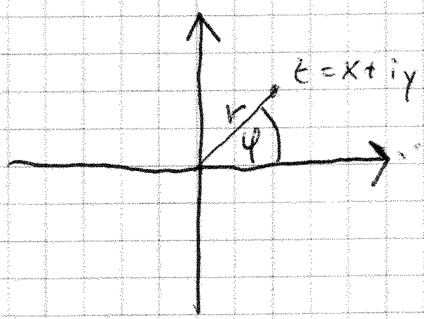
Polarni zapis

$z = x + iy = r(\cos\varphi + i\cdot\sin\varphi) = re^{i\varphi}$

$x = r\cos\varphi$

$y = r\sin\varphi$

$r = |z| = \sqrt{x^2+y^2}$



$\varphi = \text{arc tg}(y/x)$

Eulerjeva formula:

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

x	0	$\pi/6$	π/π	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin x$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
$\operatorname{tg} x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0

$$z = |z|(\cos\varphi + i\sin\varphi)$$

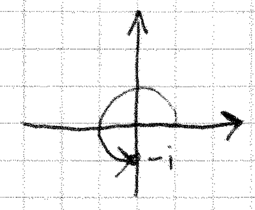
$$w = |w|(\cos\psi + i\sin\psi)$$

$$z \cdot w = |z|\,|w| \cdot (\cos(\varphi+\psi) + i\sin(\varphi+\psi))$$

$$z^2 = |z|^2(\cos 2\varphi + i\sin 2\varphi)$$

$$z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$$ DeMoivreova formula

- Zapiši kompleksna št. v polarni obliki
- DN: $z = 2i$
- $$z = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{2} = \frac{1-2i-1}{2} = -i$$



$$-i = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$ pusti se v tej obliki

$$= e^{i\cdot 3\pi/2}$$

$r = 1$

$\varphi = \operatorname{arctg}(y/x) = \operatorname{arctg}(-\infty) = 3\pi/2$

✓ $z = -1 - i\sqrt{3}$

$r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

$\varphi = atg\,(y/x) = atg(\sqrt{3}/-1) = atg\sqrt{3} = \pi/3 + \pi = 4\pi/3$

$z = 2 \cdot \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

✓ • DN $\quad z = -2 - 2i \qquad R:\ 2\sqrt{2}\, e^{i\frac{5\pi}{4}}$

✓ • $(-\sqrt{3} + 3i)^7 =$

$r = \sqrt{12} = 2 \cdot \sqrt{3}$

$\varphi = arctg\,(y/x) = arc\,tg(-3/\sqrt{3}) = atg(-\sqrt{3}) = \boxed{2\pi/3}$

$\varphi = 2\pi/3$

$z = 2\sqrt{3}\,(\cos(2\pi/3) + i\sin(2\pi/3))$

$z^7 = (2\sqrt{3})^7 (\cos(14\pi/3) + i\sin(14\pi/3))$

$= 128 \cdot 27\sqrt{3}\,(\cos(14\pi/3) + i\sin(14\pi/3))$

$= 3456\sqrt{3}\,(\cos(\cancel{4\pi} + 2\pi/3) + i\sin(\cancel{4\pi} + 2\pi/3))$ ker je perioda 2π

$= 3456\sqrt{3}\,(-\frac{1}{2} + i\,\sqrt{3}/2)$

$= 1728\sqrt{3}\,(-1 + i\sqrt{3})$

• $(1/2 - i/2)^8$

$$r = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

$$\varphi = \operatorname{arctg}(-1) = 3\pi/4 + \pi = 7\pi/4$$

$$z^8 = \left(\frac{1}{\sqrt{2}}\right)^8\left(\cos\left(8\cdot\frac{7\pi}{4}\right) + i\sin\left(8\cdot\frac{7\pi}{4}\right)\right) =$$

$$= \frac{1}{16}\left(\cos(14\pi) + i\sin(14\pi)\right) =$$

$$= \frac{1}{16}\left(\cos 0 + i\sin 0\right)$$

$$= \frac{1}{16}$$

• $(1+i)^{\frac{1}{4}} = z(\;)^4$ ~~[illegible]~~

29.10.07

1. Reši enačbe

a) $z \cdot \bar{z} = -1$

$z = x + iy$ ... $x, y \in \mathbb{R}$
$z = r(\cos\varphi + i\sin\varphi)$

$(x+iy)(x-iy) = -1$

$x^2 + y^2 = -1$ nima $\mathbb{R}$ rešitve R: $\emptyset$

b) $|z| + z = 2 + i$ R: $z = 3/4 + i$

c) $2z^2 - 3\bar{z}^2 = 10i$

$z = x + iy$

$2(x+iy)^2 - 3(x-iy)^2 = 10i$
$2x^2 + 4ixy - 2y^2 - 3x^2 + 6ixy + 3y^2 = 10i$

$\underbrace{2x^2 - 2y^2 - 3x^2 + 3y^2}_{\text{Re}} + \underbrace{4ixy + 6ixy}_{\text{im}} = 10i$

$y^2 - x^2 + 10ixy = 10i$

$\boxed{\begin{array}{l} \text{Re: } y^2 - x^2 = 0 \\ \text{im: } 10xy = 10 \end{array}} \rightarrow y = 1/x$

$\frac{1}{x^2} - x^2 = 0$

$1 - x^4 = 0$

$(1-x^2)(1+x^2) = 0$
$(1-x)(1+x)(1+x^2) = 0$

$x_1 = 1 \rightarrow y_1 = 1$
$x_2 = -1 \rightarrow y_2 = -1$

$z_1 = 1 + i$
$z_2 = -1 - i$

d) $z^3 = 1$

$z = r(\cos\varphi + i\sin\varphi)$

L: $z^3 = r^3(\cos 3\varphi + i\sin 3\varphi)$

D: $1 = 1\cdot(\cos 0 + i\sin 0)$

$r^3(\cos 3\varphi + i\sin 3\varphi) = 1(\cos 0 + i\sin 0)$

$r^3 = 1 \longrightarrow r = 1$

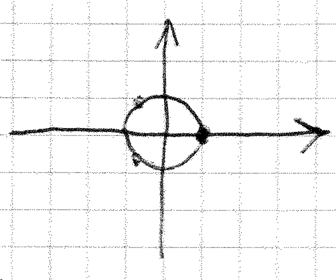
$\cos 3\varphi = \cos 0 \longrightarrow 3\varphi = 0 + \underline{2k\pi} \to \varphi = 2/3\, k\pi, \; k \in \mathbb{Z}$

$z = 1\cdot(\cos 2k\pi/3 + i\sin 2k\pi/3), \; k \in \mathbb{Z}$

3 rešitve
$k = 0 \to \varphi = 0$
$1 \to 2\pi/3$
$2 \to 4\pi/3$
$3 \to 2\pi$

$k = 0, 1, 2$

$z_0 \overset{k=0}{=} 1\cdot(\cos 0 + i\sin 0) = 1$
$z_1 \overset{k=1}{=} 1(\cos 2\pi/3 + i\sin 2\pi/3) = 1\cdot(-1/2 + i\sqrt{3}/2)$
$z_2 \overset{k=2}{=} 1(\cos 4\pi/3 + i\sin 4\pi/3) \quad 1(-1/2 - i\sqrt{3}/2)$



e) $z^3 = i$ R: $z_0 = \sqrt{3}/2 + (1/2)\, i$
$z_1 = \sqrt{3}/2 - (1/2)\, i$
$z_2 = -i$

F) $z^4 = -8 + 8i\sqrt{3}$

$z = r(\cos\varphi + i\sin\varphi)$

$L = r^4(\cos 4\varphi + i\sin 4\varphi)$

$z^4 = -8 + 8i\sqrt{3}$

$R = \sqrt{x^2 + y^2}$ $\qquad \varphi = \operatorname{arctg} \frac{y}{x}$

$R = 16$ $\qquad \varphi = \operatorname{arctg}\left(\frac{8\sqrt{3}}{-8}\right)$

$\varphi = \operatorname{arctg}(-\sqrt{3})$

$\varphi = 2\pi/3 + \pi$

2. kvadrant, koti $< \pi$ $\quad \varphi = 2\pi/3$

$-8 + 8i\sqrt{3} = 16 \cdot \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Enačba

$r^4(\cos 4\varphi + i\sin 4\varphi) = 16 \cdot \cos\left(\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$r^4 = 16 \rightarrow r = 2$

$\cos 4\varphi = \cos 2\pi/3 \rightarrow 4\varphi = 2\pi/3 + 2k\pi \rightarrow$

$\rightarrow \varphi = \pi/6 + k\pi/2$

R: $z = 2 \cdot (\cos(\pi/6 + k\pi/2) + i\sin(\pi/6 + k\pi/2))$

$z_1 =$ $\qquad \sqrt{3} + i$

$z_2 =$ $\qquad$ dn $\qquad -1 + i\sqrt{3}$

$z_3 =$ $\qquad -\sqrt{3} - i$

$z_4 =$ $\qquad 1 - i\sqrt{3}$

g) $= (1+i)^{1/4} = ?$

$(1+i)^{1/4} = z \quad / ()^4$

$1+i = z^4$

DN ... $z = 2^{1/8}\left(\cos\left(\frac{\pi}{16} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{16} + \frac{k\pi}{2}\right)\right)$

$k = 0, 1, 2, 3$

(2) Reši sisteme enačb

a) $|z-2| = 3$
$|z+1| = 3$

$z = x + iy$

$|x + iy - 2| = 3$

$|x + iy + 1| = 3$

$|(x-2) + iy| = 3$
$|(x+1) + iy| = 3$

$\sqrt{(x-2)^2 + y^2} = 3 \quad /()^2$

$\sqrt{(x+1)^2 + y^2} = 3 \quad /()^2$

$\left.\begin{array}{l}(x-2)^2 + y^2 = 9 \\ (x+1)^2 + y^2 = 9\end{array}\right) -$

$(x-2)^2 - (x+1)^2 = 0$

$-6x + 3 = 0$
$x = 1/2 \qquad y^2 = 9 - (x-2)^2 = 9 - \frac{9}{4} = 27/4$

$\rightarrow \quad y = \pm 3\sqrt{3}/2$

R: $z_1 = \frac{1}{2} + i\frac{3\sqrt{3}}{2} \qquad z_2 = \frac{1}{2} - \frac{i\,3\sqrt{3}}{2}$

DN: b) $\left|\frac{z}{z+1}\right| = 1$ R= $z = -1/2 - (1/2)i$

$\frac{z}{\overline{z}} = i$

c) $z_1 \cdot \overline{z_2} = \sqrt{2}$

$\frac{z_1}{\overline{z_2}} = i\sqrt{2} \rightarrow z_1 = i\sqrt{2} \cdot \overline{z_2}$

$i \cdot \sqrt{2}(\overline{z_2})^2 = \sqrt{2} \quad /:\sqrt{2}$

$i(\overline{z_2})^2 = 1 \quad /\cdot -i$

$(\overline{z_2})^2 = -i$

$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$(x_2 - iy_2)^2 = -i$

$x_2^2 - 2ix_2y_2 - y_2^2 = -i$

Re: $x_2^2 - y_2^2 = 0$ Im: $-2x_2y_2 = -1 \rightarrow y_2 = 1/2x_2$

$x_2^2 - \frac{1}{4x_2^2} = 0$

$4x_2^4 - 1 = 0$

$(2x_2^2 - 1)(2x_2^2 + 1)$

$(\sqrt{2}x_2 - 1)(\sqrt{2}x_2 + 1)(2x_2^2 + 1)$

$x_2 = 1/\sqrt{2} \qquad x_2 = -1/\sqrt{2}$

$y_2 = 1/\sqrt{2} \qquad y_2 = -1/\sqrt{2}$

R: $z_2^{(1)} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \qquad z_2^{(2)} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$z_1^{(1)} = i + 1 \qquad z_2^{(2)} = -i - 1$

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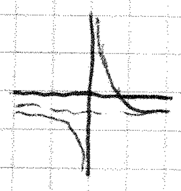
DN: d) $(2+i)z_1 + (2-i)z_2 = 6$
$(3+2i)z_1 + (3-2i)z_2 = 8$

3: Nariši podmnožice $\mathbb{C}$ ravnine

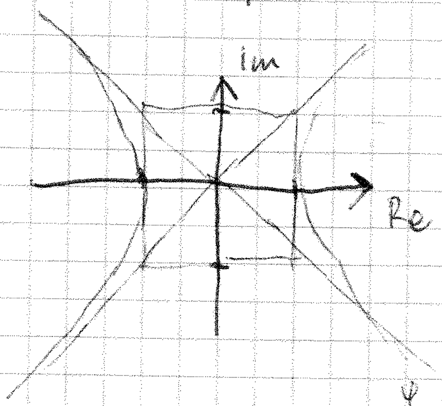
a) $Re(z) + im(z^2) = 2$ R:



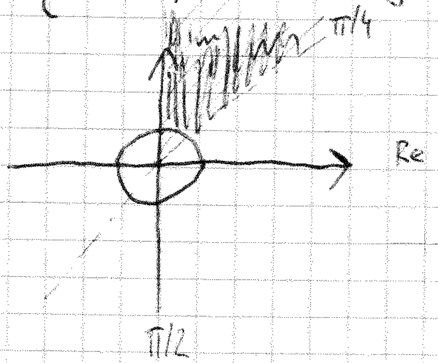
b) $Re(\bar{z}^2) = 4$

$z = x + iy$
$\bar{z} = x - iy$
$x^2 - y^2 = 4$ hiperbola

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ $a, b = 2$



c) $\{z \in \mathbb{C};\ \pi/4 < \arg(z) < \pi/2,\ |z| > 1\}$ (φ = arg, r = |z|)



d) $z \cdot \bar{z} + (1-i)z + (1+i)\bar{z} = 4$

$z = x + iy$

$(x+iy)(x-iy) + (1-i)(x+iy) + (1+i)(x-iy) = 4$

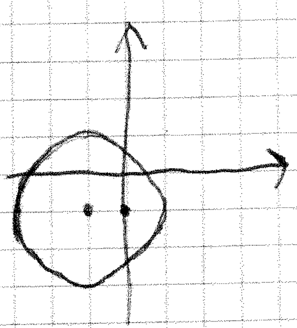
$x^2 + y^2 + x + iy - ix + y + x - iy + ix + y = 4$

$x^2 + y^2 + 2x + 2y = 4$

$(x+1)^2 + (y+1)^2 - 1 - 1 = 4$

$(x+1)^2 + (y+1)^2 = 6$

kružnica, $s = (-1,-1)$ $r = \sqrt{6}$



e) DN: $2 \leq |z - 1 - 2i| < 3$

Zaporedja

- monotonost zaporedja
  - naraščanje, padanje
    $a_{n+1} \geq a_n$ $\quad a_{n+1} \leq a_n$
  - strogo naraščanje, padanje
    $a_{n+1} > a_n$ $\quad a_{n+1} < a_n$

Kriteriji za monotonost

- $a_{n+1} - a_n \geq 0$ naršča
- $a_{n+1} / a_n \geq 1$ narašča

Omejenost zaporedja

- zgornja meja je vsako realno število M, za katero so vsi členi $a_n \leq M$ $\forall n$
- spodnja meja, ko je $a_n \geq m$ $\forall n$
- supremum je najmanjša zgornja meja
  - zgornja meja
    $\forall M' < M \; \exists \; n : a_n > M'$
- infimum je največja spodnja meja
  - spodnja meja
  - $\forall m' > m \; \exists n : a_n < m'$

sup in inf obstajata vedno $(-\infty, \infty)$!
nista člena zaporedja

- navzgor / navzdol omejeno, ko ima končno zgornjo / spodnjo mejo
- zap. je omejeno, če je navzgor in navzdol omejeno.

– Največji in najmanjši člen

- $\max_n a_n$
- $\min_n a_n$ – sta člena zaporedja

Ne obstajata vedno.

- Če $\max a_n$ obstaja, je enak sup a_n, isto za $\min a_n$

4) Določi monotonost zaporedij

a) $a_n = n^2$

$1, 4, 9, 16, 25, 36, 49 \ldots$

$a_{n+1} - a_n = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 > 0$

strogo narašča

✓ b) $a_n = \sin(n\pi)$ R: konstantno

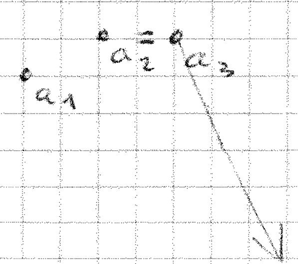
✓ c) $a_n = \frac{3^n}{n!}$

$n! = n \cdot (n-1) \cdot \ldots \cdot 1$

$0! = 1$ (po definiciji)

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$$

$$= \begin{cases} > 1, & n = 1 \\ = 1, & n = 2 \\ < 1, & n \geq 3 \end{cases}$$



$n=1$ narašča
$n \geq 2$ pada
$n > 2$ strogo pada

d) $a_n = e^{-n}$ DN: strogo pada

(5) zapiši splošni člen

a) 2, 3, 4, 5, 6 ....

$a_n = n+1$

b) 1, -4, 9, -16, 25

$a_n = n^2 \boxed{(-1)^{n+1}}$ alternirajoče

(6) Določi največji in najmanjši člen zap. (če obstajata) ter supremum in infimum

a) $a_n = \frac{n}{n+1}$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \ldots$ narašča

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = (n+1)^2 / n^2+2n = \frac{n^2+2n+1}{n^2+2n} > 1$$

zap. strogo narašča

minimum: $a_n = a_1 = 1/2$ = infimum

maximum a_n ne obstaja

Zap narašča proti 1, a se ji poljubno približa, a 1 nikoli ne doseže

$\sup_n a_n = 1$

✓ b) $a_n = 2^n / n!$ R: $\sup a_n = \max a_n = 2$
$\inf = 0$ min ne obstaja

✓ c) $a_n = (n-1)\cdot(-1)^n$

$0, 1, -2, 3, -4, \cancel{5}, -6, \cancel{7}, -8, \cancel{9}, -10$

alternira

- poz. členi (sodi indeks) strogo naraščajo čez vse meje (∞) $\Rightarrow \max_n a_n \nexists$

$\text{supremum}_n\ a_n = \infty$

- neg. členi (lihi indeksi!) strogo padajo čez vse meje ($-\infty$) $\Rightarrow \min_n a_n \nexists$

$\inf_n a_n = -\infty$

✓ d) DN $a_n = \left(1 - \frac{1}{n}\right)(-1)^n$

R: $\min a_n \nexists$, $\inf a_n = -1$
$\max a_n \nexists$, $\sup a_n = 1$

e) $a_n = \frac{1000^n}{n!}$

$10^3, 10^6/2, 10^9/6, 10^{12}/24$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^n}{n!}} = \frac{1000}{n+1} = \begin{cases} >1, & n<999 \\ =1, & n=999 \\ <1, & n>999 \end{cases}$$

$$\max_n a_n = a_{999} = a_{1000} = 1000^{1000}/1000! = \sup_n a_n$$

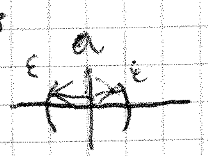
$\min_n a_n$: $\quad 1000^n/n! > 0$ $\quad$ nič ne doseže

$$\min_n a_n \nexists$$

$\text{infimum}_n \, a_n = 0$

STEKALIŠČA IN LIMITE

a je stekališče, če je v vsaki ε-okolici števila a neskončno členov zap $\{a_n\}$



$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

Število a je limita zap. $\{a_n\}$, če je v vsaki ε-okolici št a neskončno mnogo členov zap $\{a_n\}$ in izven te okolice je končno mnogo členov

$$\forall \varepsilon > 0 \;\exists\, n_0 \in \mathbb{N} : \forall n \geq n_0 : |a_n - a| < \varepsilon$$

$$a = \lim_n a_n$$

Zaporedje ki ima ~~zaporec~~ limito je konvergentno. Vsaka limita je stekališče, obratno ni nujno res.
Stekališče je limita podzaporedja.

5.11.07

Konvergenca zaporedij

- omejeno z enim stekališčem
- naraščajoče /padajoče in navzgor/navzdol omejeno

1. Določi stekališča zaporedij. Ali so konvergentna?

a) $a_n = 1 - \frac{1}{n}$ < 1

$a_1 = 0$
$a_2 = 1/2$
$a_3 = 2/3$
$a_4 = 3/4$
$a_5 = 4/5$
$a_6 = 6/5$

eno stekališče: 1 dokaži, da narašča

omejeno med 0 in 1, ima eno stekališče, torej je konvergentno.

b) $a_n = \left(1 - \frac{1}{n}\right)(-1)^n$

$a_1 = a$
$a_2 = 1/2$
$a_3 = -2/3$
$a_4 = 3/4$
$a_5 = -4/5$
$a_6 = 5/6$

alternirajoče zaporedje

$0, 1/2, 3/4, 5/6$

$0, -2/3, -4/5, -6/7$

dve stekališči; $1, -1$; zaporedje je divergentno

$\surd$ c) $a_n = \sin(n\pi/3)$ $\quad$ R: $0, -\sqrt{3}/2, \sqrt{3}/2$ $\quad$ divergentno

$\surd$ d) $0, 1/2, 1/2, 0, 1/4, 3/4, 0, 1/8, 7/8, 0, 1/16, 15/16, 0$

stekališča: $0, 1$

divergira

$\surd$ e) $a_n = \left(2 + \frac{3}{n}\right) \cos \frac{n\pi}{2}$ $\quad$ R: $-2, 0, 2$, div.

$\surd$ f) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \dots$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \longrightarrow 0$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots \rightarrow 1$

$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12} \rightarrow 1/2$

$\frac{n}{m}, \quad n<m \qquad \frac{n}{m}, \frac{2n}{2m}, \frac{3n}{3m} \longrightarrow \frac{n}{m}$

stekališča: - racionalna št. $\frac{n}{m}, n<m$
- 0
- 1

neskončno stekališč, ne konvergentno zap.

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Lastnosti limite

$$\lim_{n\to\infty} c = c \quad , \quad c = \text{konst.}$$

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

② Izračunaj naslednje limite

√ a) $$\lim_{n\to\infty} \frac{8n^2+9n-6}{2n^3+3n+1} \Big/ :n^3 = \lim_{n\to\infty} \frac{\frac{8}{n} + \frac{9}{n^2} - \frac{6}{n^3}}{2 + \frac{3}{n^2} + \frac{1}{n^3}} = \frac{0}{2} = \boxed{0}$$

√ b) $$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{1+2n} = \lim_{n\to\infty} \frac{(n^2+1)^{1/2}}{1+2n} = \lim_{n\to\infty} \frac{\frac{(n^2+1)^{1/2}}{n}}{\frac{1}{n}+\frac{2n}{n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}{\frac{1}{n}+\frac{2n}{n}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{\frac{1}{n}+2} = \frac{1}{2}$$

√ c) $$\lim_{n\to\infty} \frac{7n^2+2}{3n^2+11n-2} = \frac{7}{3}$$

√ d) $$\lim_{n\to\infty} \frac{\sqrt{n+1}}{2+n}$$ R: 0 DN

√ e) $$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \sqrt{n-1} =$$

$$\sqrt{n+1} \to \infty, \quad \sqrt{n} \to \infty, \quad \sqrt{n-1} \to \infty$$

$$(\infty - \infty) \cdot \infty$$

$\infty - \infty$ je nedoločen izraz

$0 \cdot \infty$ je —‖— nedoločeno

∞ / ∞

$\infty + \infty = \infty$

$\infty \cdot \infty = \infty$

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})\sqrt{n-1} = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})\sqrt{n-1}}{(\sqrt{n+1} + \sqrt{n})} =$$

$$\lim_{n\to\infty} (\cancel{n} + 1 - \cancel{n}) \frac{\sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \begin{matrix} /:\sqrt{n} \\ /:\sqrt{n} \end{matrix} = \lim_{n\to\infty} \frac{\sqrt{1 - 1/n}}{\sqrt{1 + 1/n} + 1} = 1/2$$

D.N.

√ f) $\lim_{n\to\infty} \frac{1}{\sqrt{n}(\sqrt{n+1} - \sqrt{n})}$ $\quad R = 2$

√ g) $\lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} \begin{matrix} /:3^n \\ /:3^n \end{matrix} = \lim_{n\to\infty} \frac{2\cdot\left(\frac{2}{3}\right)^n + 3\cdot}{\left(\frac{2}{3}\right)^n + 1} = 3$

delimo z največjo osnovo

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-n}$$

$\ln a + \ln b = \ln(a\cdot b)$
$\ln a - \ln b = \ln(a/b)$
$a \cdot \ln b = \ln b^a$

$\ln a = b \iff a = e^b$

③ Izračunaj limite

√ a) $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n+6}$ $\quad R = e$

√ b) $\lim_{n\to\infty}\left(\frac{n+5}{n+3}\right)^n = \lim_{n\to\infty}\left(1 + \frac{2}{n+3}\right)^n = \lim_{n\to\infty}\left(1 + \frac{2}{n+3}\right)^n =$

$$\lim_{n\to\infty}\left(1 + \frac{1}{\frac{(n+3)}{2}}\right)^n \quad \frac{(n+3)}{2} \to \infty \quad = \lim_{n\to\infty}\left(1 + \frac{1}{\frac{(n+3)}{2}}\right)^{\frac{n+3}{2} \cdot \frac{2}{n+3} \cdot n} =$$

$\to e$ $\quad$ limitiramo

$$= e^{\lim\limits_{n\to\infty} \frac{2n}{n+3}} = e^2$$

c) $$\lim_{n\to\infty} (n+3)\left(\ln(n+1) - \ln n\right) = \lim_{n\to\infty} (n+3) \cdot \ln\left(\frac{n+1}{n}\right) =$$

pravilo

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{(n+3)} = \ln \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{(n+3)\cdot n \cdot \frac{1}{n}} = \ln e^{\lim\limits_{n\to\infty} \frac{n+3}{n}} =$$

$$= \ln e = 1$$

d) $$\lim_{n\to\infty} \left(\frac{2n^2+6}{2n^2+5}\right)^{4n^2+3} = \lim_{n\to\infty}\left(1+\frac{1}{2n^2+5}\right)^{4n^2+3} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{(2n^2+5)\to\infty}\right)^{2n^2+5 \cdot \frac{4n^2+3}{2n^2+5}} = e^{\lim\limits_{n\to\infty} \frac{4n^2+3}{2n^2+5}} = \underline{e^2}$$

e) $$\lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n^2+n+1}\to\infty}\right)^{1-\sqrt{n^2+n+1}} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\underset{\downarrow\,\infty}{\sqrt{n^2+n+1}}}\right) \cdot \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{\sqrt{n^2+n+1}\cdot(-1)}$$

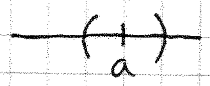
$$= \underset{\Downarrow}{1} \cdot e^{-1} = \boxed{e^{-1}}$$

f) $$\lim_{n\to\infty}\left(1+\frac{1}{\underbrace{1-n^2}_{\to -\infty}}\right)^{n^2} = \lim_{n\to\infty}\left(1-\frac{1}{\underbrace{(n^2-1)}_{\to\infty}}\right)^{n^2} =$$

$$= \boxed{\lim_{n\to\infty}\left(1-\frac{1}{n^2-1}\right)^{-(n^2-1)}}^{\,\cdot\,\frac{n^2}{(-n^2-1)}} = e^{\lim\limits_{n\to\infty}\frac{n^2}{1-n^2}} = e^{-1}$$

4. Ugotovi, od katerega člena dalje se členi zaporedja $\{a_n\}$ razlikujejo od limite za $< \varepsilon$.

a) $a_n = \dfrac{n^2+n}{n^2+1}$, $\varepsilon = \dfrac{1}{10}$



- $a = \lim\limits_{n\to\infty} a_n$
- $|a_n - a| < \varepsilon$

$$a = \lim_{n\to\infty}\frac{n^2+n}{n^2+1} = 1$$

$$|a_n - a| < \varepsilon$$

$$\left|\frac{n^2+n}{n^2+1} - 1\right| < \frac{1}{10}$$

$$\left|\frac{n^2+n-n^2-1}{n^2+1}\right| < \frac{1}{10}$$

$$\left|\frac{\overset{0}{\overset{\wedge}{n-1}}}{\underset{\underset{1}{\vee}}{n^2+1}}\right| < \frac{1}{10}$$

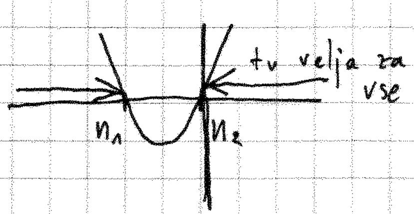
(ker so naravna št, lahko abs. brišemo)

$$\frac{n-1}{n^2+1} < \frac{1}{10} \quad /\cdot 10(n^2+1)$$

$10(n-1) < n^2 + 1$

$n^2 - 10n + 11 > 0$

$n_{1,2} = \frac{10 \pm \sqrt{56}}{2} = 5 \pm \sqrt{14}$



>0, nad x osjo

$n_2 > 5 + \sqrt{14} = 8, \ldots$

$n_0 = 9$

Od devetega člena naprej

b) DN

$a_n = \frac{n^2 + 2n}{n^2 - 2n + 3}$, $\varepsilon = 1/5$ R: $n_0 = 22$

c) $a_n = \frac{5^n - 1}{5^n}$, $\varepsilon = 25^{-25}$

- $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n - 1}{5^n} = 1$

- $|a_n - a| < \varepsilon$

$\left| \frac{5^n - 1}{5^n} - 1 \right| < \varepsilon$

$\left| \frac{5^n - 1 - 5^n}{5^n} \right| < \varepsilon$

$\left| \frac{-1}{5^n} \right| < 25^{-25}$

$\frac{1}{5^n} < 25^{-25}$

$1 < 25^{-25} \cdot 5^{n}$

$1 < 5^{-50+n}$

$n - 50 > 0$

$n > 50 \qquad n_0 = 51$

Od 51. člena naprej.

⑤ Dokaži, da je rekurzivno podano zaporedje konvergentno in izračunaj limito

a) $a_1 = 0 \qquad a_{n+1} = \frac{a_n}{3} - 2$

$0,$

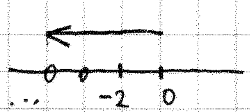
$a_2 = \frac{a_1}{3} - 2 = \frac{0}{3} - 2 = -2$

$a_3 = -8/3$

$a_4 = -26/9$

$a_5 = -80/27$

...



pada

pada in navzdol omejeno

- navzdol omejeno z -3

INDUKCIJA $\quad n=1$

$a_1 \geq -3$

$0 \geq -3$ ✓

$n \to n+1 \qquad$ I.p. $\; a_n \geq -3$

$a_{n+1} \geq -3$?

$a_{n+1} = \frac{a_n}{3} - 2 \overset{i.p.}{\geq} \frac{-3}{3} - 2 = -3$ ✓ smo dokazali

- je padajoče

$a_{n+1} - a_n = \frac{a_n}{3} - 2 - a_n = -\frac{2}{3} a_n - 2 \leq 0$

$a_n \geq -3 \quad / \cdot (-1)$

$-a_n \leq 3$

$\frac{2}{3} \cdot 3 - 2 = 0 \rightarrow$ pada $\rightarrow$ konvergentno

- limita

$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty}$

$\Downarrow$

$\lim_{n\to\infty} a_{n+1} = a$

$a_{n+1} = \frac{a_n}{3} - 2$

$a = \frac{a}{3} - 2$

$a = \frac{a}{3} - 2$

$a = -3$

b) D.N. $a_1 = 1, \quad a_{n+1} = \frac{a_n}{2} + 1 \qquad \lim = 2$

c) $a_1 = 1 \qquad a_{n+1} = \frac{1}{5} a_n^2 + 1$

$1, \frac{6}{5}, \frac{161}{125}, \ldots$

narašča? in je navzgor omejeno?

• navzgor omejeno z 2 ?

INDUKCIJA baza: $n=1$

$a_1 \leq 2$
$1 \leq 2$ ✓

$n \to n+1$ i.p. $a_n \leq 2$

$a_{n+1} \leq 2$?

$$a_{n+1} = \frac{1}{5} a_n^2 + 1 \overset{\text{i.p.}}{\leq} \frac{1}{5} \cdot 4 + 1 = \frac{9}{5} < 2 \quad ✓$$

$a_n \leq 2$
$a_n^2 \leq 4$

• narašča $a_{n+1} - a_n \geq 0$

INDUKCIJA

$\boxed{n=1}$ $a_2 - a_1 \geq 0$

$$\frac{6}{5} - 1 = \frac{1}{5} \geq 0 \quad ✓$$

ind. korak
$n \to n+1$ i.p. $a_{n+1} - a_n \geq 0$

$a_{n+2} - a_{n+1} \geq 0$?

$$a_{n+2} - a_{n+1} = \frac{1}{5}(a_{n+1})^2 + 1 - a_{n+1} = \frac{1}{5}(a_{n+1})^2 + 1 - \frac{1}{5}a_n^2 - 1 =$$

$$= \frac{1}{5}\left(a_{n+1}^2 - a_n^2\right) = \frac{1}{5}\underbrace{(a_{n+1} - a_n)}_{\text{po ip } \geq 0}\underbrace{(a_{n+1} + a_n)}_{>1 \text{ (po zvezi)}} \geq \frac{1}{5} \cdot 0 \cdot 1 = \boxed{0}$$

dokazano

• limita

imata isto limito !

$a = \lim_{n\to\infty} a_n$ $a_{n+1} = \frac{1}{5} a_n^2 + 1$

$$a = \frac{1}{5} a^2 + 1$$

$$a^2 - 5a + 5 = 0$$

$a_{1,2} = \frac{5 \pm \sqrt{5}}{2}$

zaporedje je med 1 in 2. Tam je tudi limita.

$a = (5 - \sqrt{5})/2$

$a_2 = (5 + \sqrt{5})/2$ ne leži in ni prava rešitev!

d) $a_1 = 1 \quad a_{n+1} = 2(a_n + 1)$ R: divergira, neomejeno

Geometrijska vrsta

$a + aq + aq^2 + aq^3 + \ldots = \sum_{n=0}^{\infty} aq^n = \sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}$

$|q| < 1$!

⑥ Izračunaj vsoto geo. vrste

a) $\sum_{n=1}^{\infty} \frac{37}{100^n} = \sum_{n=1}^{\infty} \underset{\substack{\shortparallel \\ a}}{37} \cdot \underset{\substack{\shortparallel \\ q}}{\left(\frac{1}{100}\right)}^n = \frac{a}{1-q} - a_0 = \frac{37}{1 - 1/100} - 37 =$

$= \frac{37 \cdot 100 - 37 \cdot 99}{99} = \boxed{37/99}$

Pazi na člen ($a_1, a_0 \ldots$)

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b) $\sum_{n=1}^{\infty} \frac{6}{10^n}$ R: $\frac{2}{3}$

c) $\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$ R: 4

d) $$\sum_{n=3}^{\infty} \frac{11}{2\cdot 5^n} = \sum_{n=3}^{\infty} \underset{a}{\underset{\|}{11}} \cdot \underset{q}{\underset{\|}{\frac{1}{2\cdot 5^n}}} = \underset{a}{\underset{\|}{\frac{11}{2}}} \cdot \underset{q}{\underset{\|}{\left(\frac{1}{5}\right)^n}} =$$

$$= \frac{a}{1-q} = \frac{\frac{11}{2}}{1-\frac{1}{5}} - \frac{11}{2} - \frac{11}{2\cdot 5} - \frac{11}{2\cdot 5\cdot 5} =$$

$$= \frac{\frac{11}{2}}{\frac{4}{5}} - \left(\frac{25\cdot 11 + 5\cdot 11 + 11}{2\cdot 5\cdot 5}\right) =$$

$$= \frac{55\cdot 25 - 11\cdot 100 - 44 - 11\cdot 20}{200} =$$

$$= \frac{11(5\cdot 25 - 100 - 4 - 20)}{200} = \frac{11(125 - 100 - 4 - 20)}{200} =$$

$$= \frac{11}{200}$$

$$\sum_{n=3}^{\infty} \frac{11}{2\cdot 5^n} = \sum_{n=3}^{\infty} \frac{11}{2\cdot 5^3 \cdot 5^{n-3}} = \sum_{n=3}^{\infty} \frac{11}{250} \left(\frac{1}{5}\right)^{n-3} =$$

$$= \frac{\frac{11}{250}}{1-\frac{1}{5}} = \frac{11}{200}$$

② Zapiši decimalno število v obliki ulomka (z uporabo geom vrste)

√ a) 0,232323 R: 23/99

√ b) 5.146146146...

$$\frac{146}{1000} \quad \frac{146}{1000^2} \quad \frac{146}{1000^3}$$

$$= 5 + \frac{146}{1000} + \frac{146}{1000^2} + \frac{146}{1000^3} =$$

$$= 5 + \sum_{n=1}^{\infty} \frac{146}{1000^n} = 5 + \sum_{n=1}^{\infty} \frac{146}{\underset{\overset{\shortparallel}{a}}{1000} \cdot \underset{\overset{\shortparallel}{q}}{1000^{n-1}}} =$$

$$= 5 + \sum_{n=1}^{\infty} \frac{146}{1000} \cdot \frac{1}{1000^{n-1}}$$

$$= 5 + \frac{\frac{146}{1000}}{1-\frac{1}{1000}} = \frac{146}{999} + 5 = \frac{4995+146}{999} = \frac{5141}{999}$$

c) $8{,}\underbrace{1238}_{\frac{1238}{10000}}1238\,123\ldots$

$$= 8+\sum_{n=1}^{\infty}\frac{1238}{10000^n} = 8+\sum_{n=1}^{\infty}\frac{1238}{10000}\cdot\left(\frac{1}{10000}\right)^{n-1} =$$

$$= 8+\frac{\frac{1238}{10000}}{1-\frac{1}{10000}} = 8+\frac{\frac{1238}{1000}}{\frac{9999}{10000}} = 8+\frac{1238}{9999} =$$

$$= 8+\frac{1238}{9999} =$$

Numerične metode

Številske vrste

$$a_1+a_2+a_3+\ldots = \sum_{n=1}^{\infty} a_n$$

$$\left.\begin{array}{l} S_1 = a_1 \\ S_2 = a_1+a_2 \\ S_3 = a_1+a_2+a_3 \\ S_n = a_1+a_2+a_3+\ldots+a_n \end{array}\right\}\text{ zaporedje delnih vsot}$$

Vrsta konvergira, če konvergira zap. delnih vsot, sicer divergira.

Če vrsta konvergira, velja, da je

$$S=\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Konvergira absolutno, če konvergira vrsta

$$\sum_{n=1}^{\infty} |a_n|$$

Pogojno konvergentne vrste – če konvergira, a ne konvergira absolutno

$$\left.\begin{array}{l}\sum a_n = \infty \\ \sum 1 = \infty\end{array}\right\} \text{ divergira}$$

③ Seštej vrsto, tako da izračunaš limito zaporedja delnih vsot

a) $\sum\limits_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)}}_{a_n}$

$S_N = a_1 + a_2 + \dots a_N$

$\lim\limits_{N\to\infty} S_N$

parcialni ulomki:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)} = \frac{A(n+1)+B\cdot n}{n(n+1)} =$$

$\Rightarrow$ izenačimo števce

$1 = A(n+1) + B(n)$

$n^1: \quad 0 = A + B$

$n^0: \quad 1 = A \longrightarrow B = -1$

$$\boxed{\frac{1}{n} - \frac{1}{n+1}} = a_n$$

$$S_n = a_1 + a_2 + \dots + a_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} s_n = \lim_{N \to \infty} \left(1 - \underbrace{\frac{1}{N+1}}_{0} \right) = 1$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

$$\downarrow$$

$$a_n$$

$$a_n = \frac{1}{4n^2 - 1} = \frac{1}{(4n-1)(4n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n+1)} = \frac{A(2n+1) + B(2n-1)}{4n^2 - 1}$$

$$1 = A(2n+1) + B(2n-1) = A2n + A + B2n - B$$

$$n^0: \quad 1 = A - B$$
$$n^1: \quad 0 = 2A + 2B$$

$$A = 1 + B$$

$$0 = 2(1+B)$$
$$B = -1/2$$
$$A = 1/2$$

$$\frac{1}{4n^2 - 1} = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} = a_n$$

$$S_N = a_1 + a_2 + \ldots a_N$$

$$S_N = \left(\frac{1}{2} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} \right) + \ldots + \left(\cancel{\frac{\frac{1}{2}}{2N-1}} - \frac{\frac{1}{2}}{2N+1} \right) =$$

$$= \frac{1}{2} - \frac{1}{4N+2}$$

$$\lim_{N \to \infty} \left(\frac{1}{2} - \frac{1}{4N+2} \right) = \frac{1}{2}$$

Kriteriji konvergence za vrste s pozitivnimi členi

- Kvocientni kriterij
- Korenski kriterij

$\sum_{n=1}^{\infty} a_n$

$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $q = \lim_{n\to\infty} \sqrt[n]{a_n}$

- $q < 1$: Vrsta konvergira
- $q > 1$: divergira
- $q = 1$: kriterij odpove

(4) S pomočjo kvocientnega ali korenskega kriterija določi konvergenco vrste

√ a) $\sum_{n=1}^{\infty} (1/n!)$ R: konvergira, kvocientni

√ b) $\sum_{n=1}^{\infty} \underbrace{(3n)!/(2n)!}_{a_n}$

– kvocientni:

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(3n+1)!}{(2n+2)!}}{\frac{(3n)!}{(2n)!}} =$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}\cancel{(2n)!}}{\cancel{(3n)!}(2n+2)(2n+1)\cancel{(2n)!}} =$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(2n+2)(2n+1)} =$$

$$= \lim_{n\to\infty} \frac{27n^3 + \ldots}{4n^2 + \ldots} \Big/ :n^3 = \lim_{n\to\infty} \frac{27 + \overbrace{\ldots}^{\to 0}}{\underbrace{\frac{4}{n}}_{\to 0} + \underbrace{\ldots}_{\to 0}} = \infty > 1$$

divergira

d) $\sum\limits_{n=1}^{\infty} \frac{2n \cdot n!}{(n+1)!}$ $\qquad q=1$, odpove

e) $\sum\limits_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$q=\lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} =$$

$$= \lim_{n\to\infty} \frac{\frac{2\cdot 2^n\cdot(n+1)(n)!}{(n+1)^n(n+1)}}{\frac{2^n \cdot n!}{n^n}} = \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}\cancel{(n+1)}\cancel{(n!)}\cancel{n^n}\, n^n}{\cancel{2^n}\cdot\cancel{n!}\cdot\cancel{(n+1)}\cdot(n+1)^n} =$$

$$= \lim_{n\to\infty} \frac{2\, n^n}{(n+1)^n} = \lim_{n\to\infty} 2 \frac{n^n}{(n+1)^n} = 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= 2\cdot \lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = 2\cdot\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n\cdot\left(-\frac{n+1}{1}\right)\left(-\frac{1}{n+1}\right)} =$$

$$= 2\lim_{n\to\infty} e^{n\cdot\left(-\frac{1}{n+1}\right)} = 2\cdot e^{\lim\limits_{n\to\infty} n\left(-\frac{1}{n+1}\right)} =$$

$$= 2\, e^{(-n)/(n+1) \quad /:n/:n} = 2e^{-1}$$

$2e^{-1} < 1$, vrsta konvergira

f) $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n \cdot 5^n}$

– korenski

$$q = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^{2n+1}}{n \cdot 5^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{9^n \cdot 3}{n \cdot 5^n}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{9^n \cdot 3}{n \cdot 5^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{9^n \cdot 3}{n \cdot 5^n}} = \lim_{n\to\infty} \frac{9 \cdot \sqrt[n]{3}}{\sqrt[n]{n} \cdot 5}$$

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1 \quad , \quad \lim_{n\to\infty} \sqrt[n]{c} = 1}$$

$= \frac{9}{5} > 1$, divergira

g) $\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$ div.

Primerjalni kriterij

Če za vsak $n \geq n_0$ velja $0 \leq a_n \leq b_n$

- Če $\underbrace{\text{vrsta } b}_{\text{MAJORANTA}}$ konvergira ⇒ konvergira tudi vrsta a

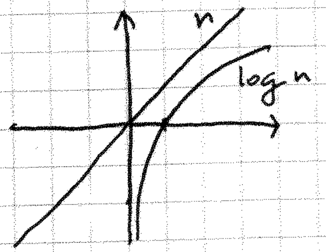
- Če $\underbrace{\text{vrsta } a}_{\text{minoranta}}$ divergira ⇒ divergira tudi vrsta b

$\sum_{n=1}^{\infty} \frac{1}{n}$ harmonična vrsta divergira

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \alpha > 1 & \text{konvergira} \\ \alpha \leq 1 & \text{divergira} \end{cases}$$

⑥ S pomočjo minorante ali majorante določi konvergenco vrste

a) $\sum_{n=1}^{\infty} \frac{1}{\log n}$



$\forall n: \ln n < n$ (indukcija

$$\frac{1}{\ln n} > \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{\ln n}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira, torej tudi

divergira

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}$ $\frac{1}{(n^3+n)^{1/2}} < 1/n^{3/2}$

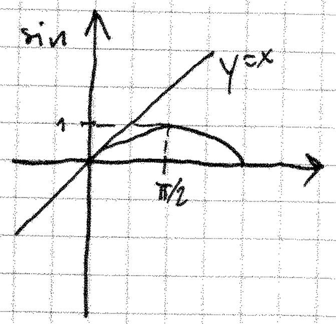
$$(n^3+n)^{1/2} \geq n^{3/2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ konvergira

torej konvergira tudi

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+n)^{1/2}}$$

c) $\sum_{n=1}^{\infty} \sin \left(\frac{1}{n^2}\right) \longrightarrow \in (0,1]$



$\sin x < x \qquad \forall x \in (0,1]$

$\sin 1/n^2 < 1/n^2 \qquad \forall n$

$$\sum_{n=1}^{\infty} \sin \frac{1}{n^2} \Leftarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{konvergira}$$

konvergira

⑥ Določi konvergenco vrste

$$\sum_{n=1}^{\infty} \frac{(2n+1)^3}{(n^3+1)^2} \qquad \text{D.N. po kvocientnem}$$

- po primerjalnem

$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

$$= \sum_{n=1}^{\infty} \frac{8n^3 + 12n^2 + 6n + 1}{n^6 + 2n^3 + 1}$$

od nekje naprej

$$\frac{8n^3 + 12n^2 + 6n + 1}{n^6 + 2n^3 + 1} \leq \frac{8n^4}{n^6} = \frac{8}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{8}{n^2} \quad \text{konvergira}$$

konvergira $\Leftarrow$

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VELJA

$$\sum_{n=1}^{\infty} \frac{p(n)}{q(n)} \begin{cases} \text{konvergira, če} & st.(q) - st(p) \geq 2 \\ \text{divergira, če} & st(q) - st(p) < 2 \end{cases}$$

Konvergenca alternirajoče vrste

$\pm\sum(-1)^n a_n$, $a_n \geq 0$

Leibnitzov kriterij: alternirajoča vrsta konvergira, če zaporedje $\{a_n\}$ od nekje naprej pada proti 0.

- od nekje naprej padajoče zap. $\{a_n\}$.
- $\lim a_n = 0$

1. Določi konvergenco alternirajoče vrste

a) $\sum_{n=1}^{\infty} (-1)^n / n(n+1)$ konvergira

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ $\rightarrow + - + - \ldots$

$\Downarrow$

$a_n = \frac{1}{2^n}$

- monotonost: $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2 \cdot 2^n}}{\frac{1}{2^n}} = \frac{1}{2} < 1 \Rightarrow$ pada

- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2^n} = 0 \rightarrow$ torej vrsta konvergira (leibniz)

c) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$, $a_n = \frac{n!}{n^n}$

- $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)\cancel{n!}}{n \cdot \cancel{n^n}}}{\frac{\cancel{n!}}{\cancel{n^n}}} = \cancel{\ldots} \left(\frac{n}{n+1}\right)^n < 1$, pada

$\bullet \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n!}{n^n} = \lim\limits_{n\to\infty} \frac{\cancel{n}\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}{\cancel{n}\cdot n\cdot\ldots\cdot n\cdot n}$ → 0 · nekaj omejenega

$\frac{n-1}{n} \leq 1$, $\frac{2}{n} \leq 1$, $\frac{1}{n} \leq 1$

$= \underbrace{\lim\limits_{n\to\infty} \frac{1}{n}}_{=0} \cdot \underbrace{\lim\limits_{n\to\infty} \frac{(n-1)\cdot\ldots\cdot 2\cdot 1}{n\cdot\ldots\cdot n\cdot n}}_{\text{omejeno}} = 0 \cdot \text{omejeno} = 0$

$\lim\limits_{n\to\infty} a_n = 0$, vrsta konvergira

② Pokaži, da vrsta $\sum\limits_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ konvergira pogojno, absolutno pa ne.

- absolutna konv: $\sum |a_n|$ konvergira
- pogojna konv: $\underbrace{\sum a_n \text{ konv}}_{\text{Leibniz}}$, $\underbrace{\sum |a_n| \text{ divergira}}_{\text{harmonična - divergira}}$

Kolokvij 1. (2003)

1. Reši neenačbo in zapiši kot interval oz. unijo intervalov

$$\left|\frac{2x-1}{x}\right| < 2$$

$$-2 < \frac{2x-1}{x} < 2 \qquad \text{ali} \qquad / \cdot |x|$$

$$\underbrace{|2x-1|} < 2\underbrace{|x|}$$

+	+
+	−
−	+
−	−

$2x-1>0$, $x>0$ $\qquad$ $2x-1>0$, $x<0$ $\qquad$ $2x-1<0$, $x>0$ $\qquad$ $2x-1<0$, $x<0$

1) $\boxed{x \geq 1/2}$, $x>0$ $\qquad$ 2) ni rešitve $\qquad$ 3) $\boxed{0<x<1/2}$ $\qquad$ 4) $\boxed{x<0}$

1) $2x-1<2x$, $-1<0$, $x \in \mathbb{R}$ $\qquad$ 3) $-2x+1<2x$, $x>1/4$ $\qquad$ 4) $-2x+1<-2x$, $1<0$ // ni rešitve

$x \in [1/2, \infty)$ $\qquad$ $x \in (1/4, 1/2)$

$$R: R_1 \cup R_2 = [1/2, \infty) \cup (1/4, 1/2) = (1/4, \infty)$$

2. Poišči vse rešitve:

$z^3 + 2 + 2i\sqrt{3} = 0$

$z^3 = -2 - 2i\sqrt{3}$

polarna oblika

$z = r \cdot (\cos\varphi + i\sin\varphi)$

$r^3(\cos 3\varphi + i\sin 3\varphi) = -2 - 2i\sqrt{3}$

$-2 - 2i\sqrt{3} \rightarrow$ polarno

$r' = \sqrt{4 + 12} = 4$

$\varphi' = \operatorname{atg} y/x = \sqrt{3} = \frac{\pi}{3} + \pi$

$D: 4(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

$r^3(\cos 3\varphi + i\sin 3\varphi) = 4(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

$r = \sqrt[3]{4}$ $\quad \cos 3\varphi = \cos\frac{4\pi}{3} + 2k\pi \quad k \in \mathbb{Z}$

$\boxed{\varphi = 4\pi/9 + 2k\pi/3} \quad k \in \mathbb{Z}$

$z = \sqrt[3]{4}\,(\cos(4\pi/9 + 2k\pi/3) + i\sin(4\pi/9 + 2k\pi))$

$k: 0, 1, 2$

$k=0: \sqrt[3]{4}\,(\cos(4\pi/9) + i\sin(4\pi/9))$
$k=1: \sqrt[3]{4}\,(\cos(10\pi/9) + i\sin(10\pi/9))$
$k=2: \sqrt[3]{4}\,(\cos(16\pi/9) + i\sin(16\pi/9))$

3) Izračunaj limito zaporedja, podanega z

$a_n = \sqrt{n}\left(\sqrt{n+7} - \sqrt{n}\right)$

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+7} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n+7} - \sqrt{n}\right)\left(\sqrt{n+7} + \sqrt{n}\right)}{\left(\sqrt{n+7} + \sqrt{n}\right)} =$$

$$/ : \sqrt{n} / \sqrt{n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\,(n+7-n)}{\sqrt{n+7} + \sqrt{n}} = \lim_{n\to\infty} \frac{7\sqrt{n}}{\sqrt{n+7} + \sqrt{n}} = \frac{7 \cdot 1}{\sqrt{1} + \sqrt{1}} = \frac{7}{2}$$

4) Ali je vrsta

$\sum_{n=1}^{\infty} 2n/(2n!)$ konvergentna?

$$q = \lim_{n\to\infty} (a_{n+1}/a_n)$$

$$q = \lim_{n\to\infty} \frac{\frac{2n+2}{(2n+2)!}}{\frac{2n}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{\cancel{(2n+2)}}{\cancel{(2n+2)}(2n+1)\cancel{(2n)!}}}{\frac{2n}{\cancel{(2n)!}}} = \frac{1}{(2n+1)2n} = 0$$

$q < 1$, konvergira

4b) $\sum_{n=1}^{\infty} \frac{2^{n-1} \cdot n}{3^{2n} \cdot 5}$

$$q = \lim_{n\to\infty} \sqrt[n]{\frac{2^{n-1} \cdot n}{3^{2n} \cdot 5}} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n \cdot n}{2 \cdot 9^n \cdot 5}} = \frac{2}{9} < 1, \text{ konvergira}$$

Kol 1 (2006)

1) Reši neenačbo

$|-2|x+1| + 1| < 1$

rešitev zapiši kot interval oz. kot unijo intervalov.

$|-2|x+1| + 1| < 1$

- $x+1 \geq 0 \rightarrow \boxed{x \geq 1}$

$|-2x-2+1| < 1$

$|-2x-1| < 1$

$-2x-1 \geq 0 \rightarrow \boxed{x \leq 1/2}$

- $-2x-1 < 1$

$2x > 2$

$\boxed{x \geq 1}$

- $-2x-1 < 0 \rightarrow \boxed{x > -1/2}$

$2x+1 < 1$

$2x < 0$

$\boxed{x < 0}$

$R_1: (-1, -1/2]$

$R_2: (-1/2, 0)$

- $x+1 < 0 \rightarrow \boxed{x < 1}$

$|2x+2+1| < 1$

$|2x+3| < 1$

- $2x+3 \geq 0$

$\boxed{x \geq -3/2}$

$2x+3 < 1$

$\boxed{x < 1}$

- $2x+3 < 0$

$\boxed{x < -3/2}$

$-2x-3 < 1$

$\boxed{x > -2}$

$R_3: [-3/2, -1)$

$R_4: (-2, -3/2)$

$R = R_1 \cup R_2 \cup R_3 \cup R_4$

3) Poišči infimum in ~~...~~ supremum ter najmanjši in največji člen.

$a_n = 4n^2 - 21n + 2006$

kdaj zap. narašča in kdaj pada

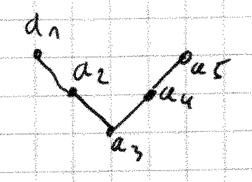
- monotonost

$a_{n+1} - a_n = 4(n+1)^2 - 21(n+1) + \cancel{2006} - 4n^2 + 21n - 2006 =$

$= 4n^2 + 8n + 4 - 21n - 21 - 4n^2 + 21n = \underbrace{\boxed{8n - 17 \geq 0}}_{\text{narašča}}$

$n \geq 17/8 = 2, \ldots$

narašča od 3jega člena naprej
pada za $n=1$, $n=2$



$\min_n = \inf_n a_n = a_3 = 4 \cdot 9 - 63 + 2006 = 1979$

$\lim_{n \to \infty} a_n = \infty = \max \nexists$, $\sup_n a_n = \infty$

4) Ali vrsta $\sum_{n=1}^{\infty} \frac{(4n)!\, n^n}{(3n)!}$

$$q = \lim_{n\to\infty} \frac{\frac{(4n+4)!\,(n+1)^{(n+1)}}{(3n+3)!}}{\frac{(4n)!\, n^n}{(3n)!}} = \lim_{n\to\infty} \frac{\frac{(4n+4)(4n+3)(4n+2)(4n+1)\cancel{(4n)!}\,(n+1)(n+1)^n}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}}{\frac{\cancel{(4n)!}\, n^n}{\cancel{(3n)!}}} =$$

$$= \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n+1)(n+1)^n}{(3n+3)(3n+2)(3n+1)\, n^n} =$$

$$= \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n+1)}{(3n+3)(3n+2)(3n+1)} \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{e} = \infty \cdot e = \infty > 1$$

~~...~~ divergira

26.11.2007

II

Funkcije

$f: D \to K$, $D \subseteq \mathbb{R}$, $K \subseteq \mathbb{R}$

kodomena

domena ali def. območje

$Z_f = \{ f(x); x \in D\}$ zaloga vrednosti

$Z_f \subseteq K$

Funkcija f narašča, če $\forall x_1 < x_2$ velja $f(x_1) \leq f(x_2)$. Če je $f(x_1) < f(x_2)$, strogo narašča. Funkcija pada, če je ravno obratno.

• Lihost in sodost funkcije.

$f(-x) = -f(x)$ liha
$f(-x) = f(x)$ soda

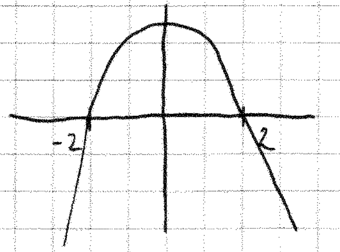
– sode so simetrične glede na y os.
– lihe so simetrične glede na koordinatno izhodišče.

① Določi df naslednjih funkcij

a) $f(x) = \log \underbrace{(4-x^2)}_{4-x^2>0} + \dfrac{1}{x^2-1} \rightarrow x^2-1 \neq 0$ pogoj za obstoj izraza

• $x \neq \pm 1$

• $(2-x)(2+x) > 0$



$x \in (-2, 2)$

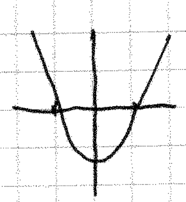
• Df: $Df = \{x; x^2-1 \neq 0 \text{ in } 4-x^2>0\} = \boxed{(-2,2) - \{-1,1\}}$

b) DN: $f(x) = x + \ln\left(\frac{x-1}{x+1}\right)$ R: $(-\infty, -1) \cup (1, \infty)$

c) $f(x) = x + \sqrt{\frac{x-1}{x+1}}$

- $x + 1 \neq 0$
  $x \neq -1$
- $\frac{x-1}{x+1} \geq 0 \quad / (x+1)^2$ da ne pride do napak

$(x-1)(x+1) \geq 0$



$x \in (-\infty, -1] \cup [1, \infty)$

- R: $x \in (-\infty, -1) \cup [1, \infty)$

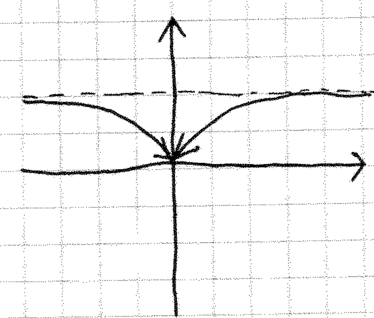
d) DN: $f(x) = \frac{x-2}{2x-1}$ R: $\mathbb{R} - \{1/2\}$

(2) Določi def. območje in Zf funkcij

a) $f(x) = 1 - x^2$ R: DF: $\mathbb{R}$, $Zf = (-\infty, 1]$

b) $f(x) = e^{-1/x^2}$

$x \neq 0$ DF: $\mathbb{R} - \{0\}$



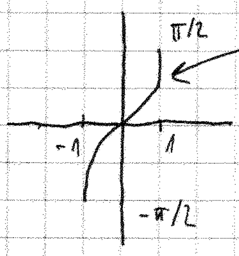
$x \to \infty \Rightarrow -1/x^2 \to 0 \Rightarrow f(x) = 1$

$x \to -\infty \Rightarrow -1/x^2 \to 0 \Rightarrow f(x) = 1$

$x \to 0 \Rightarrow -1/x^2 \to -\infty \Rightarrow e^{-1/\infty} \to 0$

$Zf = (0, 1)$

c) $f(x) = \arcsin \frac{2x}{x+1} \longrightarrow x \neq -1$ prvi pogoj



$D_f = [-1, 1]$

$$-1 \leq \frac{2x}{x+1} \qquad\qquad \frac{2x}{(x+1)} \leq 1 \quad / (x+1)^2$$

$$-(x+1)^2 \leq 2x(x+1) \qquad\qquad 2x(x+1) \leq (x+1)^2$$

$$(x+1)(-x-1-2x) \leq 0 \qquad\qquad 2x(x+1) - (x+1)(x+1) \leq 0$$

$$(x+1)(3x-1) \leq 0 \qquad\qquad (x+1)(2x-x-1) \leq 0$$

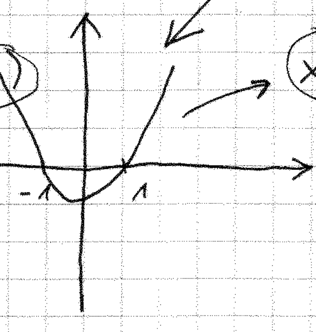
$$(x+1)(x-1) \leq 0$$

$x_1 = -1$

$x_2 = -1/3$

$x \in (-\infty, 1] \cup [-1/3, \infty)$

drugi pogoj



3. pogoj

$x \in [-1, 1]$

• $\boxed{D_f : [-1/3, 1)}$

• Zaloga vrednosti :

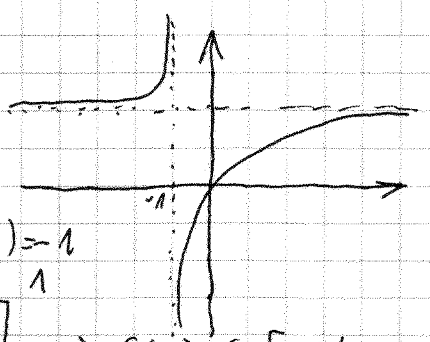
$x = 0$ ničle

poli : $x = 1$

asimptota : 2

$g : g(-1/3) = -1$

$g(1) = 1$



$$x \in \left[-\frac{1}{3}, 1\right] \Rightarrow \frac{2x}{x+1} \in [-1, 1] \Rightarrow f(x) \in [-\pi/2, \pi/2]$$

$\boxed{Z_f = [-\pi/2, \pi/2]}$

3. Določi sodost / lihost naslednjih funkcij

a) DN: $f(x)=\frac{x}{1+x^2}$ R: liha

b) $f(x)=\frac{\sin x}{x^3}$

sinx liha

Primerjaj: $-f(x)$
$--f(x)$

$f(-x)=\frac{\sin(-x)}{(-x)^3}=\frac{-\sin x}{-x^3}=\frac{\sin x}{x^3}$ $=f(x)$

f je soda

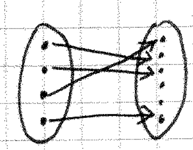
c) $f(x)=x+\sqrt{x^4+x^6}$

$f(-x)=-x+\sqrt{(-x)^4+(-x)^6}=-x+\sqrt{x^4+x^6}$

$\neq f(x)$ ni soda
$\neq -f(x)$ ni liha

ni ne liha ne soda

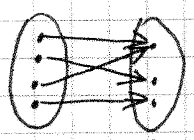
$F: A \to B$



A B

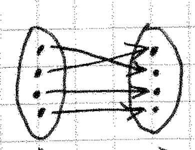
injektivna
dva različna se preslikata v dva različna

vsak element iz B je slika največ enega elt iz A



A B

surjektivna
vsak element iz B je slika vsaj enega elt iz A.



A B

bijektivna
D in K enako močni.

Vsak elt iz B je slika natanko enega elt iz A.

$\underline{\text{injektivnost}}$: $\forall x_1 \neq x_2$ velja $f(x_1) \neq f(x_2)$

$\underline{\text{surjektivnost}}$: $\forall y \in B\ \exists x \in A : y = f(x)$

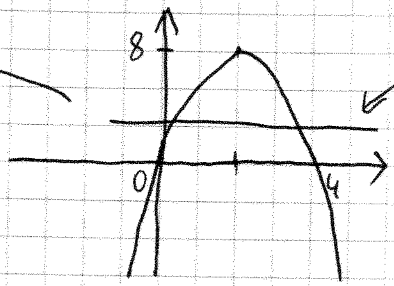
$\underline{\text{bijektivnost}}$: inj. + sur.

(4.) Ugotovi inj., sur. in bij. funkcij. Spremeni funkcijski predpis, tako da bo funkcija inj., sur. oz. bij.

✓ a) DN: $f(x) = e^x$ R: $f: \mathbb{R} \to \mathbb{R}$ inj., ni surj
$f: \mathbb{R} \to \mathbb{R}^+$ bij

✓ b) $f(x) = 8x - 2x^2$.

$Def = \mathbb{R}$
$Zf = (-\infty, 8]$
$= 2x(4-x)$



vsaka funk. vrednost je 2x dosežena, torej ni injektivna

✓ • $f: \mathbb{R} \to \mathbb{R}$: ni injektivna
ni surjektivna: nad 8 ne dobimo vrednosti
surj: $K = Zf$
ker ni inj in surj, tudi ni bijekt.

✓ • $f: \mathbb{R} \to (-\infty, 8]$ in $K = Zf$, torej surj
ni inj, torej tudi ni bij.

✓ • $f: (-\infty, 2] \to \mathbb{R}$ je funkcija injektivna, ni surjektivna in ni bijektivna

✓ • $f: (-\infty, 2] \to (-\infty, 8]$ inj, surj $\Rightarrow$ bij

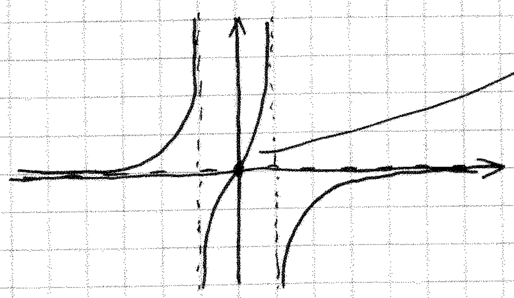
c) $f(x) = \sin x$ DN

d) $f(x) = \dfrac{2x}{1-x^2}$

• Df: $1 - x^2 \neq 0$ $\quad Df = \mathbb{R} - \{-1, 1\}$

$x \neq \pm 1$

• Zf: $\quad Zf = \mathbb{R} \cancel{- \{0\}}$



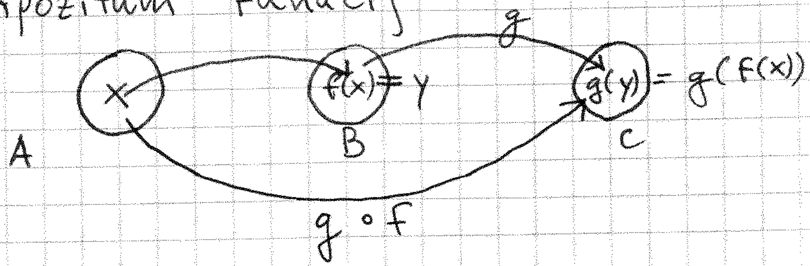
Zf = kodomena

DOMENA

• $f: \mathbb{R} - \{-1, 1\} \rightarrow \mathbb{R}$ ni injektivna, surjektivna ($Zf = \mathbb{R}$)

• $f: (-1, 1) \rightarrow \mathbb{R}$ je injektivna, surjektivna, bijektivna

Kompozitum funkcij



⑤ Dani sta funkciji f in g določi kompozituma ter def obm in zaloge vrednosti.

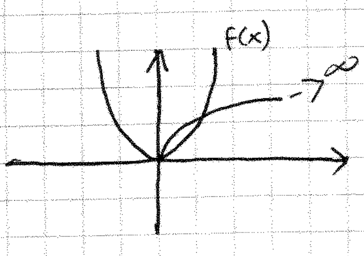
$f(x) = x^2, \quad g(x) = \sqrt{x}$

$(f \circ g)(x)$

$(g \circ f)(x)$

$Df = \mathbb{R} \quad Zf = [0, \infty)$

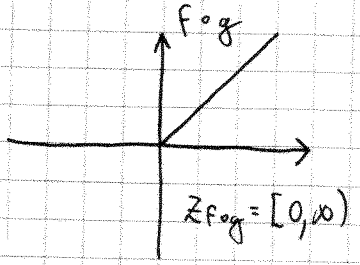
$Dg = [0, \infty) \quad Zg = [0, \infty)$

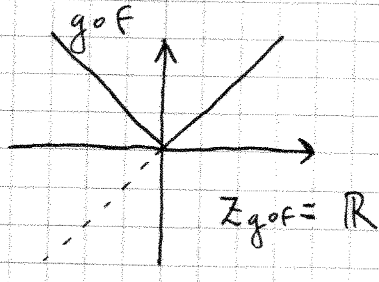


$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$
$g \circ f(x) = g(x^2) = \sqrt{x^2} = |x|$

$D_{f \circ g} = D_g = [0, \infty)$

$D_{g \circ f} = D_f = \mathbb{R}$





√ ⑥ Izračunajte kompozitume $f \circ f$, $f \circ g$, $g \circ f$ in $g \circ g$

√ a) DN $\quad f(x) = x^2 \qquad g(x) = e^x$

R: $f \circ f(x) = x^4$
$f \circ g(x) = e^{2x}$
$g \circ f(x) = e^{x^2}$
$g \circ g(x) = e^{e^x}$

√ c) $f(x) = \dfrac{1}{1+x^2}$

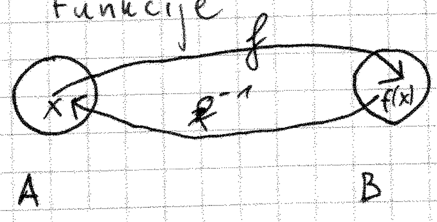
$g(x) = \sqrt{\dfrac{1-x}{x}}$

$f \circ g(x) = \dfrac{1}{1 + \frac{1-x}{x}} = x$

R: $f \circ f(x) = \dfrac{1 + 2x^2 + x^4}{2 + 2x^2 + 4} \qquad g \circ f(x) = |x|$

$g \circ g(x) = \sqrt{\dfrac{2x-1}{\sqrt{x-x^2} + 1 - x}}$

Inverz funkcije



Inverzi ne obstajajo, če funkcija ni injektivna in če ni surjektivna. Fja mora biti bijektivna

$f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$

(7.) Poišči $f^{-1}(x)$ k funkciji: $f(x)$

a) $f(x) = \dfrac{2x+3}{x-2}$ R: $f^{-1}(x) = f(x)$

✓ b) $f(x) = 1 + \operatorname{atg}(3x)$

$y = 1 + \operatorname{atg}(3x)$ ← zamenjamo vlogi x in y

$x = 1 + \operatorname{atg}(3y)$ → izrazimo y, ki je iskana funkcija

$\operatorname{atg}(3y) = x - 1$

$3y = \operatorname{tg}(x-1)$

$y = \operatorname{tg}(x-1)/3$

$f^{-1}(x) = \operatorname{tg}(x-1)/3$

√ c) $f(x) = e^x - e^{-x}$

$x = e^y - e^{-y}$ $\quad t = e^y$

$x = t - t^{-1}$

$x = \dfrac{t^2-1}{t}$

$tx - t^2 - 1 = 0$

$t_{1,2} = \dfrac{x \pm \sqrt{x^2 - 4ac}}{2a} = \dfrac{x \pm \sqrt{x^2+4}}{2}$

$t_1 = \dfrac{x + \sqrt{x^2+4}}{2}$ ✓

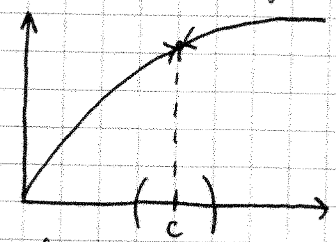
$\cancel{t_2 = \dfrac{x - \sqrt{x^2+4}}{2}}$ $> x^2$, e^y samo na poz.

74

$$e^y = \frac{x+\sqrt{x^2+4}}{2}$$

$$f^{-1}(x) = \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)$$

Limita funkcije



f definirana na (a,b), razen morda v c ($c \in (a,b)$).

$\lim_{x\to c} f(x) = L$, če za vsak $\varepsilon > 0 \ \exists \ \delta > 0$, tako

da

$|f(x) - L| < \varepsilon$ za $|x-c| < \delta$

leva limita: $\lim_{x\uparrow c} f(x)$ desna limita $\lim_{x\downarrow c} f(x)$

- $\lim_{x\to c} (f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- $\lim_{x\to c} (f(x) g(x)) = \lim_{x\to c}(f(x)) \lim_{x\to c}(g(x))$
- $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim(f(x))}{\lim(g(x))}$ samo, če $g(x) \neq 0 \ \forall x$
- $\lim_{x\to c} A f(x) = A \cdot \lim_{x\to c} f(x)$

Velja:

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

- $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = 0$ ← omejena

- $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^{x} = e$

- $\lim\limits_{x\to \infty} \left(1 - \dfrac{1}{x}\right)^{-x} = e$

- $\lim\limits_{x\to \infty} P(x)e^{-x} = 0$

  ↑ polinom

  e^x je od nekje večji od vsakega polinoma

(8) Izračunaj naslednje limite:

a) DN $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$ R: 3

√ b) $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx} = \lim\limits_{x\to 0} \dfrac{\overbrace{\dfrac{\sin ax}{ax}}^{\to 1} \cdot a\cancel{x}}{\underbrace{\dfrac{\sin bx}{bx}}_{\to 1} \cdot b\cancel{x}} = \dfrac{a}{b}$

$\dfrac{0}{0}$ nedoločeno

√ c) $\lim\limits_{x\to \infty} \left(\dfrac{x^2+1}{x^2-1}\right)^{x^2} = \lim\limits_{x\to \infty} \left(1 + \dfrac{2}{x^2-1}\right)^{x^2} = \lim\limits_{x\to \infty} \left(1 + \dfrac{2}{x^2-1}\right)^{x^2 \cdot \frac{x^2-1}{2} \cdot \frac{2}{x^2-1}} =$

$= e^{\lim\limits_{x\to\infty} \frac{2x^2}{x^2-1} \, /:x^2} = e^2$

3.12.07

d) $$\lim_{x\to 1}\frac{x^m-1}{x^n-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x^{m-1}+x^{m-2}+\ldots+x+1)}{\cancel{(x-1)}(x^{n-1}+x^{n-2}+\ldots+x+1)} = \frac{m}{n}$$

$\frac{0}{0}$ nedoločeno

m enic toliko členov
n enic

1. Izračunaj limite

a) $$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to 4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{x-4} =$$

$\to 0/0$ nedoločeno $\to 0/0$

$$= \lim_{x\to 4}\frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \lim\frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} =$$

$$= \lim_{x\to 4}\frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{8}{6} = \boxed{\frac{4}{3}}$$

b) $$\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}} =$$ potence $x^1, x^{2/3}, x^1, x^{4/5}$

$/ :x$
$:x$

$$= \lim_{x\to\infty}\frac{\sqrt{\frac{x^2+1}{x^2}}-\sqrt[3]{\frac{x^2+1}{x^3}}}{\sqrt[4]{\frac{x^4+1}{x^4}}-\sqrt[5]{\frac{x^4+1}{x^5}}} = \lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt[3]{\frac{1}{x}+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}}-\sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}} =$$

$\to 0$ $\to 0$

$= 1$

$(a+b)(a-b)=a^2-b^2$
$(a-b)(a^2+ab+b^2)=a^3-b^3$
$(a+b)(a^2-ab+b^2)=a^3+b^3$

c) $$\lim_{x\to\infty}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right) =$$

a b

$\infty - \infty$ nedoločeno

$$= \lim_{x\to\infty}\frac{\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)\left(\sqrt[3]{(x+1)^4}+\sqrt[3]{(x+1)^2(x-1)^2}+\sqrt[3]{(x-1)^4}\right)}{\left(\sqrt[3]{(x+1)^4}+\sqrt[3]{(x+1)^2(x-1)^2}+\sqrt[3]{(x-1)^4}\right)} =$$

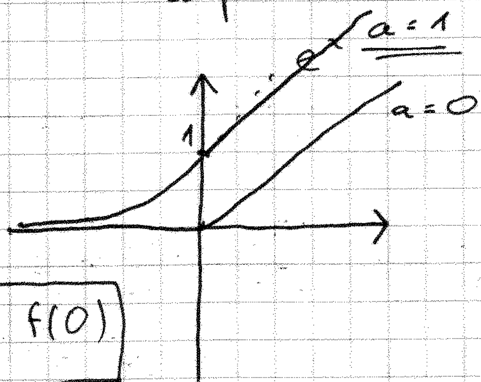
$$= \lim_{x\to\infty} \frac{(x+1)^2-(x-1)^2}{\sqrt[3]{(x+1)^4}+\sqrt[3]{(x+1)^2(x-1)^2}+\sqrt[3]{(x-1)^4}} = \lim_{x\to\infty} \frac{4x}{\sqrt[3]{(x+1)^4}+\sqrt[3]{(x+1)^2(x-1)^2}+\sqrt[3]{(x+1)^4}} \quad \infty/\infty$$

$$= x^1,\ x^{4/3},\ x^{4/3},\ x^{4/3} \Big/ \frac{x^{4/3}}{x^{4/3}}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{x^{1/3}} \to 0}{\sqrt[3]{\left(\frac{x+1}{x}\right)^4}+\sqrt[3]{\frac{(x-1)^2(x+1)^2}{x^2\ \cancel{x^4}\ x^2}}+\sqrt[3]{\left(\frac{x-1}{x}\right)^4}} = \frac{0}{3} = \boxed{0}$$

(2) Določi parameter a, tako da bo podana funkcija zvezna

a) $f(x) = \begin{cases} e^x; & x<0 \\ a+x, & x \geq 0 \end{cases}$



$a=1$

$$\boxed{\text{Zvezna} \quad \lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x) = f(0)}$$

$$\lim_{x\uparrow 0} f(x) = \lim_{x\uparrow 0} e^x = 1$$

$$\lim_{x\downarrow 0} f(x) = \lim_{x\downarrow 0}(a+x) = a$$

$f(0) = a \quad \Rightarrow a = 1$

DN b) $f(x) = \begin{cases} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}}; & x \neq 0 \\ a; & x=0 \end{cases}$ $\leftarrow \boxed{x - \sin 1/x} \rightarrow$ du2

R: $a=0$

3) V točkah, kjer funkcija ni zvezna, določi levo in desno limito:

DN:

$$F(x) = \begin{cases} 2\sqrt{x}; & 0 \leq x < 1 \\ 4-2x; & 1 \leq x \leq 5/2 \\ 2x-7; & 5/2 \leq x \leq 4 \end{cases}$$

V stičiščih

R: zvezna v 1, ni zvezna v 5/2

4) Določi funkcijo $f(x)$:

a) $f(1+x) = 2 - 3x + x^2 =$

$= (x+1)^2 - 5x + 1$ ✓ $= (x+1)^2 - 5(x+1) + 6 =$

$\underbrace{(x+1)^2}_{x^2+2x+1}$, $\underbrace{5(x+1)}_{-5x-5}$

$\Rightarrow \boxed{f(x) = x^2 - 5x + 6}$

! b) $f\left(\frac{x}{1+x}\right) = x^2$

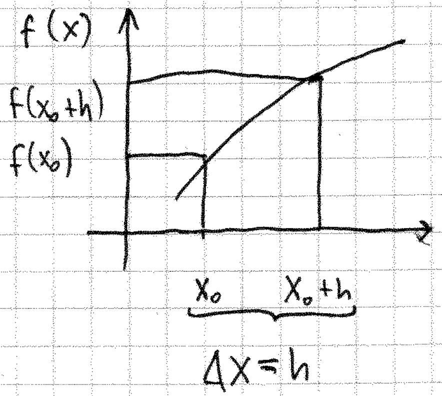
nova spremenljivka; $y = x/(x+1)$ $/\cdot(1+x)$

$y(1+x) - x = 0$
$x(y-1) = -y$
$x = -y/(y-1) = \frac{y}{1-y}$

$f(y) = \left(\frac{y}{1-y}\right)^2 =$

$= \frac{y^2}{1-2y+y^2}$

ODVOD



$$\Delta y = f(x_0+h) - f(x_0)$$

$$k = \Delta y / \Delta x = \frac{f(x_0+h) - f(x_0)}{h} =$$ smerni koef oz naklonski kot sekante (x_0) in (x_0+h)

tangenta v točki:

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} =$$ smerni koef tangente v x_0, to je odvod v x_0

Pravila za odvajanje

$$(A \cdot f(x))' = A \cdot f'(x)$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Tabela elementarnih funkcij

$c' = 0$

$(e^x)' = e^x$

$(x^n)' = n x^{n-1}$

$(a^x)' = a^x \ln a$

$(\ln x)' = 1/x$

$(\sin x)' = \cos x$

$(\log_a x)' = \dfrac{1}{x \ln a}$

$(\cos x)' = -\sin x$

$tg(x)' = \dfrac{1}{\cos^2 x} = 1 + tg^2 x$

$(ctg\, x)' = -\dfrac{1}{\sin^2 x} = -1 - ctg^2 x$

$(atg\, x)' = \dfrac{1}{1+x^2}$

$(actg\, x)' = \dfrac{-1}{1+x^2}$

$(a\sin x)' = \dfrac{1}{\sqrt{1-x^2}}$

$(a\cos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

⑤ Odvajaj funkcije

√ a) $f(x) = x\sqrt{1+x^2} = x \cdot \underbrace{(1+x^2)^{1/2}}$

$f'(x) = 1 \cdot (1+x^2)^{1/2} + x \cdot 1/2\,(1+x^2)^{-1/2} \cdot (1+x^2)' =$

$= (1+x^2)^{1/2} + x \cdot \cancel{1/2}\,(1+x^2)^{-1/2} \cdot \cancel{2}x =$

$= \sqrt{1+x^2} + \dfrac{x^2}{\sqrt{1+x^2}} = \boxed{\dfrac{1+2x^2}{\sqrt{1+x^2}}}$

√ b) $f(x) = \dfrac{x^2-5x-1}{x^3} = \dfrac{(2x-5)x^3 - 3x^2(x^2-5x-1)}{x^6} =$

$= \dfrac{2x^4 - 5x^3 - 3x^4 + 15x^3 + 3x^2}{x^6} = \boxed{\dfrac{-x^2+10x+3}{x^4}}$

c) $f(x) = \arccos\frac{1}{x} \neq \left(-\frac{1}{\sqrt{1-\frac{1}{x^2}}}\right)\cdot(-1x^{-2}) =$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}} = \frac{1\,\sqrt{x^2}}{x^2\sqrt{x^2-1}} = \frac{|x|}{x^2\sqrt{x^2-1}} =$$

$$= \boxed{\frac{1}{|x|\sqrt{x^2-1}}}$$

d) $f(x) = \ln^3(x^2) = (\ln(x^2))^3 = (2\ln x)^3$

$$f'(x) = \frac{6\ln^2(x^2)}{x} = \boxed{\frac{24}{x}\ln^2(x)}$$

e) $f(x) = \sqrt{\frac{x^2-1}{x^2+1}} = \left(\frac{x^2-1}{x^2+1}\right)^{1/2}$

$$f'(x) = \frac{1}{2}\left(\frac{(x^2-1)}{(x^2+1)}\right)^{-1/2}\cdot\frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} =$$

$$= \frac{1}{2}\left(\frac{(x^2+1)}{(x^2-1)}\right)^{1/2}\cdot\frac{2x(\cancel{x^2}+1-\cancel{x^2}+1)}{(x^2+1)^2} =$$

$$= \frac{1}{2}\sqrt{\frac{(x^2+1)}{(x^2-1)}}\cdot\frac{4x}{(x^2+1)^2} = \frac{1}{2}\sqrt{\frac{1}{(x^2-1)}}\;\frac{4x}{(x^2+1)^{3/2}} =$$

$$= \boxed{\frac{2x}{(x^2+1)\sqrt{x^4-1}}}$$

f) $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

$-(\sin x - \cos x)^2$

$$f'(x) = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\cos x + \sin x)^2\cancel{(\sin x + \cos x)}}{(\sin x - \cos x)^2} =$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} =$$

$$= \frac{-\sin^2 x + \cancel{2\sin x\cos x} - \cos^2 x - \sin^2 x - \cancel{2\sin x\cos x} - \cos^2 x}{\sin^2 x - 2\sin x\cos x + \cos^2 x}$$

$$= \frac{-2\cancel{(\sin^2 x + \cos^2 x)}\,1}{1 - 2\sin x\cos x} = \frac{-2}{1-2\sin x\cos x} = \boxed{\frac{-2}{1-\sin 2x}}$$

sinus dvojnega kota

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$

g) $f(x) = \ln|x| = \begin{cases} \ln x; & x \geq 0 \\ \ln(-x); & x < 0 \end{cases}$

$$f'(x) = \begin{cases} 1/x \\ \frac{1}{-x}\cdot(-1); & x<0 \end{cases} = \boxed{\frac{1}{x}}$$

h) $f(x) = \ln(\ln^2(\ln^3(x)))$

$$f'(x) = \frac{1}{\ln^2(\ln^3(x))}\cdot 2\cancel{\ln(\ln^3(x))}\cdot\frac{1}{\ln^3(x)}\cdot 3\cancel{\ln^2(x)}\cdot\frac{1}{x}$$

$$= \frac{2}{\ln(\ln^3(x))}\cdot\frac{3}{\ln(x)}\cdot\frac{1}{x} = \boxed{\frac{6}{\ln(\ln^3(x))\cdot\ln(x)\cdot x}}$$

lažje: $2\ln(3\ln(x))$ poenostavimo

i) $f(x) = \ln\sqrt{\frac{e^{2x}}{1+e^{2x}}} = \frac{1}{2}\ln\left(\frac{e^{2x}}{1+e^{2x}}\right) =$

$$f'(x) = \frac{1}{2}\cdot\frac{1}{\left(\frac{e^{2x}}{1+e^{2x}}\right)}\cdot\frac{e^{2x}\cdot 2(1+e^{2x}) - e^{2x}e^{2x}\cdot 2}{(1+e^{2x})^2} =$$

$$= \frac{1}{2}\cdot\frac{1+e^{2x}}{e^{2x}}\cdot\frac{2e^{2x}(1+\cancel{e^{2x}}-\cancel{e^{2x}})}{1+2e^{2x}+e^{4x}} =$$

$$= \frac{1}{2}\cdot\frac{\cancel{1+e^{2x}}}{\cancel{e^{2x}}}\cdot\frac{2\cancel{e^{2x}}}{\cancel{1+e^{2x}+e^{4x}}\;(e^{2x}+1)^{\cancel{2}}} =$$

$$= \frac{1\cancel{2}}{\cancel{2}\;\dots} = \boxed{\frac{1}{e^{2x}+1}}$$

j) $f(x) = x^{\frac{1}{x}} =$

- x^n — x konst potenca
- a^x — x konst osnova

$\Rightarrow$! $\boxed{x = e^{\ln x}}$

$= e^{\ln x^{\frac{1}{x}}} = e^{\frac{1}{x} \cdot \ln x}$

$f'(x) = \underbrace{e^{\frac{1}{x}\ln x}}_{x^{1/x}} \cdot \left(-\frac{1}{x^2} \cdot \ln x + \frac{1}{x} \cdot \frac{1}{x}\right) =$

$= \frac{1}{x^2} \cdot x^{1/x} (1 - \ln x) = \boxed{x^{1/x - 2}(1 - \ln x)}$

(7) Dana je funkcija $f(x) = \frac{1}{x+2} + \frac{1}{x^2+1}$. Določi odvod v $f'(0)$ in $f'(-1)$

R: $f'(0) = -1/4$, $f'(-1) = 1/2$

(8) Odvajaj implicitno podane funkcije

a) DN $x + y = x^2 + x^3$ R: $y' = 2x + 3x^2 - 1$

b) $\ln(x^2 + y^2) = \mathrm{atg}\,\frac{y}{x}$ $/ ()'$

$\frac{1}{x^2 + y^2} \cdot (x^2 + y^2)' = \frac{1}{1 + (y/x)^2} \cdot \left(\frac{y}{x}\right)'$

$\frac{1}{x^2 + y^2}(2x + 2yy') = \frac{\cancel{x^2}}{x^2 + y^2} \cdot \frac{y' \cdot x - y}{\cancel{x^2}}$ $/(x^2 + y^2)$

$2x + 2yy' = y'x - y \Rightarrow 2yy' - y'x = -y - 2x$

$y' = \frac{-y - 2x}{2y - x} = \boxed{\frac{y + 2x}{x - 2y}}$

⑨ Pokaži, da je razlika funkcij odsekoma konstantna in jo določi

$f(x) = \operatorname{atg}(1/x)$

$g(x) = \operatorname{atg} \frac{x+1}{-1+x}$

$f(x) - g(x) = C \iff (f(x) - g(x))' = 0 \iff f'(x) = g'(x)$

$f'(x) = \frac{1}{1+(1/x)^2} \cdot \left(\frac{1}{x}\right)' = \frac{\cancel{x^2}}{1+x^2} \cdot \left(-\frac{1}{\cancel{x^2}}\right) = \boxed{-\frac{1}{1+x^2}}$

$g'(x) = \frac{1}{1+\left(\frac{x+1}{-1+x}\right)^2} \cdot \left(\frac{x+1}{-1+x}\right)' = \frac{(-1+x)^2}{(-1+x)^2+(1+x)^2} \cdot \frac{(-1+x)-(x+1)}{(-1+x)^2} =$

$= \frac{-2}{(-1+x)^2+(x+1)^2} = \frac{-2}{2x^2+2} = \boxed{-\frac{1}{1+x^2}}$

$f'(x) = g'(x) \Rightarrow f(x) - g(x)$ je odsekoma konstantna

določi razliko: vstavimo eno točko

$\left.\begin{array}{l} f(-1) = \operatorname{atg}(-1) = -\operatorname{atg}(1) = -\pi/4 \\ g(-1) = \operatorname{atg}(0) = 0 \end{array}\right\}$ $f(x) - g(x) = f(-1) - g(-1)$ ker je konstantno

razlika $-\pi/4$

L'Hospitalovo pravilo

Če računamo limite nedoločenih izrazov oblike

$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}$, tedaj velja

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

(10) Z uporabo L'Hospitalovega pravila izračunaj limite.

a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \left(\frac{0}{0}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to 0} \frac{\sin x}{2x} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\cos x}{2} =$

$= \boxed{\frac{1}{2}}$

b) $\lim_{x\to 0} \frac{2\arcsin x}{3x} = \lim_{x\to 0} \frac{2\left(-\frac{1}{\sqrt{1-x^2}}\right)}{3} = \boxed{\frac{2}{3}}$

c) DN: $\lim_{x\to 0} \frac{1-\cos x}{x}$ R: 0

d) DN: $\lim_{x\to 1} \frac{x^m - 1}{x^n - 1}$ R: $\frac{m}{n}$

(11) Poišči drugi odvod funkcije

a) DN: $f(x) = x\sqrt{1+x^2}$ R: $f''(x) = \frac{2x^3+3x}{(1+x^2)^{3/2}}$

b) $f(x) = e^{\sin x} \cdot \cos x$

$f'(x) = e^{\sin x} \cdot \cos^2 x \cancel{+} e^{\sin x} \sin x$

$f'(x) = e^{\sin x}(\cos^2 x - \sin x)$

$f''(x) = e^{\sin x} \cdot \cos x(\cos^2 x - \sin x) + e^{\sin x}(2\cos x(-\sin x) - \cos x) =$

$= e^{\sin x}(\cos^3 x - \sin x\cos x \cancel{+} - \cos x - 2\cos x \sin x) =$

$= e^{\sin x}\cos x(\cos^2 x - 3\sin x - 1) =$

$= e^{\sin x}\cos x(-3\sin x - \sin^2 x) = \boxed{e^{\sin x}\cos x \sin x(-3-\sin x)}$

10. 12. 2007

2. kolokvij 7. 1. 2008, 19.00

2. DN: petek

① Poišči drugi odvod funkcije

a) DN: $f(x) = x\sqrt{1+x^2}$

b) $F(x) = e^{\sin x} \cos x$

$$f'(x) = e^{\sin x} \cos x \cos x + e^{\sin x}(-\sin x)$$
$$f'(x) = e^{\sin x}(\cos^2 x - \sin x)$$

$$f''(x) = e^{\sin x}\cos x(\cos^2 x - \sin x) + e^{\sin x}(2\cos x(-\sin x) - \cos x) =$$
$$= e^{\sin x}(\cos^3 x - \sin x \cos x - 2\sin x \cos x - \cos x) =$$
$$= e^{\sin x}(\cos x \sin x(-3 - \sin x))$$

c) DN: $f''(x) = (-2/x)\sin(\ln x)$

$$f'(x) = x(\sin(\ln x) + \cos(\ln x))$$

d) $x = \ln(1+y)$

$$e^x = 1 + y$$
$$y = e^x - 1$$
$$y' = e^x$$
$$y'' = e^x$$

2) Poišči n-ti odvod funkcije $f(x) = e^{-3x}$

$$f'(x) = e^{-3x} \cdot (-3)$$
$$f''(x) = e^{-3x} \cdot (-3)^2$$
$$f'''(x) = e^{-3x} \cdot (-3)^3$$
$$f^{(n)}(x) = e^{-3x} \cdot (-3)^n$$

DN: n-ti odvod $F(x) = \frac{1}{x^2-1}$ tako, da jo zapišeš s parcialnimi ulomki

R: $f^{(n)}(x) = \frac{1}{2}(-1)^n \cdot n! \cdot \left(\frac{1}{(x-1)^{(n+1)}} - \frac{1}{(x+1)^{(n+1)}}\right)$

④ Določi a in b, da bo $f(x)$ zvezno odvedljiva

$$f(x)=\begin{cases} e^x+x^2+1 \ ; & x<0 \\ ax+b \ ; & x\geq 0 \end{cases}$$

Kar je v odvodu je še zmeraj odvedljivo.

$\rightarrow f(0)=b$

• zveznost :

$$\lim_{x\uparrow 0} f(x) = \lim_{x\uparrow 0} e^x+x^2+1 = \lim_{x\uparrow 0} e^0+0+1 = \underline{\underline{2}}$$

$\rightarrow \boxed{b=2}$

$$f'(x)=\begin{cases} e^x+2x & x<0 \\ a & x\geq 0 \end{cases}$$

$$\left.\begin{array}{l} f'(0)=a \\ \lim\limits_{x\uparrow 0} e^x+2x = 1 \end{array}\right] \rightarrow \boxed{a=1}$$

Monotonost funkcij

$f(x)$ narašča, če je $f'(x)>0$, pada, če $f'(x)<0$.

Pri $f'(x)=0$ je to stacionarna točka funkcije $f(x)$.

⑤ Določi intervale, kjer je funkcija $f(x)$ naraščajoča in intervale, kjer je padajoča

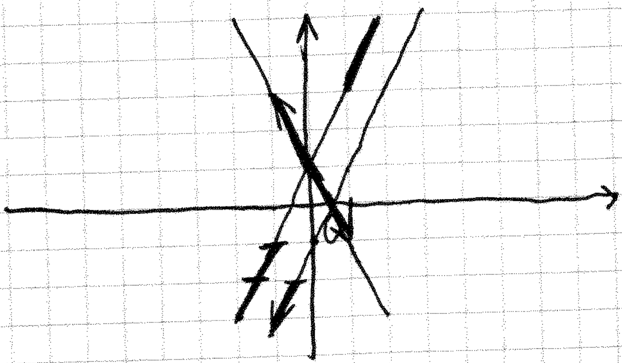
a) $f(x)= 2x/(1+x^2)$

R: narašča $x\in[-1,1]$
pada $x\in[-\infty,-1]\cup[1,\infty)$

b) $f(x)=|x^2-1|+|x+2| = \ldots\ldots$ pogoji

$$=\begin{cases} x^2+x+1 \ ; & -2\leq x\leq -1 \quad \text{ali} \quad 1\leq x<\infty \\ x^2-x-3 \ ; & \infty<x<-2 \\ -x^2+x+3 \ ; & -1<x<1 \end{cases}$$

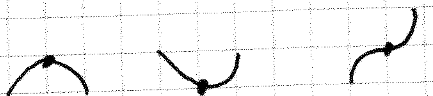
$$f'(x) = \begin{cases} 2x+1; & -2 \leq x \leq -1 \text{ ali } 1 \leq x < \infty \\ 2x-1; & -\infty < x < -2 \\ -2x+1; & -1 < x < 1 \end{cases}$$



$f'(x) > 0$, ko je $x \in (-1, 1/2) \cup [1, \infty)$

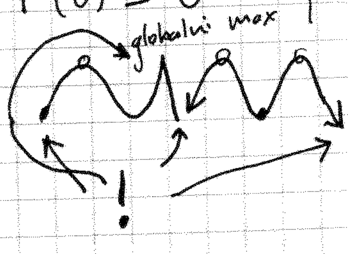
$f'(x) < 0$, $x \in (-\infty, -1] \cup (1/2, 1)$

Ekstremi



$f'(a) = 0$ ..... a je stacionarna točka in je kandidat za ekstrem

$f''(a) > 0$ lokalni minimum
$f''(a) < 0$ lokalni maksimum
$f''(0) = 0$ "prevoj"



⑥ Določi ekstreme funkcij

a) DN: $f(x) = 8x - 2x^2$ R: $x = 2$ lok. max.

b) DN: $f(x) = x \ln x$ R: $x = e^{-1}$ lok. min.

c) $f(x) = x^2 \cdot e^{-x}$

$f'(x) = 2x e^{-x} - x^2 e^{-x} = 0$

$x e^{-x}(2-x)$

$x_1 = 0$
$x_2 = 2$ stacionarni točki

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x} =$

$= e^{-x}(2 - 4x + x^2)$

$f''(0) = ?, \quad f''(2) = ?$

$f''(0) = 2 > 0 \quad f''(2) = e^{-2}(-2) < 0$

$\Downarrow \qquad \Downarrow$

V $x_1 = 0$ lok. min   V $x_2 = 2$ je lok. max

⑦ Določi točke, kjer ~~k~~ funkcija $f(x)$ doseže največjo oz. najmanjšo vrednost na intervalu I.

a) $f(x) = \dfrac{x^2 - 3x}{x + 1}$ $I = [0, 4]$

- ekstremi
- točke neodvedljivosti
- točke nezveznosti
- krajišča intervala

} kandidati za globalni min in max

• ekstremi

$f'(x) = \dfrac{(2x-3)(x+1) - (x^2-3)(1)}{(x+1)^2} = \boxed{\dfrac{x^2+2x-3}{(x+1)^2} = 0}$

$\rightarrow x^2 + 2x - 3 = 0$
$(x-1)(x+3) \rightarrow x_1 = 1, \cancel{x_2 = -3}$ ni v intervalu

$\boxed{F(1) = -1}$

• točke nezveznosti

~~x = -1~~ ni v I

• točke neodvedljivosti: $x=-1$, ni v I ✗

• krajišča:

$$\boxed{\begin{aligned} f(0) &= 0 \\ f(4) &= 4/5 \end{aligned}}$$

maksimum pri $x=4$, $f(4)=4/5$
min $x=1$, $f(1)=-1$
} globalni ekstremi.

b) $f(x) = 2\,tg\,x - tg^2 x \qquad I = [0, \pi/2)$

$$f'(x) = 2 \cdot \frac{1}{\cos^2 x} - 2\cdot tg\,x \cdot \frac{1}{\cos^2 x}$$

$$= \frac{2}{\cos^2 x} - \frac{2\sin}{\cos^3 x} = \frac{2}{\cos^2 x}\left(1 - tg\,x\right)$$

$tg\,x = 1$, $x = \pi/4 + k\pi \rightarrow \boxed{x_1 = \pi/4}$

$\boxed{f(\pi/4) = 1}$

točke nezveznosti: $\pi/2 + k\pi$ ni v I.

točke neodvedljivosti: $\cos^2 x = 0$, $x = \pi/2 + k\pi$
ni v int.

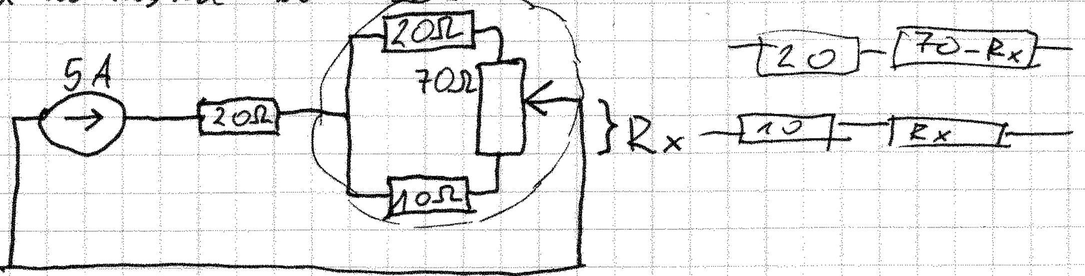
krajišča: $f(0) = 0$

$$\lim_{x\uparrow \pi/2} f(x) = \lim_{x \uparrow \frac{\pi}{2}} (2\,tg\,x - tg^2 x) = +\infty - \infty = \text{nedoločeno}$$

$$= \lim_{x\uparrow \pi/2} tg\,x\,(2 - tg\,x) = \infty(-\infty) = \underline{\underline{-\infty}}$$

globalni maksimum $x=\pi/4$ $f(\pi/4) = 1$
globalni minimum ne obstaja

8) V kateri legi drsnika oz. kateri vrednosti R_x spodnjega dela drsnega upora, bo el. moč v vseh uporih skupaj maksimalna in kolikšna bo moč.



$$R_{nad} = 20 + R$$

$$\frac{1}{R} = \frac{1}{20+70-R_x} + \frac{1}{10+R_x} = \frac{10+R_x+90-R_x}{(90-R_x)(10+R_x)} = \frac{100}{(90-R_x)(10+R_x)}$$

$$R_n = 20 + \frac{1}{100}(90-R_x)(10+R_x) =$$

$$= \frac{1}{100}\left(2900 + 80R_x - R_x^2\right)$$

$$R_{nad}'(R_x) = \frac{1}{100}\left(80 - 2R_x\right) \Rightarrow \boxed{R_x = 40}\,\Omega$$

$$R_{nad}''(R_x) = \frac{1}{100}(-2) = -1/50 < 0, \text{ lok. max}$$

$$P_{max} = I^2 \cdot R_{nad} = (25A)^2 \cdot 45\Omega = \underline{1125W}.$$

9) DN: Kolikšna mora biti naravno število n, da bo funkcija

$$f(x) = \begin{cases} x^n \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

a) zvezna v $x=0$
b) odvedljiva v $x=0$
c) zvezno odvedljiva v $x=0$

R: a) $n \geq 1$
b) $n \geq 2$
c) $n \geq 3$

Tangente in normale

- smerni koeficient tangente v x_0

$k_t = f'(x_0)$

- normala

$k_n = -1/k_t = -1/f'(x_0)$

(10) Zapiši enačbi tangente in normale na krivuljo v točki T

- a)! $y = x^{\cos x}$, $x_0 = \pi$ DN

$\boxed{e^{\ln x^{\cos x}} = e^{\cos x \ln x}}$

R: tang. $y = -1/\pi^2 x + \frac{2}{\pi}$

norm. $y = \pi^2 x - \pi^3 + 1/\pi$

b) DN: $y^2 - y - 6x = 0$, $T(1, -3)$

c) $y = x^3 + 2x^2 - 4x - 3$ $T(-2, 5)$

$y' = 3x^2 + 4x - 4$

$k_t = 12 - 8 - 4 = 0 \rightarrow y = 5$

$k_n = -1/k_t = \infty$ $x = -2$

d) $y = e^{1-x^2}$ T = presečišče s premico $y = 1$

$1 = e^{1-x^2} \rightarrow x = 1, -1$ $(1 - x^2 = \ln 1)$

$T_1(1, 1)$ $T_2(-1, 1)$

$y' = e^{1-x^2} \cdot (-2x)$: T_1: $k_{t_1} = -2$

T_2: $k_{t_2} = 2$

$y = kx + n$ $1 = -2 + n$ $n = 3 \rightarrow$ $y = -2x + 3$

$1 = -2 + n$ $n = 3 \rightarrow$ $y = 2x + 3$

$k_{n_1} = -\frac{1}{k_{t_1}} = \boxed{1/2}$

$k_{n_2} = \boxed{-1/2}$

DN: (11) Poišči tisto točko na intervalu od [0, 3], v kateri je tangenta na krivuljo $f(x) = x^2$ vzporedna premici skozi točko

$A(0, f(0))$ in $B(3, f(3))$.

R: $T(3/2, 9/4)$

(12) Določi kot pod katerim se sekata krivulji $y_1 = \sin x$ in $y_2 = \sin 2x$ v izhodišču.

$$y'_1 = \cos x$$
$$y'_2 = 2\cos 2x$$

$T(0, 0)$

$$k_1 = y_1' = \cos 0 = 1$$
$$k_2 = y_2' = \cos 0 \cdot 2 = 2$$

$$tg\,\varphi = \frac{|k_2 - k_1|}{1 + k_1 k_2}$$

$$tg\,\varphi = \frac{1}{3} \rightarrow \varphi = 18^\circ 26'$$

(13) Razstavi število 36 na produkt dveh faktorjev, tako da bo vsota njunih kvadratov najmanjša

$$\left.\begin{array}{l} 36 = xy \\ x^2 + y^2 \quad \min \end{array}\right\} \text{nima maksimuma}$$

$$36 = M \cdot \frac{1}{M}, \quad M^2 + \frac{1}{M^2} >> \ldots$$

$$y = 36/x$$

$$f(x) = x^2 + \left(\frac{36}{x}\right)^2 = x^2 + 36x^{-2}$$

$$f'(x) = 2x + 36^2(-2x^{-3}) = 2x - \frac{2 \cdot 36^2}{x^3} = 0$$

$$2x^4 - 2 \cdot 36^2 = 0 \quad /:2$$

$$x^4 - 36^2 = 0$$
$$(x^2 - 36)(x^2 + 36) = 0$$

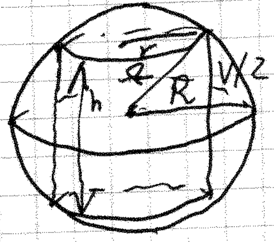
$$(x+6)(x-6)\underbrace{(x^2+36)}_{\text{kompleksni}} = 0$$

$\boxed{\begin{aligned} x_1 &= 6 \\ x_2 &= -6 \end{aligned}}$ stacionarni točki

$36 = 6 \cdot 6 = (-6)\cdot(-6) \Rightarrow$ v x_1 in x_2 ima $f(x)$ min!

Razcep

(14) Krogli z radijem R včrtaj valj z največjim volumnom. Določi radij r valja.



$$R^2 = r^2 + \frac{v^2}{4} \rightarrow r^2 = R^2 - \frac{v^2}{4}$$

$$V = \pi r^2 v \quad \text{max.}$$

$$V(v) = \pi\left(R^2 - v^2/4\right) v = \pi R^2 v - \frac{\pi}{4} v^3$$

$$V'(v) = \pi R^2 - \frac{\pi}{4} 3v^2 \rightarrow \boxed{=0}$$

$$v^2 = \frac{4R^2}{3} \qquad v = (+)\frac{2R}{\sqrt{3}} = \boxed{\frac{2R}{\sqrt{3}}} \quad \text{maksimum}$$

$$r^2 = R^2 - \frac{v^2}{4} = R^2 - \frac{R^2}{3} = 2R^2/3$$

$$r = \sqrt{2/3} \cdot R$$

17.12.07

Taylorjeva vrsta

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 +$$

$$+ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$ napaka

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ T.V. okrog točke a

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

najboljši približek (polinomski) n-tega reda

Približek z diferencialom

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) \quad dF|_a$$

① Razvij funkcijo e^x v Taylorjevo vrsto okrog točke $a=0$! Izračunaj približek $e^{-0.03}$ z diferencialom.

$f(x) = e^x \qquad a=0$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$f'(x) = f''(x) = e^x$

$f'(0) = f''(0) = \ldots = f^{(n)}(0) = 1$

$$f(x) = e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \ldots =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$ To je taylorjeva vrsta funkcije e^x okrog $a=0$

$e^x = f(x) \approx 1 + x$ približek z diferencialom

$e^{-0,03} \approx 1 - 0,03 = 0,97$

GRAFI FUNKCIJ

(1) Nariši grafe polinomov

D.N.

a) $f(x) = 3 - |x| = \begin{cases} 3 - x \\ 3 + x \end{cases}$

b) $f(x) = x^4 - x^3 - 2x^2$

ničle, ekstremi, predznak pri polinomih

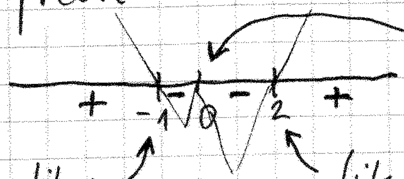
• ničle

$f(x) = 0$

$x^4 - x^3 - 2x^2 = x^2(x^2 - x - 2) = x^2(x-2)(x+1)$

$x_1 = 0$ ničla druge stopnje
$x_2 = 2$ prve st.
$x_3 = -1$ prve st.

• predznak



soda
poiščemo en predznak

• ekstremi

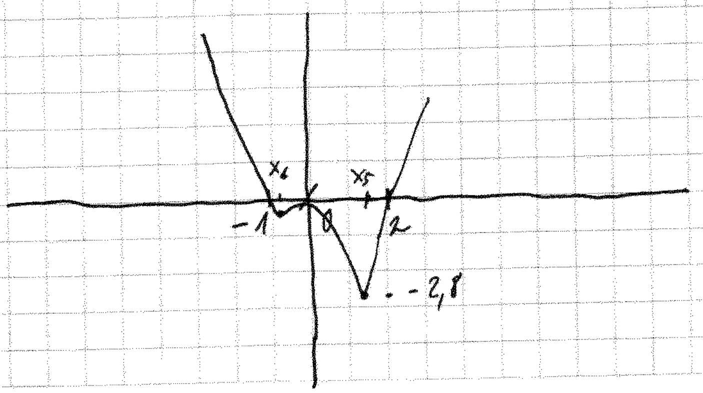
$f'(x) = 0$
$f'(x) = 4x^3 - 3x^2 - 4x = x(4x^2 - 3x - 4)$

$x_4 = 0$
$x_5 = 1{,}4$
$x_6 = -0{,}7$

$\frac{-b \pm \sqrt{D}}{2a} = \frac{3 \pm \sqrt{9+64}}{8}$

globini grafa: $f(x_5) \doteq -2{,}8$
$f(x_6) \doteq -0{,}4$

$\max(x_4) = 0$



3. Nariši grafe racionalnih funkcij:

a) $f(x) = \frac{2x}{1-x^2}$ ničle, poli, asimptota, ekstremi, začetna vrednost, predznak

- ničle: $2x = 0$, $x_1 = 0$ prve stopnje
- poli: $1-x^2=0$, $(1-x)(1+x)=0$

  $x_2 = 1$, $x_3 = -1$ prvih stopenj
- asimptota: $st(št) < st(im) \Rightarrow y = 0$
- začetna vrednost:

  $f(0) = 0$
- predznak

  določanje z ničlami in poli

$$\begin{array}{ccccccc} & + & & + & & & \\ \hline & & -1 & - & 0 & 1 & - \end{array}$$

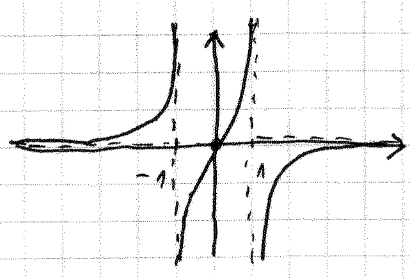
- ekstremi

  $f'(x) = 0$

  $f'(x) = \frac{2-2x^2+4x^2}{(1-x^2)^2} = \frac{2x^2+2}{(1-x^2)^2}$

  $2x^2+2=0$

  $x^2 = -1$ ni $\mathbb{R}$ rešitev, ni $\mathbb{R}$ ekstremov



b)

$$f(x) = \frac{4(x^2-x^4)}{1-4x^2}$$

• ničle: $4(x^2-x^4) = 0$

$4x^2(1-x^2 = 0$

$4x^2(1-x)(1+x) = 0$

$x_1 = 0$ 2. st

$x_2 = 1$ 1. st

$x_3 = -1$ 1. st.

• poli: $1-4x^2 = 0$

$(1-2x)(1+2x) = 0$

$\left.\begin{array}{l} x_4 = 1/2 \\ x_5 = -1/2 \end{array}\right\}$ 1. st.

• asimptota: st(št) ≥ st(im)

$4(x^2-x^4) : (1-4x^2) =$

$-4x^4+4x^2 : -4x^2+1 = x^2 - 3/4$

$-(-4x^4+x^2)$

$3x^2$

$-(3x^2 - 3/4)$ ost. 3/4

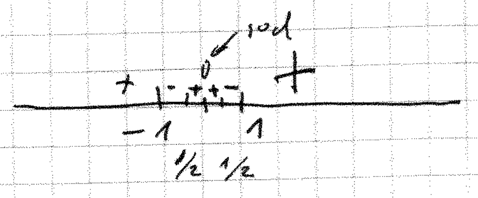
če so v ostanku x_i, izenačimo ~~ost~~ = 0, rešitve so presečišča.

$\underline{y = x^2 - 3/4}$

• začetna vrednost

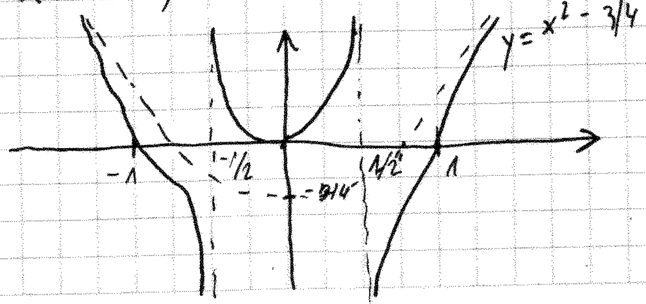
$f(0) = 0$

• predznak



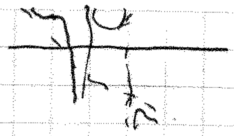
• ekstremi $f'(x) = 0$

$x = 0$, neracionalne st. točke

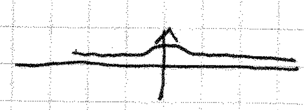


99

DN:

c) $f(x) \quad \frac{1+x^3}{x-x^2}$ R: 

④ Nariši grafe funkcij

D.N. a) $f(x) = e^{-x^2}$ R 

b) $f(x) = x^3 e^{-x}$

- ničle, ekstremi, intervali naraščanja in padanja, obnašanje funkcije v $\pm\infty$.

- ničle $f(x)=0$

$$x^3 e^{-x} = 0$$

$$x_1 = 0, \quad 3.\ st.$$

- ekstremi; $f'(x)=0$

$$f'(x) = 3x^2 e^{-x} + (-x^3 e^{-x}) = x^2 e^{-x}(3-x) = 0$$

$$\left.\begin{array}{l} x_2 = 0 \\ x_3 = 3 \end{array}\right\} \text{stacionarni točki}$$

$$f''(x) = 6xe^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x} =$$

$$= xe^{-x}(6 - 6x + x^2)$$

$f''(0) = 0$ — $f'''(0) \neq 0$ prevoj

$f''(0) < 0$ lok. maksimum

- intervali nar./pad.

$$f'(x) > 0$$

$$x^2 e^{-x}(3-x)$$

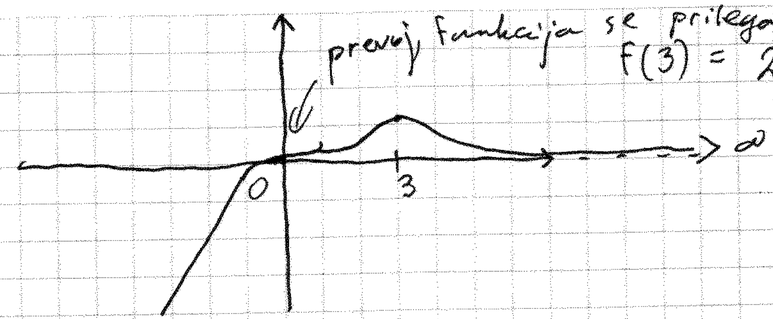
x^2: poz, e^{-x}: poz $\quad 3-x>0 \quad x<3$ narašča

$x>3$ pada

- obnašanje v $\pm\infty$ narašča hitreje x^3/e^x dokaz L'Hospital

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3 e^{-x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 e^{-x} = (-\infty)^3 \cdot e^{-\infty} = -\infty \cdot \infty = -\infty$$



c) $F(x) = 2\sin\left(\frac{x}{2} - \frac{2\pi}{3}\right) = 2\sin\left(\frac{1}{2}\left(x - \frac{4\pi}{3}\right)\right)$

premik v desno za $4\pi/3$

razteg v smeri y osi

razteg po x osi

• ničle: $\frac{1}{2}\left(x - \frac{4\pi}{3}\right) = 0 + k\pi \quad , k \in \mathbb{Z}$

$x - \frac{4\pi}{3} = 2k\pi$

$x = \frac{4\pi}{3} + 2k\pi \quad , k \in \mathbb{Z}$

$-\frac{8\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16}{3}\pi$

• ekstremi: max(2)
min(-2)

max: $\frac{1}{2}\left(x - \frac{4\pi}{3}\right) = \frac{\pi}{2} + 4k\pi \quad , k \in \mathbb{Z}$

$x = \frac{7\pi}{3} + 4k\pi, \quad k \in \mathbb{Z}$

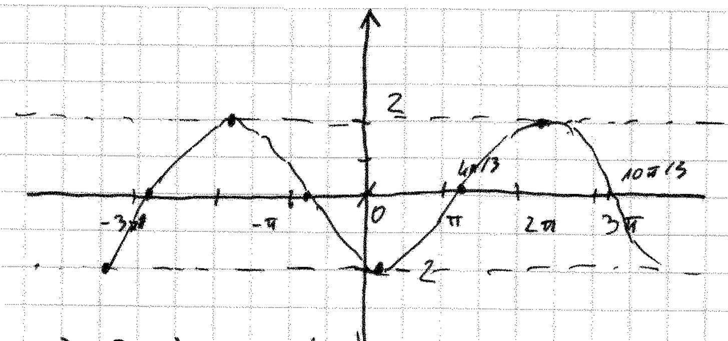
$-\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{19\pi}{3}$

minimumi (-2)

$\frac{1}{2}\left(x - \frac{2\pi}{3}\right) = -\frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{3} + 4k\pi \quad k \in \mathbb{Z}$

$-\frac{11\pi}{3}, \frac{\pi}{3}, \frac{11\pi}{3}$



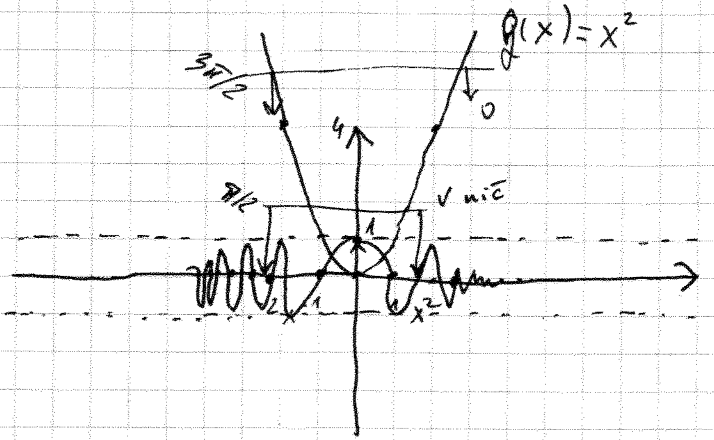
d) $f(x) = \cos(x^2)$

DN: ničle, max, min !

kot kompozitum

$g(x) = x^2$
$h(x) = \cos x$

$f(x) = h \circ \boxed{g(x)}$ 1.



Integral

$$\int f'(x)\,dx = f(x) + C$$

$$\int (f(x) \pm g(x))\,dx = \int f(x) \pm \int g(x)$$

$$\int c(f(x))\,dx = c\int f(x)\,dx$$

Integracija po delih

$$\int u\,dv = u \cdot v - \int v\,du$$

u se mora lepo odvajat

dv se lepo integrira

Uvedba nove spremenljivke

$$\int f(x)\,dx \underset{x=g(t)}{=} \int f(g(t)) \cdot g'(t)\,dt$$

$dx - g'(t)\,dt$

Tabela elementarnih integralov

TO JE ESENCIALNEGA POMENA

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$$

$$\int \frac{1}{x} \, dx = \log|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\log a} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{dx}{1+x^2} = \operatorname{atg} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} = \operatorname{asin} x + C$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \log\left(x + \sqrt{x^2+1}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \log\left(x + \sqrt{x^2-1}\right) + C$$

$$\int \frac{dx}{x^2-1} = \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| + C$$

$$\int \operatorname{sh} x \, dx = \operatorname{ch} x + C \qquad \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$\int \operatorname{ch} x \, dx = \operatorname{sh} x + C \qquad \operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

⑤ Izračunaj integrale

a) DN: $\int (1-x^2)(1-x)\,dx$ R: $\frac{x^4}{4} - \frac{x^3}{3} \quad \frac{x^2}{2} + x + C$

b) $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx = \int (1-x^{-2})\underbrace{x^{1/2}\;x^{1/4}}_{x^{3/4}}\,dx =$

$= \int \left(x^{3/4} - x^{-5/4}\right)dx =$

$= \frac{x^{7/4}}{\frac{7}{4}} - \frac{x^{-1/4}}{\frac{-1}{4}} + C = \frac{4}{7}x^{7/4} + 4x^{-1/4} + C$

c)

$\int \frac{x^4}{1+x^2}\,dx =$

$x^4 : (1+x^2) = x^2 - 1 + \frac{1}{1+x^2}$
$-(x^4+x^2)$
$-x^2$
$-(-1-x^2)$
1 ostane

$\int \frac{x^4}{1+x^2} = \int \left(x^2 - 1 + \frac{1}{1+x^2}\right)dx = \frac{x^3}{3} \cancel{+} - x + \operatorname{atg} x + C$

d) $\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx =$

$1-\cos^2 x$

$= \int \left(\frac{1}{\cos^2 x} - 1\right)dx = \operatorname{tg} x - x + C$

6. Izračunaj integrale z uvedbo nove spremenljivke

√ a) $\int \frac{x}{3x+2}\,dx = \int \frac{(t-2)/3}{t}\,\frac{dt}{3} = \frac{1}{9}\int \left(1 - 2\cdot\frac{1}{t}\right) dt =$

$t = 3x+2 \rightarrow x = (t-2)/3$

$dt = 3\,dx \rightarrow dx = \frac{dt}{3}$

$= \frac{1}{9}\left(t - 2\log|t|\right) + C$

$= \frac{1}{9}(3x+2) - \frac{2}{9}\log|3x+2| + C$

√ b) DN:

$\int \sqrt[3]{1-3x}\,dx$ R: $-\frac{1}{4}(1-3x)^{4/3} + C$

√ c) $\int \left(e^{-2x} + 3e^{3x}\right) dx =$

$= \int e^{-2x}\,dx + \int 3e^{3x}\,dx = \int e^{t}\left(-\frac{dt}{2}\right) + \int 3e^{u}\frac{du}{3} =$

$t = -2x$ $u = 3x$ $x = u/3$

$dt = -2\,dx \rightarrow dx = -\frac{dt}{2}$ $du = 3\,dx$ $dx = du/3$

$\rightarrow x = -t/2$

$= -\frac{1}{2}e^{t} + e^{u}\,du = -\frac{1}{2}e^{-2x} + e^{3x} + C$

√d) DN: $\int \frac{e^x}{1+e^{2x}}\,dx$ R: $\operatorname{atg} e^x + C$

√e) $\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{t}\,dt = -\log|t| = \boxed{\log(\cos x)+c}$

$t = \cos x$

$dt = -\sin x\,dx \to dx = -dt/\sin x$

√f) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx =$

$\frac{1-\cos 2x}{2}$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}}$$

$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\left(x - \int \cos t \frac{dt}{2}\right) =$

$t = 2x$

$dt = 2\,dx$

$= \frac{1}{2}x - \frac{1}{4}\int \cos t\,dt = \frac{1}{2}x - \frac{1}{4}\sin t + c$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + c}$

√g) DN: $\int e^{\sin x} \cdot \cos x\,dx$ R: $e^{\sin x} + C$

√h) $\int \frac{\log^2 x}{x}\,dx = \int t^2\,dt = \frac{t^3}{3} + C = \boxed{\frac{\log^3 x}{3} + C}$

$t = \log x$

$dt = \frac{1}{x} \cdot dx$

√ i) $\int \frac{dx}{x \cdot \log x \cdot \log(\log x)} = \int \frac{dt}{t} = \log(\log|\log x|) + c$

$t = \log(\log x)$

$dt = \frac{1}{\log x} \cdot \frac{1}{x}\,dx$

21.12.07

j) DN: $\int \frac{\mathrm{atg}\sqrt{x}}{\sqrt{x}(1+x)}\,dx$ R: $\mathrm{atg}^2\sqrt{x}+C$

k) DN: $\int x^3\sqrt{1-x^2}\,dx$ R: $-\frac{1}{2}(1-x^2)^{3/2}+\frac{1}{5}(1-x^2)^{5/2}+C$

l) $\int \frac{\sin x\cos^2 x}{1+\cos^2 x} = \int \frac{t^2}{1+t^2}(-dt) = \int\left(1-\frac{1}{1+t^2}\right)dt =$

$t=\cos x$
$dt=-\sin x\,dx$

$t^2:(1+t^2)=1-\frac{1}{1+t^2}$
$-(1+t^2)$
-1

$=-t+\mathrm{atg}\,t+C = -\cos x+\mathrm{atg}(\cos x)+C$

vaje kolokvij petek, 4.1.08

① Z integracijo per partes izračunaj

a) $\int x^2 e^x\,dx$ $\int u\,dv = u\cdot v-\int v\,du$

v se lepo integrira

u se lepo odvajat

polinom niža stopnjo $u=x^2 \xrightarrow{'} du=2x\,dx$
$dv=e^x dx \xrightarrow{\int} v=e^x$

$x^2\cdot e^x-\int 2xe^x dx =$

$u=2x \rightarrow du=2\,dx$
$dv=e^x dx\rightarrow v=e^x$

$x^2e^x-2xe^x+\int 2e^x dx = x^2e^x-2xe^x+2e^x+C=$

$=\underline{\underline{e^x(x^2-2x+2)+C}}$

b) $\int x \ln(x-1)\,dx$

$u = \ln(x-1) \rightarrow du = \frac{1}{x-1}\,dx$

$dv = x\,dx \rightarrow v = x^2/2$

$\ln(x-1)\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x-1}\,dx =$

$x^2 : (x-1) = x + 1 + \frac{1}{x-1}$

$-(x^2 - x)$

$x - 1$

1 ost

$\ln(x-1)\cdot\frac{x^2}{2} - \frac{1}{2}\int\left(x+1+\frac{1}{x-1}\right)dx = \underline{\underline{\ln(x-1)\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{2}+x+\ln|x-1|\right)+C}}$

c) DN: $\int \arcsin x\,dx$ R: $x\arcsin x + \sqrt{1-x^2} + C$

d) $\int \operatorname{arctg}\sqrt{x}\,dx$

$u = \operatorname{arctg}\sqrt{x} \longrightarrow du = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,dx$

$dv = dx \longrightarrow v = x$

$x\cdot\operatorname{arctg}\sqrt{x} - \int\frac{x}{(x+1)2\sqrt{x}}\,dx = x\cdot\operatorname{arctg}\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{x+1}\,dx =$

$t = \sqrt{x} \rightarrow t^2 = x \rightarrow x+1 = t^2+1$

$dt = \frac{1}{2\sqrt{x}}\,dx \longrightarrow dx = 2\sqrt{x}\,dt$

$dx = 2t\,dt$

$= x\operatorname{arctg}\sqrt{x} - \frac{1}{\cancel{2}}\int\frac{t^2+1-1}{t^2+1}\cancel{2}t\,dt = x\operatorname{arctg}\sqrt{x} - \int\left(1-\frac{1}{t^2+1}\right)dt$

$t^2 : (t^2+1)$

$= x\operatorname{arctg}\sqrt{x} - t + \operatorname{arctg} t + C = \underline{x\operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + C}$

e) DN $\int (2x\sin 2x + 2x^2\cos 2x)\,dx$ R: $x^2 \sin 2x + c$

f) $I = \underset{\sim\sim\sim}{\int e^{ax}\cos bx\,dx}$ ni pomembno, kaj je u in dv

$u = \cos bx \rightarrow -\sin bx \cdot b$
$dv = e^{ax}\,dx \rightarrow$

$$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + c$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos ax + c$$

$$\cos bx \frac{1}{a} e^{ax} + \frac{b}{a}\int e^{ax}\sin bx\,dx =$$

$u = \sin bx \longrightarrow du = b\cos bx\,dx$
$dv = e^{ax}dx \longrightarrow v = 1/a\ e^{ax}$

$$= \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a}\left(\sin bx \cdot \frac{1}{a} e^{ax} - \underline{\int \frac{1}{a} e^{ax}\cdot b\cos bx\,dx}\right) =$$

$$= \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a^2}\sin bx\, e^{ax} - \frac{b^2}{a^2}\underbrace{\int e^{ax}\cos bx\,dx}_{I}$$

Iz enačbe izrazimo I

$$I + \frac{b^2}{a^2} I = \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a^2}\sin bx\, e^{ax}$$

$$I = \frac{\frac{1}{a^2} e^{ax}(a\cos bx + b\sin bx)}{1 + \frac{b^2}{a^2}} = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

② Izračunaj integrale racionalnih funkcij

a) DN: $\int \frac{1}{x^2-x-2}$ R: $\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$

b) $\int \frac{x}{(x+1)(x+2)(x+3)}\,dx$

st. števca $\geq$ st. imenovalca : delimo
st. števca $<$ st. imenovalca : parcialni ulomki, substitucija

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)} =$$

$$= \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

$$x = A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2)$$

x^2: $0 = A+B+C$ $/\cdot 5$, $:6$
x^1: $1 = 5A+4B+3C$ $-$ $\quad -1 = B+2C$ $+$ $\quad 2 = B \quad C = -\frac{3}{2},\ A = -\frac{1}{2}$
x^0: $0 = 6A+3B+2C$ $\quad 0 = 3B+4C$

$$= \int \frac{-1/2}{x+1} + \frac{2}{x+2} + \frac{3/2}{x+3} = \underline{-\frac{1}{2}\ln|x+1| + 2\ln|x+2| - 3/2\ln|x+3| + C}$$

c) DN $\int \frac{1}{(x^2+1)(x^2+2)}\,dx$ R: $\operatorname{arctg} x - \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} + C$

d) $\int \frac{dx}{x^3+x^2+2x+2} = \int \frac{dx}{x^2(x+1)+2(x+1)} = \int \frac{dx}{(x+1)(x^2+2)}$

$$\frac{A}{x+1} + \frac{Bx+C}{x^2+2} = \frac{A(x^2+2)+(Bx+C)(x+1)}{(x+1)(x^2+2)}$$

$$1 = Ax^2 + 2A + Bx^2 + Bx + Cx + C$$

x^2: $0 = A+B$ $-$ $\quad A - C = 0 \quad A = \frac{1}{3},\ C = \frac{1}{3},\ B = -\frac{1}{3}$
x^1: $0 = B+C$ $\quad 2A+C = 1$
x^0: $1 = 2A+C$ $\quad 3A = 1$

$$\int \left(\frac{1/3}{x+1} + \frac{-1/3x+1/3}{x^2+2}\right)dx = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{x\,dx}{x^2+2} + \frac{1}{3}\int\frac{dx}{\underbrace{x^2+2}_{a^2}}$$

$t = x^2+2$
$dt = 2x\,dx$

$$\int \frac{1}{x+a}\,dx = \ln|x+a| + C$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{dt}{2t} + \frac{1}{3}\,\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2+2| + \frac{1}{3\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} + C$$

e) DN $\int \frac{x\,dx}{1-x^8}$ R: $\frac{1}{8}\ln|1-x^2| + \frac{1}{8}\ln|1+x^2| + \frac{1}{4}\operatorname{arctg}x^2 + C$

f) $$\int \frac{x^3}{x^8+3}\,dx = \int\frac{dt}{4(t^2+3)} = \frac{1}{4}\,\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}} + C$$

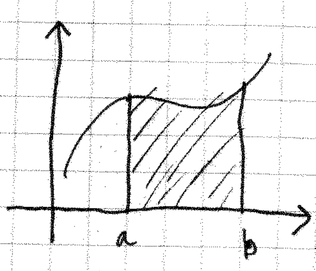
$t = x^4 \qquad dt = 4x^3\,dx$

$$= \frac{1}{4\sqrt{3}}\operatorname{arctg}\frac{x^4}{\sqrt{3}} + C$$

Določeni integral

$$\int f(x)\,dx = F(x) + C$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

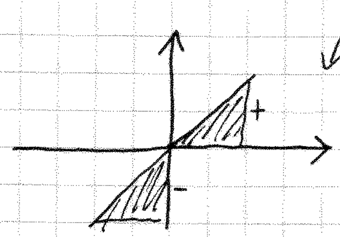


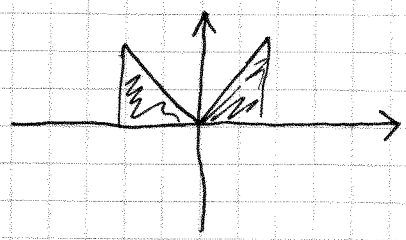
3) Izračunaj določene integrale

$$\int_{-2}^{2} x^2 \sin x \, dx = 0$$

soda liha

f je liha, integral je 0 na simetričnem intervalu je 0





$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

soda

b) $$\int_1^{\infty} \frac{dx}{(1+x^2)(3+x^2)}$$ ← posplošeni integrali

$$\int \frac{dx}{(1+x^2)(3+x^2)} \overset{DN}{=} \ldots = \frac{1}{2}\operatorname{arctg} x - \frac{1}{6\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}} + C$$

$$= \left[\frac{1}{2}\operatorname{arctg} x - \frac{1}{6\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\right]\Big|_1^{\infty}$$

$$= \underbrace{\qquad\qquad}_{} - \underbrace{\left(\frac{1}{2}\operatorname{arctg} 1 - \frac{1}{6\sqrt{3}}\operatorname{arctg}\frac{1}{\sqrt{3}}\right)}_{\text{spodnja meja}}$$

$$\lim_{x\to\infty}\left(\frac{1}{2}\operatorname{arctg} x - \frac{1}{6\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\right) = \frac{1}{2}\frac{\pi}{2} - \frac{1}{6\sqrt{3}}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4} + \frac{1}{6\sqrt{3}}\frac{\pi}{6} =$$

$$= \frac{\pi}{8} - \frac{\pi}{18\sqrt{3}}$$

c) $I = \int_1^2 \frac{dx}{x^2\sqrt{1+x^2}}$

$$\int_1^2 \frac{dx}{x^2\sqrt{1+x^2}} \qquad 1+(\mathrm{tg}\,x)^2 = \left(\frac{1}{\cos x}\right)^2$$

$$x = \mathrm{tg}\,t \rightarrow 1+x^2 = 1+\mathrm{tg}^2 t = \frac{1}{\cos^2 t}$$

$$dx = \frac{1}{\cos^2 t}\,dt$$

$$= \int \frac{\overset{dx}{\frac{dt}{\cos^2 t}}}{\frac{1}{\cos t}\cdot \mathrm{tg}^2 t} = \int \frac{dt\,\cancel{\cos^2 t}}{\cancel{\cos t}\cdot \sin^2 t} = \int \frac{\cos t\,dt}{\sin^2 t} =$$

$$\underset{\substack{u=\sin t\\ du=\cos t\,dt}}{=} \int \frac{du}{u^2} = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{\sin t} + t = -\frac{1}{\sin(\mathrm{arctg}\,x)} + C$$

$x = \mathrm{tg}\,t \quad t = \mathrm{arctg}\,x$

$$I = \left[-\frac{1}{\sin(\mathrm{arctg}\,x)}\right]_1^2 = -\frac{1}{\sin(\mathrm{arctg}\,2)} + \frac{1}{\sin(\underset{\pi/4}{\mathrm{arctg}\,1})}$$

$\sqrt{2}/2$

Integrali z $\sqrt{1-x^2}$, $\sqrt{x^2-1}$, $\sqrt{1+x^2}$ se prevedejo na integrale trigonometričnih funkcij.

$R(x, \sqrt{1+x^2}) \rightarrow x = \mathrm{tg}\,t$ ali $x = \sin t$

$R(x, \sqrt{1-x^2}) \rightarrow x = \sin t$ ali $x = \cos t$

$R(x, \sqrt{x^2-1}) \rightarrow x = 1/\cos t$

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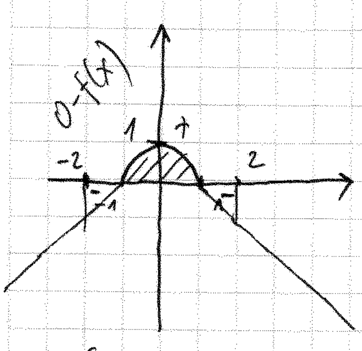
4) Izračunaj integral

$\int_{-2}^{2} f(x)\,dx$, kjer je $f(x)\begin{cases}1-x^2, & |x|\le 1\\ 1-|x|; & |x|>1\end{cases}$

$$=\begin{cases}1-x^2, & |x|\le 1\\ 1-x, & x>1\\ 1+x, & x<-1\end{cases} \qquad x\in[-1,1]$$



$$\int_{-2}^{2} f(x)\,dx = \int_{-1}^{1}(1-x^2)\,dx+\int_{1}^{2}(1-x)\,dx + \int_{-2}^{-1}(1+x)\,dx$$

$$=\left[x-x^3/3\right]_{-1}^{1}+2\cdot\left[x-\frac{x^2}{2}\right]_{1}^{2}=$$

$$=(1-1/3)-(-1+1/3)+2((2-2)-(1-\tfrac{1}{2}))=$$

$$=\frac{2}{3}+\frac{2}{3}-1=\boxed{\frac{1}{3}}$$ to je integral

ploščina

$$\int_{-2}^{-1}(0-(1+x))\,dx+\int_{-1}^{1}(1-x^2-0)\,dx+\int_{1}^{2}(0-(1-x))\,dx$$

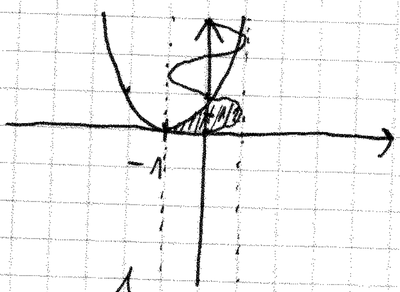
$\boxed{R: 7/3}$

5) DN Izračunaj ploščino lika med grafom funkcije $f(x)=\sin x+\cos x+x$, abscisno osjo ter premicama $x=0$ in $x=\pi$

R: $\frac{4\pi^2}{2}$

⑥ Izračunaj ploščino lika, omejen $y=(x+1)^2$ $x=\sin \pi y$ in $y=0$

skrčen za π



$S = S_1 + S_2$

$$S_1 = \int_{-1}^{0} (x+1)^2\,dx = \left[\frac{x^3}{3} + \frac{2x^2}{2} + x\right] =$$

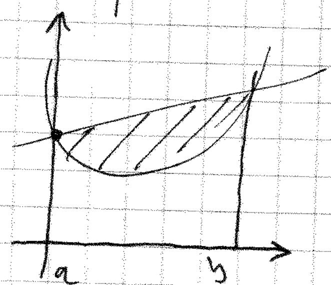
$$= 0 - \left(-\frac{1}{3} + 1 - 1\right) = \frac{1}{3}$$

$$S_2 = \int_0^1 (\sin \pi y - 0)\,dy = \left[-\frac{1}{\pi}\cos \pi y\right]_0^1 = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$S = \frac{1}{3} + \frac{2}{\pi}$$

⑦ DN Izračunaj ploščino območja med premico in parabolo

$$y = \frac{x}{3} + \frac{8}{3}, \quad y = x^2 - \frac{14x}{3} + \frac{20}{3}$$



Ploščine območij v ravnini

$\int_a^b (\text{zg.kr.} - \text{sp.kr.})\,dx$ v kartezičnem

$\frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi$ polarne koordinate $r = r(\varphi)$

$\frac{1}{2}\int_a^b (x \cdot \dot{y} - y \cdot \dot{x})\,dt$ parametrična oblika $x = x(t)$, $y = y(t)$

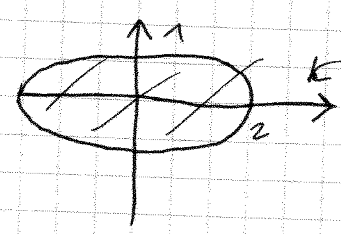
⑧ DN Izračunaj ploščino območja $r = \sin^2\varphi$, $0 < \varphi, \pi$

R: $3\pi/16$

⑨ Ploščina zanke $x = 2\cos t$, $y = \sin t$

$[0, 2\pi]$ zanka

$$S = \frac{1}{2}\int_0^{2\pi} (x\dot{y} - y\dot{x})\,dt = \frac{1}{2}\int_0^{2\pi} \underbrace{(2\cos^2 t + \sin^2 t)}_{\cancel{2}}\,dt = [t]_0^{2\pi}$$



krivulja $\sin^2 t + \cos^2 t = 1$

$y^2 + (x/2)^2 = 1$

Dolžina loka

$$s = \int_a^b \sqrt{1 + (y')^2}\,dx \qquad \text{kart. koor.}$$

$$s = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\,d\varphi \qquad \text{polarne}$$

$$s = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt$$

⑩ Izračunaj dolžino loka krivulje $y = \operatorname{ch} x$ za $-1 < x < 1$

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

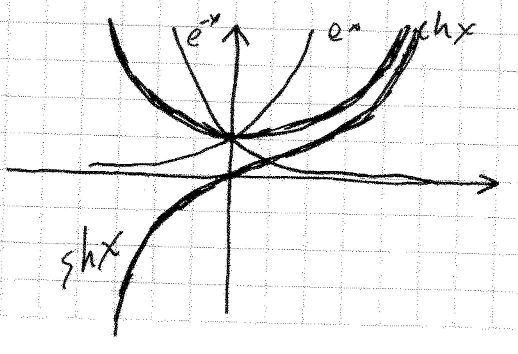
$\operatorname{ch}' x = \operatorname{sh} x$

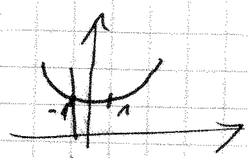
$\operatorname{sh}' x = \operatorname{ch} x$

$1 + \operatorname{sh}^2 x = \operatorname{ch}^2 x$

$\int \operatorname{ch} x\,dx = \operatorname{sh} x + C$

$\int \operatorname{sh} x\,dx = \operatorname{ch} x + C$





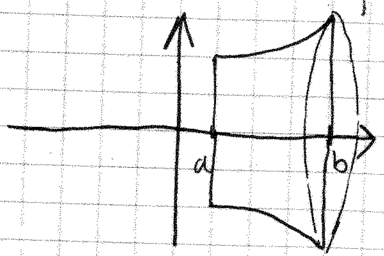
$$s = \int_{-1}^{1} \sqrt{1 + (y')^2}\,dx =$$

$$= \int_{-1}^{1} \sqrt{1 + \operatorname{sh}^2 x}\,dx = \int_{-1}^{1} \operatorname{ch} x\,dx = [\operatorname{sh} x]_{-1}^{1} = \boxed{e - e^{-1}}$$

(11) DOMA: Izračunaj dolžino zanke

$r = a(1+\cos\varphi)$, $a>0$ R: $8a$

Volumen rotacijskega telesa



$$V = \pi \int_a^b f^2(x)\,dx \quad \text{kart. koor.}$$

$$V = \pi \int_\alpha^\beta r^2\,d\varphi \quad \text{pol. koor}$$

(10) Površina rot. telesa

$$P = 2\pi \int_a^b y\,ds \qquad ds = \sqrt{1+(y')^2}\,dx \quad \text{kart. koor.}$$

$$V = \pi \int_\alpha^\beta y(t)^2\,\dot{x}(t)\,dt \quad \text{param. koor}$$

$$P = 2\pi \int_\alpha^\beta y(t)\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\,dt$$

(12) Izračunaj volumen telesa, ki nastane, če krivuljo

$r = \sqrt{\cos 2\varphi}$, $0<\varphi<\pi/4$ zavrtimo okrog x osi

$$V = \pi \int_0^{\pi/4} \cos 2\varphi\,d\varphi = \left[\pi \cdot \frac{1}{2}\sin 2\varphi\right]_0^{\pi/4} = \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \underline{\underline{\frac{\pi}{2}}}$$

(13) Površina telesa, če funkcijo $f(x) = \sqrt{1-x^2}$, $0<x<1$, zavrtimo okoli x osi R: 2π

7.1.08

$$f(x)=\frac{x^2+2x+1}{x^2-4}$$

a) ničle, poli, asimptota, ekstremi funkcije in nariši

b) določi DF $g(x)=\sqrt{f(x)}$ in skiciraj

ničle: $x^2+2x+1=0$ poli: $(x-2)(x+2)=0$

$(x+1)^2$ $x_3=-2$ $x_2=2$ (1. st.)

$x=-1$ (2. st.)

asimptota: $x^2+2x+1 : x^2-4 = 1 \rightarrow y=1$

$-x^2+4$

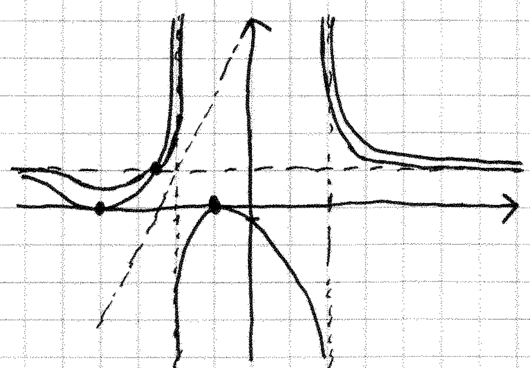
$2x+5=0$ presečišče

$x=-5/2$

ekstremi: $f'(x)=\frac{(2x+2)(x^2-4)-(x^2+2x+1)(2x)}{(x^2-4)^2}=\frac{2(x+1)(\cancel{x^2}-4-\cancel{x^2}-x)}{(x^2-4)^2}=$

$=\frac{-2(x+1)(x+4)}{(x^2-4)^2}=0$

$x_4=-1$ $x_5=-4$ stacionarni točki



+ -2 − -1 − 2 +

-1 je lok max

$f(-4)=3/4$

b) $f(x)\geq 0$

$\frac{x^2+2x+1}{x^2-4}\geq 0$ na grafu: $D_g: (-\infty,-2)\cup\{-1\}\cup(2,\infty)$

Valjasta posoda s pokrovom ima $s = 8\pi$. Določi r in h tako, da bo V največja

$$P = 2\pi r^2 + 2\pi r v = 8\pi \rightarrow v = \frac{8\pi - 2\pi r^2}{2\pi r} = \frac{4-r^2}{r}$$

$V max$

$$V = \pi r^2 \cdot v = \pi r^2 \cdot \frac{4-r^2}{\cancel{r}} = \pi r(4-r^2) = 4\pi r - \pi r^3$$

$$V = V(r)$$

$$V'(r) = 4\pi - 3\pi r^2 = \pi(4-3r^2) = 0$$

$$4 = 3r^2 \rightarrow r = \sqrt{4/3} = \boxed{\frac{2}{\sqrt{3}}}$$

$$V = \frac{8}{3}\frac{\sqrt{3}}{2} = \boxed{4/\sqrt{3}}$$

Izračunaj integral

$$\int x\left(\ln(2x) + \frac{1}{x(x^2-x-2)}\right) dx =$$

$$\int x \cdot \ln 2x \, dx = \quad \Bigg\| \quad \int \frac{1}{x^2-x-2} dx$$

$$= / \ PP: \ u = \log 2x \rightarrow du = \frac{1}{2x} \cdot 2 \, dx = \frac{dx}{x}$$
$$dv = x\,dx \rightarrow v = x^2/2$$

$$= \frac{x^2}{2} \ln 2x - \int \frac{x^{\cancel{2}}}{2} \frac{dx}{\cancel{x}} = \frac{x^2}{2} \ln 2x - \frac{1}{2}\frac{x^2}{2} + C_1$$

$$\frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$1 = Ax + A + Bx - 2B$$

$$\left.\begin{array}{l} x^1: \ 0 = A + B \\ x^0: \ 1 = A - 2B \end{array}\right\} \quad \begin{array}{l} -A = 3B \\ B = -1/3 \end{array} \quad , A = 1/3$$

$$\int \frac{1/3}{x-2} - \frac{1/3}{x+1} = \frac{1}{3}\log(x-2) - \frac{1}{3}\log(x+1) + C$$

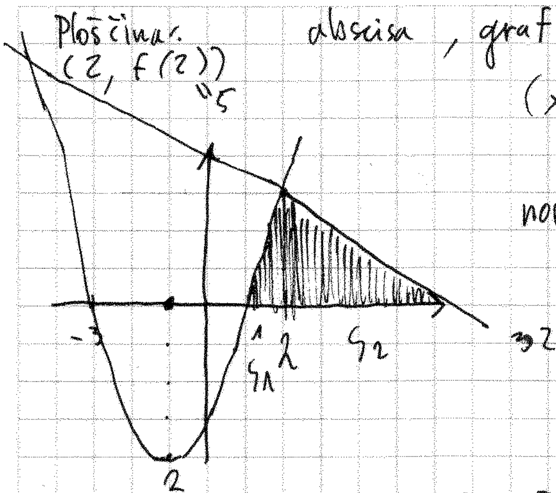
$$\boxed{\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a}}$$

seštejemo

Ploščina: abscisa, graf $f(x) = x^2 + 2x - 3$ in normala

$(2, f(2))$

$(x-1)(x+3)$



normala $\quad f'(x) = 2x + 2 \quad f'(2) = 6$

$k_n = -1/6$

$y = -1/6x + n$

$5 = -1/3 + n \rightarrow n = \frac{16}{3}$

$y = -1/6x + 16/3$

$$S_1 = \int_1^2 (x^2 + 2x - 3)\,dx = \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$$= \frac{8}{3} + 4 - 6 - \frac{1}{3} - 1 - 3 = \boxed{7/3} = S_1$$

$$S_2 = 30 \cdot 5 / 2 = \boxed{75}$$

$$S = 77 + 1/3$$

Izračunajte: dolžino loka $x = \sqrt{5}t^3 \quad y = 2 - 2t^3$

ki leži v prvem kvadrantu

$$S = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2}$$

$x, y \geq 0$

$x \geq 0 \quad x = \sqrt{5} \cdot t^3$

$t^3 \geq 0$

$t \geq 0$

$\dot{x} = 3\sqrt{5}t^2$

$\dot{y} = -6t^2$

$y \geq 0$

$2 - 2t^3 \geq 0$

$\underbrace{2(1-t)}_{1-t \geq 0}\underbrace{(1 + t + t^2)}_{>0} \geq 0$

$t \leq 1$

$$\int_0^1 \sqrt{(3\sqrt{5}t^2)^2 + (-6t^2)^2}\,dt$$

$$\int_0^1 \sqrt{81t^4}\,dt = \int_0^1 9t^2\,dt =$$

$$= 9\frac{t^3}{3}\Big|_1^0 = \underline{3}$$

$$\int_0^{\pi/4} \frac{tg^2x+1}{\cos^2 x}\,dx = \left[\frac{1}{3}tg^3x + tg\,x\right]_0^{\pi/4} = \frac{1}{3}+1-0 = \boxed{4/3}$$

$$\int \frac{tg^2x+1}{\cos^2 x}\,dx = \int (t^2+1)\,dt = \frac{t^3}{3}+t+C$$

$t = tg\,x \qquad dt = \frac{1}{\cos^2 x}\,dx$

$$\frac{1}{3}tg^3x + tg\,x + C$$

③ Dana je f.

$$f(x) = x\,e^{-x/2}$$

limita, $x \to \infty$ in $x \to -\infty$

poiščite ekstrem in prevoj, ter narišite graf funkcije $y = f(x)$

$$\lim_{x\to\infty} x \cdot e^{-x/2} = \lim_{x\to\infty} \frac{x}{e^{x/2}} \quad \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{e^{x/2}\cdot 1/2} = 0$$

$$\lim_{x\to-\infty} x = -\infty$$

$$f'(x) = e^{-x/2} + x\cdot e^{-x/2}\cdot\left(-\frac{1}{2}\right) = \frac{1}{2}e^{-\frac{x}{2}}(2-x)$$

$x_1 = 2$ edina st. točka

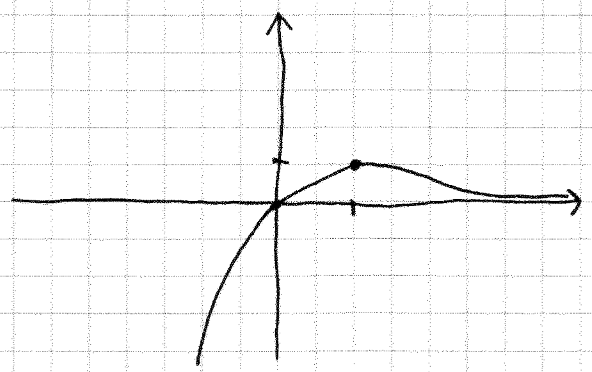
$$f''(x) = -\frac{1}{4}e^{-\frac{x}{2}}(2-x) - \frac{1}{2}e^{-x/2} = -\frac{1}{4}e^{-\frac{x}{2}}(4-x)$$

$f''(2) = \ < 0$ max $f(2) = 2/e < 1$

ničle: $x_2 = 2$

nar/pad $f'(x) = \frac{1}{2}e^{-\frac{x}{2}}(2-x) > 0$

$x < 2$ narašča



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veliko uspeha pri izpitih