MATEMATIKA II
zapiski z avditornih vaj

Šolsko leto	2007 / 2008
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UREJANJE DOKUMENTA

VERZIJA 01 **REVIZIJA** 01
DATUM 1. 3. 2009

ZADNJI POPRAVLJAL /
PREGLEDAL Blaž Potočnik, Aljoša Praznik

OPOMBE

POPRAVKI

DETERMINANTE

MAT II-V
20. 2. 2008
ŽITKO

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ - kvadratna shema, podobno pri matrikah (za razliko od determinanti so pomembna vsa števila)

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2\cdot 5 - 3\cdot 4 = -2$$

trovrstna determinanta

$$\begin{vmatrix} 2 & -1 & 3 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2\cdot 2\cdot 4 \quad \overset{\text{vmes predznaki}}{\neq} \quad -1 \cdot (-3)\cdot 4 \qquad 3 \quad 2 \quad 1$$

$$\text{ali } 2\cdot 1\cdot 2 \qquad -1\cdot(1)\cdot 1 \qquad 3\cdot (-3)\cdot 2$$

v kristoru prepišemo prva dva stolpca:

$$\left|\begin{matrix} 2 & -1 & 3 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{matrix}\right|\begin{matrix} 2 & -1 \\ -3 & 2 \\ 1 & 2 \end{matrix} = 2\cdot 2\cdot 4 + (-1)\cdot 1\cdot 1 + 3(-3)\cdot 2 - 3\cdot 2\cdot 1 - 2\cdot 1\cdot 2 - (-1)\cdot(-3)\cdot 4$$

$$= 16 - 1 - 18 - 6 - 4 - 12 = -25$$

operiranje z indeksi:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{vmatrix} = \sum \pm a_{1\cdot i_1} \cdot a_{2\cdot i_2} \ldots a_{n\cdot i_n}$$

sigma – vse permutacije

stolpci a $1, 2 \ldots n$ – vse možne permutacije

$i_1, i_2 \ldots i_n$ - permutacija indeksov od 1 do n.

število permutacij je $n! = 1\cdot 2\cdot 3\cdot 4\cdot 5$

Računanje determinant:

$$\begin{vmatrix} + & + & + & + \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \end{vmatrix} = -\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 6 & 9 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 3 \end{vmatrix}$$

determinanta se ne spremeni

$$\underset{\text{i-ti vrstici}}{v_i} \leftarrow v_i + k\,\underset{\text{večkratnik j-te vrstice}}{v_j}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} +2v_3 = \begin{vmatrix} 4 & 3 & 11 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & 1 \\ 5 & 2 & 4 \end{vmatrix}$$

$v_1 \Leftarrow v_1 + 2 \cdot v_3$

isto pravilo velja za stolpce

$s_i \leftarrow s_i + k s_j$
$s_1 \leftarrow s_1 + 2 s_2$

Razviti determinanto po isti vrstici

$$\rightarrow i \begin{vmatrix} a_{i1} & a_{i2} & a_{in} \end{vmatrix} = a_{i1} \cdot A_{i1} + a_{i2} A_{i2} + a_{in} A_{in} +$$

kofaktor $A_{ij} = (-1)^{i+j} \begin{vmatrix} & \\ & \end{vmatrix}$ i

j — i-to vrstico in stolpec se preeda in to je $(n-1)$ vrstna det.

primer razvoj po 1 vrstici

$$\begin{vmatrix} 2 & -1 & 3 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2 \cdot A_{11} + (-1) A_{12} + 3 A_{13}$$
$$= 2\,6 + (-1)(+13) + 3 \cdot (-8)$$
$$= -25$$

$A_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 6$

$A_{12} = (-1)^3 \begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} = +13$

$A_{13} = (-1)^4 \begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} = -8$

lahko razvijemo po 2 stolpcu.

$$D = -1 \cdot (-1)^3 \cdot (-13) + 2 \cdot (-1)^4 (5) + 2 \cdot (-1)^5 \cdot 11$$
$$= -13 + 10 - 22 = -25$$

večjo determinanto

$v_i + k v_j$ cilj da dobimo čim več 0, razviti po vrstici z 0-mi

$$D = \begin{vmatrix} 2 & -1 & 3 \\ -3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -3 & 2 & 7 \\ 1 & 2 & 10 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 2 & 7 \\ 5 & 2 & 10 \end{vmatrix} = -1(-1)(10-35) = -25$$

$+3v_2$ $\quad +2v_2$

1.
$$\begin{vmatrix} 1 & -3 & 1 & 4 \\ -1 & 6 & 2 & -3 \\ 2 & -2 & 6 & 3 \\ 0 & 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -7 & 1 & 4 \\ -1 & 8 & 2 & -3 \\ 2 & -5 & 6 & 3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -7 & 1 & 6 \\ -1 & 8 & 2 & 1 \\ 2 & -5 & 6 & 15 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$+2V_3$

$$= -1\,(-1)^{4+3} \begin{vmatrix} 1 & -7 & 6 \\ -1 & 8 & 1 \\ 2 & -5 & 15 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 7 \\ -1 & 8 & 1 \\ 0 & 11 & 17 \end{vmatrix}$$

$$= (-1)(-1)^{2+1} \begin{vmatrix} 1 & 7 \\ 11 & 17 \end{vmatrix} = 17 - 77 = -60$$

2.
$$\begin{vmatrix} 2 & -5 & 1 & 2 \\ -3 & 7 & -1 & 4 \\ 5 & -9 & 2 & 7 \\ 4 & -6 & 1 & 2 \end{vmatrix} \begin{matrix} \\ +V_1 \\ -2V_1 \\ -V_1 \end{matrix} \quad \begin{vmatrix} 2 & -5 & 1 & 2 \\ -1 & 2 & 0 & 6 \\ 1 & 1 & 0 & 3 \\ 2 & -11 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & -11 & 0 \end{vmatrix} -2V_2$$

$$= \begin{vmatrix} -3 & 0 & 0 \\ 1 & 1 & 3 \\ 2 & -11 & 0 \end{vmatrix} = 3(-1) \begin{vmatrix} -3 & 0 \\ 2 & 11 \end{vmatrix}$$

$$= -9$$

Reši enačbo:

$$-(\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

ena rešitev je 1

$$= \begin{vmatrix} x-1 & 1-x & 0 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x-1) \begin{vmatrix} 1 & -1 & 0 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x+1 & 1 \\ 1 & 2 & x \end{vmatrix} =$$

$$= 1 \begin{vmatrix} x+1 & 1 \\ 2 & x \end{vmatrix} (x-1)$$

$$= (x+1)(x-1)x - 2(x-1)$$

$$= (x-1)(x^2+x-2) = 0$$

$x=1$

$$(x-1)(x+2)(x-1)$$

$x=1 \quad x=1 \quad x=-2$

Števila 169, 273 in 390 so deljiva s 13. Pokaži, da je tudi determinanta deljiva s 13.

$$\begin{vmatrix} 1 & 6 & 9 \\ 2 & 7 & 3 \\ 3 & 9 & 0 \end{vmatrix} = 6\cdot3\cdot3 + 9\cdot2\cdot9 - 9\cdot7\cdot3 - 6\cdot3\cdot3\cdot9\cdot1$$

ali

$$\begin{vmatrix} 1 & 6 & 9 \\ 2 & 7 & 3 \\ 3 & 9 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 69 \\ 2 & 7 & 73 \\ 3 & 9 & 90 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 169 \\ 2 & 7 & 273 \\ 3 & 9 & 390 \end{vmatrix} = \overbrace{\begin{vmatrix} 1 & 6 & 13 \\ 2 & 7 & 21 \\ 3 & 9 & 30 \end{vmatrix}}^{\text{je}} \cdot 13$$

je cel je cel

3. Izračunaj n-vrstno determinanto !!! - glej sliko

$$D_n = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & \vdots \\ 1 & 1 & 1 & \ddots 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 1 & & \dots & 0 \\ 1 & 1 & 0 & 1 & & & 1 \\ 1 & 1 & 1 & 0 & & \dots & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & 1 & \dots & 1 \\ \vdots 1 & \vdots 2 & & 0 & \dots & 0 \end{vmatrix}$$ ni pravilno

še enkrat
$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} n-1, & n-1, & n-1, & \dots & n-1 \\ 1 & 0 & 1 & & 1 \\ 1 & 1 & 0 & & 1 \\ \vdots & 1 & 1 & \dots & 0 \end{vmatrix}$$

$$= (n-1) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$ n vrstna

$$= (n-1) \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & & 1 \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$

$$= -(1)(n-1)\cdot(1) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$ n-1 vrstna

$$= (-1)^{n-1}(n-1)$$

4. $$\begin{vmatrix} 5x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 1 & 2 & x & 3 \\ x & 1 & 2 & 2x \end{vmatrix} = \text{polinom 4 stopnje} = \underset{10x^4}{a_4} x^4 + \underset{-5x^3}{a_3} x^3 + \dots$$

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$$= 5x \begin{vmatrix} x & 1 & 2 \\ 2 & x & 3 \\ 1 & 2 & 2x \end{vmatrix} - 1 \begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ x & 2 & 2x \end{vmatrix} + 2 \begin{vmatrix} x & x & 2 \\ 1 & 2 & 3 \\ x & 1 & 2x \end{vmatrix} \overset{!!!}{(-1)} 3 \begin{vmatrix} x & x & 1 \\ 1 & 2 & x \\ x & 1 & 2 \end{vmatrix}$$

$-2x^3$ $\quad -3x^3$

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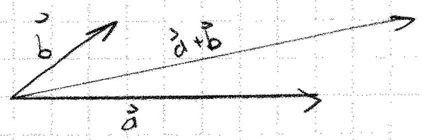
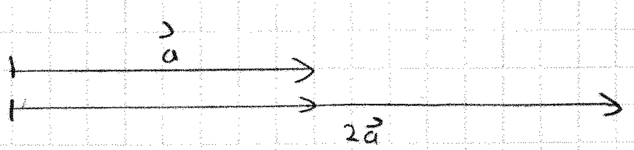
VEKTORJI

MAT II-V
27. 2. 2008

$\vec{a}+\vec{b}$
$k\vec{a}$
$\vec{a}\,\vec{b}$ = skalarno število
$\vec{a}\times\vec{b}$ = vektor

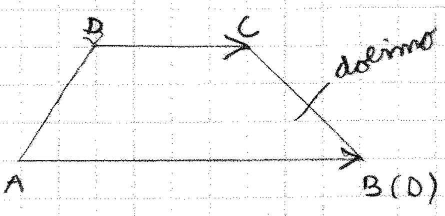


$2\vec{a}$



$-1/2\,\vec{a}$

1. Stranice četverokotnika so podane z $\vec{AB}=\vec{a}+2\vec{b}$, vektor od $\vec{BC}$ $=-4\vec{a}-4\vec{b}$, vektor od $\vec{CD}=2\vec{a}+\vec{b}$. Pokaži, da je četverokotnik trapez



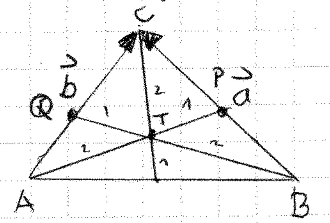
dobimo ga tako, da gremo po daljši poti

$\vec{AB}=k\cdot\vec{CD}$
ni res

drugače bodo orientirani

$\vec{AD}=k\cdot\vec{BC}$
$\vec{AB}+\vec{BC}+\vec{CD}=k\cdot\vec{BC}$
$\vec{a}+2\vec{b}-4\vec{a}-4\vec{b}+2\vec{a}+\vec{b}=k\cdot(-4\vec{a}-4\vec{b})$
$-\vec{a}-\vec{b}=k\cdot(-4\vec{a}-4\vec{b})$
$k=\frac{1}{4}$
je trapez. $\quad\vec{BC}=4\cdot\vec{AD}$

2. Dokaži, da se težiščnice Δ sekajo v razmerju 1:2



2 osnovna vektorja $\vec{a},\vec{b}$

$\vec{a},\vec{b}$ - linearno neodvisna

$\vec{AT}=\frac{2}{3}\vec{AP}$ $\quad\Rightarrow \vec{AT}=x\,\vec{AP}$ $\quad x$ je neznanka
$=x\,(\vec{b}-\frac{1}{2}\vec{a})$

$\vec{AT}+\vec{TC}+\vec{CA}=0$

$\vec{TC}=y\,\vec{RC}=y\frac{1}{2}(\vec{BA}=\frac{1}{2}y\,(\vec{a}-\vec{b})$
$\Rightarrow y\,(\vec{RA}+\vec{AC})=y\,(\frac{1}{2}(\vec{a}-\vec{b})+\vec{b})$

$\vec{CA}=-\vec{b}$

$x\,(\vec{b}-\frac{1}{2}\vec{a})+\frac{1}{2}y\,(\vec{a}+\vec{b})-\vec{b}=0$
$x\vec{b}-\frac{x}{2}\vec{a}+\frac{1}{2}y\vec{a}+\frac{1}{2}y\vec{b}-\vec{b}=0$ $\quad x=2$

$\vec{a}\,(-\frac{1}{2}x+\frac{1}{2}y)+(x+\frac{1}{2}y-1)\vec{b}=0$

1 + 2 => $\frac{x}{2}-1=0$
$x=2$
$y=2$

$\frac{1}{2}y=\frac{1}{2}x$
$x=y$

$x-\frac{1}{2}y-1=0$
$y-\frac{1}{2}x-1=0$
$\frac{x}{2}=1$

$y=2/3$

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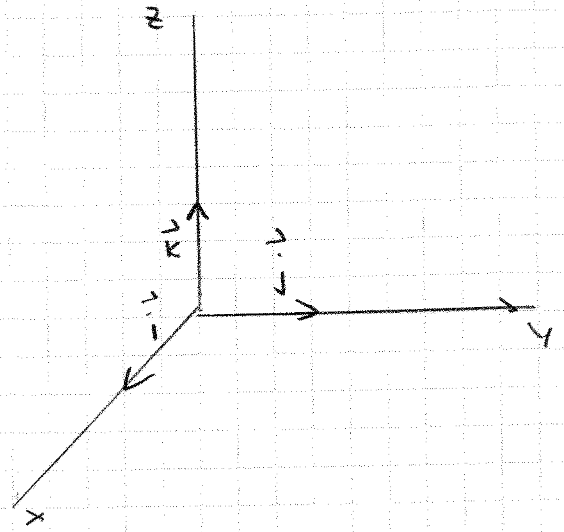
Dokaz da se sekajo v točki T – vzamemo eno izmed še vzetih še 2 neznano (2 in 3 ter 1. in 3.)

Bazni vektorji



V prostoru so dane točke:
A (1, -2, 0)
B (2, 1, 3)
C (2, 0, 5)
so zaporedna oglišča paralelograma, poišči 4 oglišče D.
Krajevni vektor r_A

$A = \vec{i} - 2\vec{j}$
$B = 2\vec{i} + 1\vec{j}$

$\vec{BA} = \vec{r_A} - \vec{r_B}$

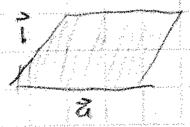
$\vec{AB} = \vec{DC}$
$\vec{r_D} = r_C + \vec{CD}$
$= r_C + \vec{BA} =$
$= (-2, 0, 5) + (-1, -3, -3)$
$= (-3, -3, 2)$
$= -3\vec{i} + 3\vec{j} + 2k$

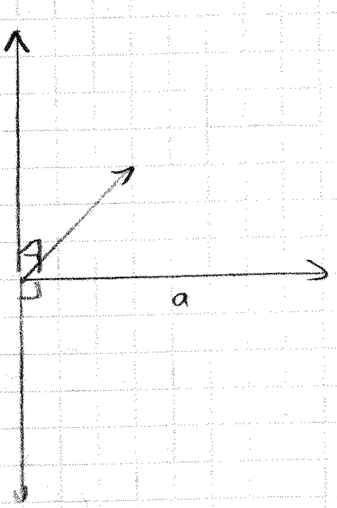
$\vec{a} \cdot \vec{b} = |a||b| \cos \varphi$

$\vec{a} \times \vec{b} = \vec{c}$

$|\vec{c}| = |a||b| \sin \varphi$ – ploščina paral

$c \perp \vec{a}$ in $\vec{b}$





$\cos \varphi = \frac{a \cdot b}{|a||b|}$

$\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$
$\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$

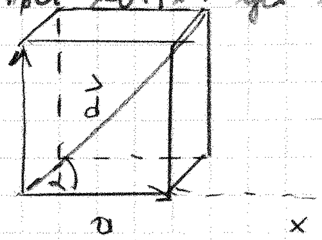
Izračunaj skalarni in vektorski produkt

$\vec{a}\cdot\vec{b} = (2i + 5j + \dots)(i + \dots) = 2 + 5 + 7 = 19$ $\qquad \vec{i}\cdot\vec{i} = 1$, $\vec{i}\cdot\vec{j} = 0$

$\vec{a}\times\vec{b} = (\quad)\times(\quad)$ $\qquad i \times i = 0$, $i \times j = \vec{k}$, $j \times i = -\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & 5 & 7 \\ 1 & 2 & 1 \end{vmatrix} = i(5-14) - j(2-7) + k(4-5) = -9i + 5j - \vec{k}$$

Poišči kot, ki ga oklepa rob kocke s telesno diagonalo:



$$\cos\varphi = \frac{\vec{a}\cdot\vec{d}}{|\vec{a}|\cdot|\vec{d}|} = \frac{1}{\sqrt{3}}$$ $\qquad a = \vec{i}$, $d = \vec{i} + \vec{j} + \vec{k}$

$d = \sqrt{1+1+1} = \sqrt{3}$

$\alpha = \arccos\frac{1}{\sqrt{3}}$

Imamo 2 osnovna vektorja $\vec{p}, \vec{q}$, $|p| = 2, |q| = 3$, kot med njima je $\pi/3$. Izračunaj dolžino vektorja $|p - 2q|$

$|p - 2q| = \sqrt{(p - 2q)\cdot(p - 2q)}$ $\qquad |\vec{a}|^2 = \vec{a}\cdot\vec{a}$

$= \sqrt{\vec{p}\vec{p} - 4\vec{p}\vec{q}\cdot\cos\frac{\pi}{3} + 4\vec{q}\vec{q}}$

$= \sqrt{2^2 - 4\cdot 2\cdot 3\cdot \frac{1}{2} + 4\cdot 3^2} = \sqrt{4 - 12 + 36} = \sqrt{28} = 2\sqrt{7}$

Kolikšna je ploščina Δ z oglišči $A(1,1,1)$, $B(2,3,4)$, $C(4,3,2)$

$P = \frac{1}{2}\cdot|\vec{AB}\times\vec{AC}|$ $\qquad AB \times AC = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = i(-4) - j(-8) + k(-4)$

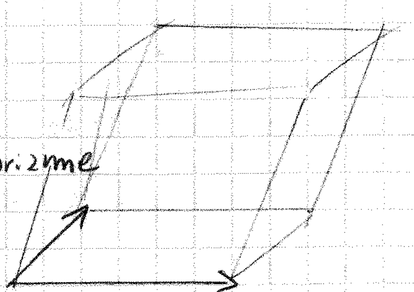
$(-4, 8, -4)$

$= \frac{1}{2}\sqrt{16 + 64 + 16}$

$= \frac{1}{2}\sqrt{96} = 2\sqrt{6}$

MEŠANI PRODUKT

$(\vec{a}\times\vec{b})\cdot\vec{c}$ - volumen prizme



$= \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$

43. Kolikšna je višina piramide V_0, če so podatki $A(2,-1,2)$, $B(1,2,1)$, $C(2,3,0)$, $D(5,0,-6)$

piramida je 1/6 prizme

$$= \frac{1}{6}(\vec{AB}, \vec{AC}, \vec{AD})$$

$$V = \frac{1}{3} \text{ pl. os. ploskve} \cdot h$$

$$h = \frac{3V}{\text{pl. osn.}} = \frac{3 \cdot \frac{1}{6}(\vec{AB}, \vec{AC}, \vec{AD})}{\frac{1}{2}|\vec{AB} \times \vec{AC}|}$$

$$AB, AC, AD = \begin{vmatrix} -1 & 3 & -1 \\ 0 & 4 & -2 \\ 3 & 1 & -8 \end{vmatrix} \underset{+3V_1}{=} \begin{vmatrix} -1 & 3 & -1 \\ 0 & 4 & -2 \\ 0 & 10 & -11 \end{vmatrix} = -1(-44+20) = 24$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 3 & -1 \\ 0 & 4 & -2 \end{vmatrix} = \cancel{-2i-2} \quad (-2, -2, -4)$$

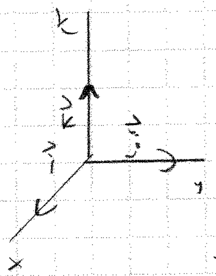
$$|c| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

$$h = \frac{24}{2\sqrt{6}} = \frac{24\sqrt{6}}{6 \cdot 2} = 2\sqrt{6}$$

do 34 je prav, potem ramah

Dokaži, da točke ležijo v isti ravnini – napaka 44

Dokaži da vektorji $e_1 = (1,1,1)$, $e_2 = (1,1,2)$, $e_3 = (1,2,3)$ tvorijo bazo prostora $\mathbb{R}^3$. Izrazi vektor $\vec{a} = (6,9,14)$ v tej bazi?



·Baza poljuben vektor se da izraziti z 3 baznimi.

$$\vec{a} = \alpha_1 e_1 + \alpha_2 (e_2) + \alpha_3 e_3$$

lin. odvisni
$e_1 = i$
$e_2 = j$
$e_3 = i + j$

$i + j - i - j = 0$

Kdaj tvorijo bazo?
– so linearno neodvisni
vsak vektor prostora se da izr. kot lin. komb. baznih vektorjev.

Vsaka baza ima 3 vektorje

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$$
$$\alpha_1 (1,1,1) + \alpha_2 (1,1,2) + \alpha_3 (1,2,3) = 0$$
$$\cancel{\alpha_1 (1+1+1) + \alpha_2 (1+1+2) + \alpha_3 (1+2+3) = 0}$$

$$(\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 + 3\alpha_3) = 0$$

$$-\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \end{cases}$$

$$\alpha_2 + \alpha_3 = 0$$

$\alpha_3 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$ – edina rešitev!!!

kar pa je def linearne neodv.

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$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = (6, 9, 14)$$

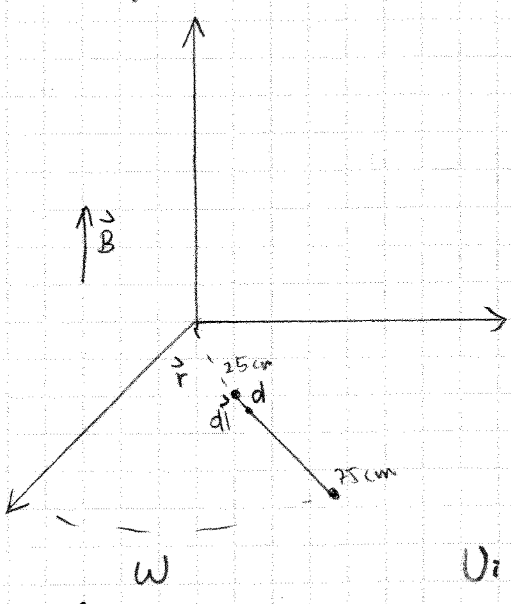
$$(\alpha_1 + \alpha_2 + \alpha_3, \ \alpha_1 + \alpha_2 + 2\alpha_3, \ \alpha_1 + 2\alpha_2 + \alpha_3)$$

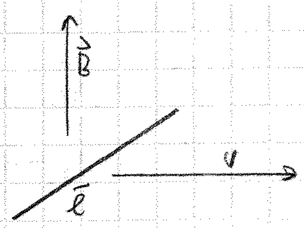
$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 6 \\ \alpha_1 + \alpha_2 + 2\alpha_3 = 9 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = 14 \end{cases} \qquad \begin{aligned} \alpha_3 &= 3 \\ \alpha_2 &= 2 \\ \alpha_1 &= 1 \end{aligned}$$

$$\alpha_2 + \alpha_3 = 5$$
$$\alpha_2 + 3 = 5$$
$$\alpha_2 = 2$$

$\vec{a} = e_1 + 2e_2 + 3e_3$ 1,2,3 so komp. vektorja $\vec{a}$ v bazi e_1, e_2, e_3

Homogeno magnetno polje $B = 0{,}5\,T$ ima navpično smer. MAT II - V 5.3.2008
V polju se po vodoravni ravnini vrti polmetrska palica s frekvenco $\nu = 2\,s^{-1}$ tako da kroži 1 krajišče po radiju 25 cm, drugo pa 75 cm. Kakšna napetost se inducira med krajiščema.





$$U_i = \vec{v} \cdot \vec{l} \times \vec{B}$$
$$U_i = \vec{v} \cdot (\vec{B} \times \vec{l})$$
$$dU_i = \vec{v} \cdot (\vec{B} \times d\vec{l})$$
$$U_i = \int_{25cm}^{75cm} \vec{v} \cdot (\vec{B} \times d\vec{l})$$

$\varphi = \omega t$

$\omega = 2\pi\nu$

$r(t) = (r \cos \varphi, r \sin \varphi, 0)$, $\varphi = \omega t$

$d\vec{r} = (dr \cos\varphi, dr \sin\varphi, 0)$

$\vec{B} = (0, 0, B)$

$v = \dot{\vec{r}}(t) = (-r\omega \cos\omega t, r\omega \cos\omega t$

$\dot{v} \cdot (B \times d\vec{r})$

$$\begin{vmatrix} -r\omega \sin\omega t & r\omega\cos\omega t & 0 \\ 0 & 0 & B \\ dr\cos\omega t, & dr \sin\omega t & 0 \end{vmatrix} =$$

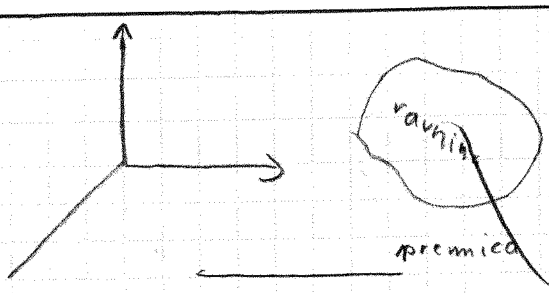
$$B \cdot (-1) \cdot \begin{vmatrix} -r\omega \sin\omega t & r\omega \cos\omega t \\ dr\cos\omega t & dr \sin\omega t \end{vmatrix}$$

$$= -B(-r \cdot dr\,\omega(\sin^2\omega t + \cancel{r\,dr\,\omega} \cos^2\omega t))$$

$$= B r\, dr\, \omega$$

$$U_i = \int B \cdot \omega \cdot r \cdot dr$$

$$U_i = \omega \cdot B \cdot \frac{r^2}{2}\Big|_{0,25}^{0,75} = 0{,}5\,T \cdot 2\pi \cdot 2 \cdot = \cancel{1{,}5}\,V \quad 0.5\pi\,V$$



premica

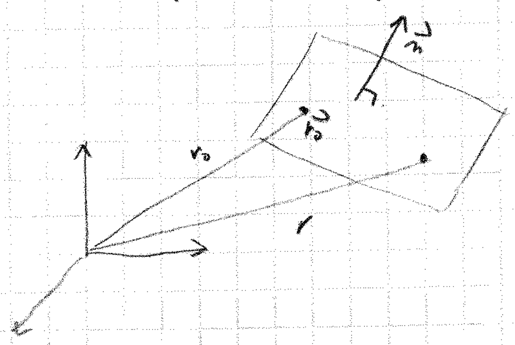
$y = kx + n$

npr. $y = 2x + 3$ $\qquad$ $T(1,5)$ - leži na premici

$(1,6)$ ni na premici

enačba ravnine $\rightarrow$

lega 2 vektorja: $\vec{r_0}$ in $\vec{n}$



$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

$$\Rightarrow ax + by + cz = d$$

Zapiši enačbo ravnine, ki vsebuje točke:

$A(-1, 6, 3)$

$B(3, -2, -5)$

$C(1, 4, 1)$

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$\vec{n} = (4, -8, -8) \times (2, -2, -2)$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 4 & -8 & -8 \\ 2 & -2 & -2 \end{vmatrix} = (\cancel{2}\,0, -8, 8)$$

$$[(x, y, z) - (-1, 6, 3)] \cdot (0, -8, 8) = 0$$

$$(x+1, y-6, z-3) \cdot (0, -8, 8)$$

$$-8(y-6) + 8(z-3) = 0$$

$$-8y + 8z + 24 = 0 \quad /:-8$$

$$y - z - 3 = 0$$

$T(7, 3, 0)$ je v ravnini

Poišči razdaljo med ravninama

$4x - 4y + 2z + 17 = 0$ $\qquad$ ravnini morata biti vzporedni

$4x - 4y + 2z + 2 = 0$

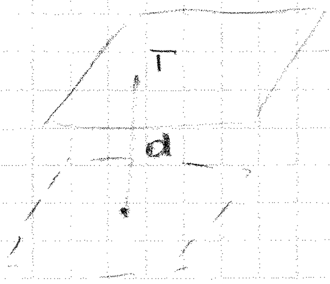
$\vec{n} = (4, -4, 2)$

pomagamo si z razdaljo med:

TOČKAMA

RAVNINO TOČKA

PREMICAMA



na eni ravnini si izberemo poljubno točko na 2 ravnini

$T_1(0,0,-1)$

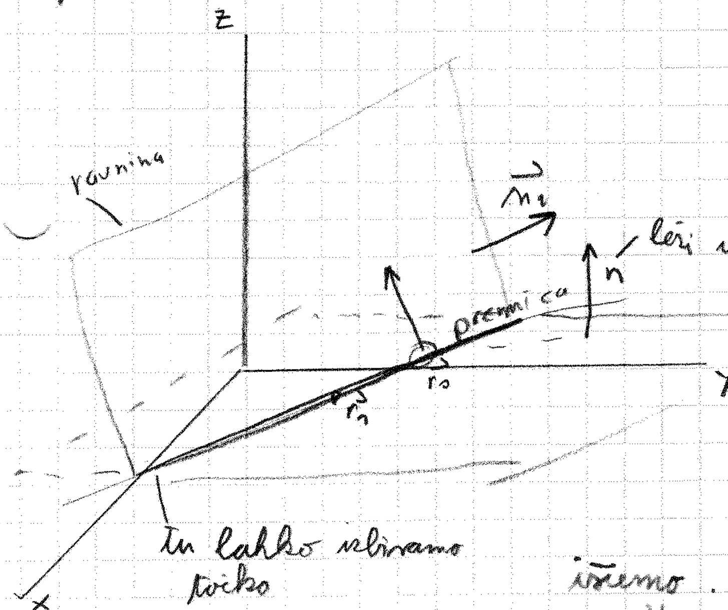
$T_2(0,0,1)$ $\quad \vec{n}(4,-4,2)$

$$\frac{\vec{n}}{|\vec{n}|} = \sqrt{16+16+4} = 6$$

$$\frac{4x-4y+2z+2}{6} = 0$$

$$d = \left|\frac{(r_1 - r_2)\cdot\vec{n}}{|\vec{n}|}\right| \qquad d = \frac{4\cdot 0 - 4\cdot 0 + 2z + 17}{6} = \frac{15}{6} = \frac{5}{2}$$

Zapiši enačbo ravnine, ki ravnino $5x - y + 3z - 2 = 0$ seka v premici, ki leži v ravnini (x,y), pod pravim kotom



$(\vec{r}-\vec{r}_0)\cdot\vec{n} = 0$

$r_0(x,y,0) \quad r_1$

$5x - y + 3\cdot 0 - 2 = 0$

leži v ravnini $\quad 5x - y - 2 = 0$

enačba premice

$T(0,-2,0) \quad r_1(1,3,0)$

$\vec{n}_1 = (5,-1,3)$ je normala na dano ravnino

iščemo $n \perp \vec{n}_1$

-||- $\quad n \perp$ premico $\vec{r}_1 - \vec{r}_0$

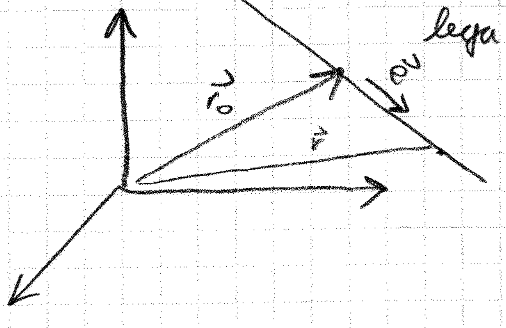
$$\vec{n} = \vec{n}_1 \times (\vec{r}_1 - \vec{r}_0) = \begin{vmatrix} i & j & k \\ 5 & -1 & 3 \\ 1 & 5 & 0 \end{vmatrix} = (-15, 3, 26)$$

enačba ravnine

$(\vec{r} - \vec{r}_0)\cdot\vec{n} = 0$

$[(x,y,z) - (0,-2,0)]\cdot(-15,3,26) = 0$

$-15x + 3y + 26z + 6 = 0$

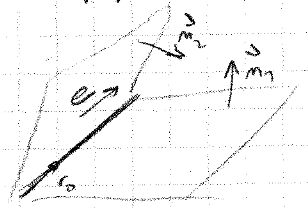


lega: $\vec{n}_0$ $\qquad \vec{r} = \vec{r}_0 + \lambda\vec{e} \quad \lambda \in \mathbb{R}$

$$\frac{x-x_0}{e_x} = \frac{y-y_0}{e_y} = \frac{z-z_0}{e_z}$$

Poišči presečišče ravnin $6x+2y-z=9$ in $3x+2y+2z=12$

$\vec{n}_1 \nparallel \vec{n}_2$



odštejemo eno od druge

$r = r_0 + te$

$r_0 \doteq (0,5,1)$ $\qquad e_1 = k \cdot e_2$

$$\vec{e} = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 6 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = (6,-15,6)$$

$$\frac{x-0}{6} = \frac{y-5}{-15} = \frac{z-1}{6}$$

t_0:

$\vec{e} = (0,5,1) + t(6,-15,6)$

t_1:

t_2:

Določi m tako da se premici sekata:

$$\underbrace{\frac{x-1}{2} = \frac{y-7}{1}} = \frac{z-m}{4}$$

$$\underbrace{\frac{x-6}{3} = \frac{y+1}{-2}} = \frac{z}{1}$$

$P(x,y,z)$ $\quad$ 3 neznanke

$x - 1 = 2y - 14 \quad / \cdot 2$ $\qquad x = -3$

$-2x + 12 = 3y + 3$ $\qquad z = -3$

$+10 = 7y - 25$

$35 = 7y$

$y = 5$

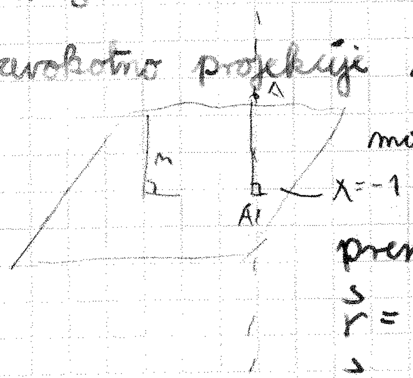
$$\frac{5-7}{1} = \frac{-3-m}{4}$$

$-8 = -3 - m$

$-5 = -m$

$m = 5$

Poišči pravokotno projekcijo $A(4,3,1)$ na ravnino $x+2y-z=3$



moramo iti v smeri normale

premica skozi A in R

$\vec{r} = (4,3,1) + \lambda \cdot \vec{n}$

$\vec{r} = (4,3,1) + \lambda \cdot (1,2,-1)$

Presek premice z ravnino

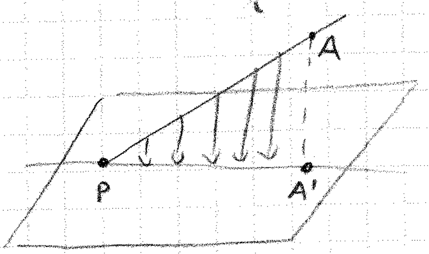
$x = 4 + \lambda$
$y = 3 + 2\lambda$
$z = 1 + \lambda$

$x + 2y - z = 3$
$4 + \lambda + 6 + 4\lambda - 1 + \lambda = 3$
$6\lambda = 3 - 9$
$\lambda = -1$

$x' = 3$
$y' = 1$
$z' = 2$

lahko pa tudi $r = r_A + \overline{AA'}$
$r = r_A + d \cdot \frac{\vec{n}}{|\vec{n}|}$

Poišči pravokotno projekcijo premice z enačbo
$\frac{x-4}{1} = \frac{y-3}{3} = \frac{z-1}{1} = t$ na isto ravnino kot v prejšnji nalogi. ($x + 2y - z = 3$



poiščemo presečišče premice in ravnine

$z = t + 1$
$y = 3t + 3$
$x = t + 4$

$t + 4 + 2(3t + 3) - t - 1 = 3$
$t + 4 + 6t + 6 - t - 1 = 3$
$6t = -6$
$t = -1$

$P(x, y, z)$ $x = 3$
$y = 0$
$z = 0$

enačba premice

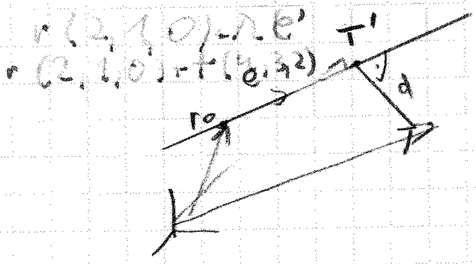
$r = (3, 0, 0) + t\vec{e}$
$r = (3, 0, 0) + t(0, 1, 2)$

$e = \vec{r}_{A'}$ $A'(3, 1, 2)$

$\frac{x-3}{0} = \frac{y}{1} = \frac{z}{2}$
$x = 3$

Kolikšna je oddaljenost točke $P(7, 9, 7)$ od premice
$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z}{2}$

$d(T, P) = \frac{|r_T - r_0 \times \vec{e}|}{|\vec{e}|}$ $|e| = \sqrt{16 + 9 + 4} = \sqrt{29}$



ravnina skozi T ⊥ na premico

$(\vec{r} - (7, 9, 7)) \cdot \vec{n} = 0$ $r =$

$\vec{n} = (2, 1, 0) + t(4, 3, 2)$

$(r - (7, 9, 7)) \cdot (4, 3, 2)$

Presek s premico T' $[(2, 1, 0) + t(4, 3, 2) - (7, 9, 7)] \cdot (4, 3, 2) = 0$

$d(T, T')$

$(-5, -8, -7) \cdot (4, 3, 2) + t \cdot 29 = 0$

$29t = 62$
$t = \frac{62}{29}$

$T' = (2 + \frac{4 \cdot 62}{29}, \ldots, \ldots)$

16 -20
9 -24
4 -14
29 -58

13

MAT II - V MATRIKE
12.3.2008

A+B

$$A=\begin{bmatrix}3&-2&1\\-1&1&-1\\2&0&-2\end{bmatrix}\qquad B=\begin{bmatrix}1&2&3\\2&4&6\\1&2&3\end{bmatrix}$$

$$A+B=\begin{bmatrix}4&0&4\\1&5&5\\3&2&1\end{bmatrix}\qquad \begin{bmatrix}2&4\\5&7\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}= \text{ni def}$$

kA

$$7\begin{bmatrix}2&4\\5&7\\6&1\end{bmatrix}=\begin{bmatrix}14&28\\35&49\\42&7\end{bmatrix}$$

$$\underset{m\times n}{A}\cdot\underset{n\times p}{B}=\underset{m\times p}{C}\qquad c_{ij}=\sum_{k=1}^{n} a_{ik}\cdot b_{jk}$$
$i=1,2,3,\dots,m$
$j=1,2,3,\dots,p$

$$\begin{bmatrix}2&3&4\\1&2&3\end{bmatrix}\cdot\begin{bmatrix}8\\9\\10\end{bmatrix}=\begin{bmatrix}\ \end{bmatrix}$$

A·B

$$A=\begin{bmatrix}3&-2&1\\-1&1&-1\\2&0&-2\end{bmatrix}\qquad B=\begin{bmatrix}1&2&3\\2&4&6\\1&2&3\end{bmatrix}$$

$$A\cdot B=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$c_{11}=a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{31}=3\cdot1+(-2)\cdot2+1\cdot1=0$$

$$B\cdot A=\begin{bmatrix}1&2&3\\2&4&6\\1&2&3\end{bmatrix}\begin{bmatrix}3&-2&1\\-1&1&-1\\2&0&-2\end{bmatrix}=\begin{bmatrix}7&0&-7\\14&&\\7&0&-7\end{bmatrix}=\begin{bmatrix}7&0&-7\\14&0&-14\\7&0&-7\end{bmatrix}$$

1. $A\cdot B=0$ 2. $A\cdot B=B\cdot A$

$X^2-3X+2=0$
$(X-I)(X-2I)=0$

$X_1=I$

$X_2=2\cdot I$

$X^2 = I \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad x^2 = 1 \quad x_{1,2} = \pm 1$

$X_{1,2} = \pm I$

$X = \begin{bmatrix} x & y \\ u & v \end{bmatrix}. \qquad X^2 = \begin{bmatrix} x & y \\ u & v \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} x^2+yu & xy+yv \\ ux+vu & uy+v^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1 $x^2 + yu = 1$ $\qquad x^2 = 1 \qquad$ 1. reš. $\quad y = 0 \quad x^2 = 1 \quad v = \pm 1$

$u(x+v) = 0$

2 $xy + yv = 0 \qquad y(x+v) = 0$

$y = 0 \qquad x+v = 0$

3 $ux + vu = 0$ $\qquad$ 1 a) $y = 0 \quad u = 0 \qquad x = \pm 1 \quad v = \pm 1$

4 $uy + v^2 = 1$

b $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad$ c $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

1 d $\quad u \neq 0$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ali $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} y = 0 \\ u \neq 0 \\ u = \pm 1 \\ v = \mp 1 \end{matrix}$

2. $\boxed{x+v=0}$ $\quad y$ polj.

eliminiramo v

1 $x^2 + yu = 1$

4 $uy + x^2 = 1$ $\quad$ isto $\Rightarrow x + v = 0$

nešk. mnogo rešitev

$X = \begin{bmatrix} x & y \\ u & -x \end{bmatrix} \qquad v = -x \qquad X = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

kjer je $\det(X) = -1$

Dana je matrika in poišči rang matrike

$A = \begin{bmatrix} 1 & 2 & 1 & -1 & 7 \\ 2 & 4 & 3 & 0 & 6 \\ 3 & 6 & 3 & -3 & 21 \\ 4 & 8 & 6 & 0 & 12 \end{bmatrix}$

= dimenzija največje poddet. $\neq 0$, vse poddet.

matriko predelamo v matriko s čim več ničlami - osnovne operacije

imata isti rang $\sim \begin{bmatrix} & & \\ & \emptyset & \\ & & \end{bmatrix}$

$= \begin{array}{c} \\ -2V_1 \\ -3V_1 \\ -4V_1 \end{array}\begin{bmatrix} 1 & 2 & 1 & -1 & 7 \\ 0 & 0 & 1 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 & 7 \\ 0 & 0 & 1 & 2 & -8 \\ 0 & 0 & 2 & 4 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot (-2V)$

$\sim \begin{bmatrix} 1 & 2 & 1 & -1 & 7 \\ 0 & 0 & 1 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad$ det Δ matrike $\quad \begin{vmatrix} 5 & 6 & 9 \\ 0 & 6 & 3 \\ 0 & 0 & 7 \end{vmatrix} = 5 \cdot 6 \cdot 7$

$= 0$

rang je 2.

Rang matrike

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -6 \\ 0 & -5 & 5 & -6 \\ 0 & 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ rang je 2

LINEARNI SISTEM ENAČB

$$x+3y-z=-2$$
$$2x+4y+3z=3$$
$$3x-2y+5z=13$$

Koliko so x, y, z

Gaussova eliminacija

$$x+3y-z=-2$$
$$-2y+5z=7 \quad /\cdot 11$$
$$-11y+8z=19 \quad /\cdot(-2) \quad +$$

$$x+3y-z=-2$$
$$-2y+5z=7$$
$$39z=39$$

$$z=1$$
$$y=-1$$
$$x=2$$

$B=\begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix}$

Lahko pa kot matrike

, matrika koeficientov

$$\underbrace{\begin{bmatrix} 1 & 3 & -1 & \vdots & -2 \\ 2 & 4 & 3 & \vdots & 3 \\ 3 & -2 & 5 & \vdots & 13 \end{bmatrix}}_{\text{RAZŠIRJENA MATRIKA} = M} \sim \Big[$$

B

rang (A) : rang (M)

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B=\begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix}$$

Dana je matrika

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \sim \Big\{ \qquad B=\begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \end{bmatrix}$$

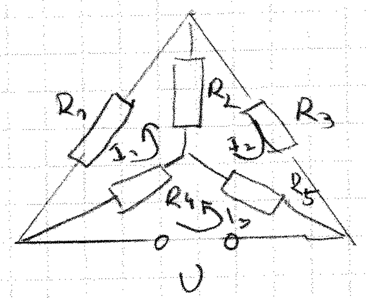
$$\underset{4\times 4}{A}\cdot\underset{4\times 1}{X}=\underset{4\times 1}{B}$$

$$X=$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 11 \\ 2 & 3 & 4 & 1 & \vdots & 12 \\ 3 & 4 & 1 & 2 & \vdots & 13 \\ 4 & 1 & 2 & 3 & \vdots & 14 \end{bmatrix}$$

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & 11 \\ 0 & -1 & -2 & -7 & -10 \\ 0 & \boxed{-2} & -8 & -10 & -20 \\ 0 & \boxed{-7} & -10 & -13 & -30 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 11 \\ 0 & -1 & -2 & -7 & \vdots & -10 \\ 0 & 0 & -4 & 4 & \vdots & 0 \\ 0 & 0 & 4 & 36 & \vdots & 40 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 11 \\ 0 & -1 & -2 & -7 & \vdots & -10 \\ 0 & 0 & -4 & 4 & \vdots & 0 \\ 0 & 0 & 0 & 40 & \vdots & 40 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ +0 \\ +1 \\ 1 \end{bmatrix}$$



$R_1 = 1\Omega$
$R_2 = 2\Omega$
$R_3 = 2\Omega$
$R_4 = 2\Omega$
$R_5 = 3\Omega$
$U = 1,5V$

I_1, I_2, I_3

$$\cancel{E_1}: \quad (R_1+R_2+R_3) I_1 - R_2 I_2 - R_3 \cdot I_3 = 0$$

$$-I_1 R_2 + I_2 (R_2 + R_3 + R_5) - R_5 I_3 = 0$$

$$-I_1 \cdot R_4 - I_2 \cdot R_5 \qquad + (R_4+R_5) \cdot I_3 = U$$

$$\begin{bmatrix} 5 & -2 & -2 & \vdots & 0 \\ -2 & 7 & 3 & \vdots & 0 \\ -2 & -3 & 5 & \vdots & 1,5 \end{bmatrix} \sim \begin{bmatrix} 5 & -2 & -2 & \vdots & 0 \\ 0 & 31 & -19 & \vdots & 0 \\ 0 & -10 & 8 & \vdots & 1,5 \end{bmatrix} \begin{array}{c} \\ +10 \\ +31 \end{array}$$

$$\sim \begin{bmatrix} 5 & -2 & -2 & 0 \\ 0 & 31 & -19 & 0 \\ 0 & 0 & 58 & 46,5 \end{bmatrix}$$

$$I_3 = \frac{46,5}{58}$$

$$I_2 = \frac{283,5}{31} = 28,5 \, A \, 58$$

$$I_1 = \cancel{30 A} \quad \frac{150}{58 \cdot 5} = \frac{30}{58} = \frac{15}{29}$$

$M = [A \vdots B) \qquad AX = B$

$X =$

$$x = \begin{bmatrix} 0 \\ 2 \\ 5/3 \\ \cdot \end{bmatrix}$$

$-\frac{25}{3} - \frac{20}{3} = -\frac{45}{3} = -9$

$= 2$

$+\frac{8}{3} = 1$

$= -\frac{5}{3}$

$$M = \begin{bmatrix} 2 & -1 & +1 & -1 & \vdots & 1 \\ 2 & -1 & 0 & -3 & \vdots & 2 \\ 3 & 0 & -1 & 1 & \vdots & -3 \\ 2 & 2 & -2 & 5 & \vdots & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 & -1 & \vdots & 1 \\ 0 & 0 & -1 & -2 & \vdots & 1 \\ 0 & 3 & -5 & 5 & \vdots & -9 \\ 0 & 3 & -3 & 6 & \vdots & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 1 & -1 & \vdots & 1 \\ 0 & 3 & -5 & 5 & \vdots & -9 \\ 0 & 0 & -1 & -2 & \vdots & 1 \\ 0 & 0 & 2 & 1 & \vdots & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 & -1 & \vdots & 1 \\ 0 & 3 & -5 & 5 & \vdots & -9 \\ 0 & 0 & -1 & -2 & \vdots & 1 \\ 0 & 0 & 0 & -3 & \vdots & 4 \end{bmatrix}$$

$3x-2y-3z=5$
$x-3y+4z=1$
$7x-7y-2z=0$

$$\begin{bmatrix} 3 & -2 & -3 & \vdots & 5 \\ 1 & -3 & 4 & \vdots & 1 \\ 7 & -7 & -2 & \vdots & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & -3 & \vdots & 5 \\ 0 & 7 & -15 & \vdots & 2 \\ 0 & -7 & 15 & \vdots & -35 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & -3 = 5 \\ 0 & 7 & -15 = 2 \\ 0 & 0 & 0 = -33 \end{bmatrix}$$

$0 \cdot z = -33$ Ni re.
z

Rang (A) ≤ rang (M) razširjena matrika

Če sta ranga različna je sistem nerešljiv. Če pa sta ranga enaka pa je sistem rešljiv.
= število neodvisnih enačb.

Število neznank - rang = število poljubnih neznank

rang (A) = 2
rang (M) = 3

$$M = \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 2 \\ 3 & -2 & -1 & -1 & \vdots & -1 \\ 5 & -3 & -4 & -2 & \vdots & -4 \\ 7 & -4 & -7 & -5 & \vdots & -7 \end{bmatrix}$$

$AX = B$

$X = ?$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -7 & -7 & \vdots & -7 \\ 0 & 2 & -14 & -12 & \vdots & -14 \\ 0 & 3 & -21 & -19 & \vdots & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 2 \\ 0 & 1 & -7 & -7 & \vdots & 7 \\ 0 & 0 & 0 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 2 & \vdots & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$1 \cdot \cancel{x_2 = -7}$ $\quad x_2 - 7x_3 - 7x_4 = 0$ $\quad x_3 = \alpha$
$2x_4 = 0$ $\quad x_4 = 0$ $\quad x_2 = 7\alpha - 7$

$x_1 - 7\alpha + 7 + 2\alpha = 2$

$x_1 = 5\alpha - 7$

$$X = \begin{bmatrix} 5\alpha - 5 \\ 7\alpha - 7 \\ \alpha \\ 0 \end{bmatrix} \quad \alpha \in \mathbb{R}$$

Reši sistem enačb:

$x + ay + z = 4$
$x + y + bz = 3$
$x + y + 2bz = 4$

$$\begin{bmatrix} 1 & a & 1 & \vdots & 4 \\ 1 & 1 & b & \vdots & 3 \\ 1 & 1 & 2b & \vdots & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 1 & 4 \\ 0 & 1-a & b-1 & -1 \\ 0 & 0 & b-1 & 1 \end{bmatrix}$$

reši za $a, b \in \mathbb{R}$

$b=1$

$b \cdot z = 1$

1. $z = \frac{1}{b}$ $\quad b \in 0 \quad$ če $b \neq 0$ $\qquad$ 2. $b = 0$

NI REŠITEV

$(1-a)y + (b-1)z = -1$

$(1-a)y = -1 - \frac{b-1}{b}$

če $a \neq 1$ $\quad y = \frac{-2b+1}{b(1-a)}$

$x + a\,\frac{-2b+1}{b(1-a)} + \frac{1}{b} = 4$

$x = 4 - \frac{2b-1-(1-a)}{b(1-a)}$

$x = \frac{4b - 4ba - a + 2ab - 1 + a}{b(1-a)}$

$x = \frac{4b - 2ba - 1}{b(1-a)}$

3. $a = 1 \quad b \neq 0 \quad z = 1/b$

$0 \cdot y = -1 - \frac{b-1}{b}$

$1 = -\frac{b-1}{b}$

$-b = b - 1$

$1 = 2b$

$b = 1/2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right]$$

$x_1 + y + z = 4$

$x + y + 2 = 4 \qquad z = 2$

$x + y = 2$

y - poljuben

$x = 2 - y$

4. $b \neq 0 \quad a = 1$

$b \neq 1/2$

NI REŠITVE

Homogen sistem enačb vedno rešljiv z 1 rešitvijo

$A \cdot X = 0 \qquad X = 0$ je ena rešitev

če je $\det(A) \neq 0$ imamo rešitev. Reši se na običajen način

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & 0 \\ 4 & 5 & -2 & 3 & 0 \\ 3 & 8 & 24 & -19 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -3 & -18 & 15 & 0 \\ 0 & 2 & 12 & -10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rang 2

nez. rang

$4 - 2 = 2$ - polj. neznank

$-x_2 - 6x_3 + 5x_4 = 0$

$x_3 = \alpha$

$x_4 = \beta$

$x_2 = 5x_4 - 6x_3$

$x_1 + 2x_2 + 4x_3 - 3x_4 = 0$

$x_1 = 5\alpha - 6\beta - 4\alpha + 3\beta$

$x_1 = \alpha - 3\beta$

$$X = \begin{bmatrix} 8x_3 - 7x_4 \\ 5x_4 - 6x_3 \\ x_3 \\ x_4 \end{bmatrix} \qquad x_3, x_4 \in \mathbb{R}$$

Inverzna matrika:

$7\,(7^{-1}) = 1$ $\qquad A \cdot A^{-1} = I$ $\qquad AB = I$, $B \cdot A \neq I$

premalo:

$$A^{-1} \cdot A = A \cdot A^{-1} = I$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} & & \\ & \tilde{A}_{ij} & \\ & & \end{bmatrix}^{T}$$

če det(A) različna od 0

$\tilde{A}_{ij}$ – kofaktorji $= (-1)^{i+j} \underset{j}{\left| A \right|}$

Naloga

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{vmatrix}$$

$\det = -1$

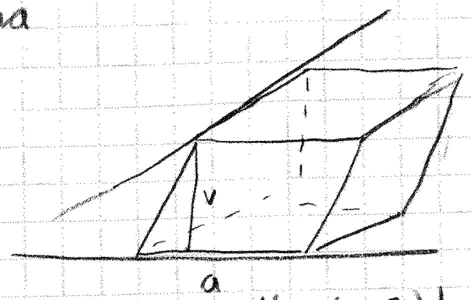
$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -1 & 1 \\ 4 & +5 & +6 \\ 3 & 3 & -4 \end{bmatrix}^{T} = -1 \begin{bmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & +6 & +4 \end{bmatrix}$$

SLED MATRIKE – VSOTA VSEH LASTNIH VREDNOSTI

* Poišči razdaljo med premicama

$$\frac{x-4}{1} = \frac{y-2}{4} = \frac{z-3}{4} \quad e_1$$

$$\frac{x-4}{1} = \frac{y-3}{3} = \frac{z-3}{1} \quad e_2$$



$$v = \frac{\text{Volumen}}{\text{Pl osn.}} = \frac{|(a,b,c)|}{|a \times b|} \qquad a = e_1, \; b = e_2$$

$$= \frac{(\vec{e}_1, \vec{e}_2, \vec{r}_2 - \vec{r}_1)}{|e_1 \times e_2|}$$

$$= \begin{vmatrix} 1 & 4 & 4 \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1(1-4) = 3$$

$$|e_1 \times e_2| = \begin{vmatrix} i & j & k \\ 1 & 4 & 4 \\ 1 & 3 & 1 \end{vmatrix} = -8, 3, -1$$

$$d = \sqrt{64+9+1} = \sqrt{74}$$

$$v = \frac{3}{\sqrt{74}} = \frac{3}{\sqrt{2 \cdot 37}}$$

* Kateremu pogoju morajo zadoščati a, b in c, da bo sistem rešljiv

$$\begin{aligned} x + y + 2z &= a \\ -2x - z &= b \\ x + 3y + 5z &= c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ -2 & 0 & -1 & b \\ 1 & 3 & 5 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & 3 & 2a+b \\ 0 & 2 & 3 & c-a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 2 & 3 & 2a+b \\ 0 & 0 & 0 & c-b-2a \end{array}\right] \Rightarrow c-b-3a=0$$

Za izbiro $a=1$, $b=1$ in $c=4$ sistem tudi reši

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\cancel{2x}$ $\quad 2y + 3z = 3 \qquad x + y + 2z = 1$

$$y = \frac{3-3z}{2} \qquad x + \frac{3-3z}{2} + 2z = 1$$

$$-x = \frac{-2+3-3z+4z}{2}$$

$$-x = \frac{1+z}{2}$$

$$x = \frac{-z-1}{2}$$

Določi x, da ne ostaja A^{-1}

$$A = \begin{bmatrix} 2 & x+3 & 4 \\ x+1 & x+2 & x+3 \\ 3 & 4 & 5 \end{bmatrix} \qquad A \cdot A^{-1} = I$$

$$A = \frac{1}{\det}\left[A_{ij} \right]^T \qquad \det A = 0$$ da nima inverzne matrike

$$\begin{vmatrix} 2 & x+3 & 4 \\ x+1 & x+2 & x+3 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & x+3 & 4 \\ x+1 & x+2 & x+3 \\ 1 & 1-x & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & x+3 & 2 \\ x+1 & x+2 & 2 \\ 1 & 1-x & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & x+3 & 2 \\ x-1 & -1 & 0 \\ 1 & 1-x & 0 \end{vmatrix}$$

$$\cancel{= 2 \cdot \begin{vmatrix} 2 & x+3 \\ x-1 & -1 \end{vmatrix} = 2 \cdot (-2)}$$

$$= 2\,(x-1)(1-x) + 1 =$$
$$= 2\,(\cancel{x} + x - x^2 - 1 + x) + 1$$
$$= -2x^2 + 4x = 2\,(x - x^2)$$

$$x_1 = 0$$
$$x_2 = 2$$

Ali je dana matrika ~~otrgonalno~~ ortogonalna

ko

$$A^{-1} = A^T$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

1. $A^{-1} = \frac{1}{\det}\left[A_{ij} \right] \stackrel{?}{=} A^T$

2. $A^T =$ in $\underbrace{A^{-1}A} = A^{-1}A^{-1} = I$

$$A^T A = I = A \cdot A^T$$

ali je

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A \cdot A^T \overset{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{6}\right)$$

$$A^T \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ je ortogonalna

LINEARNE PRESLIKAVE

$$\vec{x} \longrightarrow \vec{y} \qquad \vec{y} = f(\vec{x})$$

1. $f(\alpha \vec{x}) = \alpha f(\vec{x})$
2. $f(\vec{x}_1 + \vec{x}_2) = f(\vec{x}_1) + f(\vec{x}_2)$

$f(i)$, $f(j)$, $f(k)$ — lahko dobimo sliko kateregakoli vektorja

Linearna preslikava je množenje z matriko A.

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$\vec{x} \longmapsto$ ~~axx~~ $\vec{a} \times \vec{x}$ Poišči matriko te linearne preslikave?

$$\vec{a} \times (\vec{x}_1 + \vec{x}_2) = \vec{a} \times \vec{x}_1 + \vec{a} \times \vec{x}_2$$
$$\vec{a} \times (k\vec{x}) = k(\vec{a} \times \vec{x})$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \vec{a} \times \vec{x} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = (3x_3 - x_2)i - (2x_3 - x_1)j + (2x_2 - 3x_1)k$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 - x_2 \\ x_1 - 2x_3 \\ 2x_2 - 3x_1 \end{bmatrix}$$

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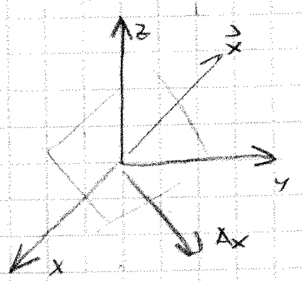
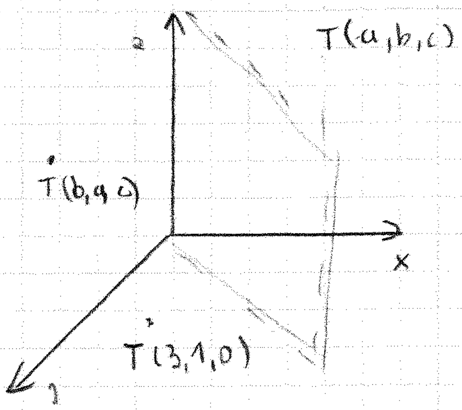
$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}$$

Naj bo A transformacija v $\mathbb{R}^3$ $A: \mathbb{R}^3 \to \mathbb{R}^3$, ki predstavlja zrcaljenje preko ravnine, ki vsebuje koordinatno izhodišče.

$x - y = 0$ Poišči matriko te linearne preslikave



$\vec{n} = (1,-1,0)$ $\qquad$ $T(0,0,0)$ je v ravnini
$\qquad\qquad\qquad\qquad\quad$ $T(0,0,z)$



$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{bmatrix}$$ poišči lastne vrednosti in lastne vektorje matrike.

LASTNE VR.

$AX = \lambda X$, kjer $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_n \end{bmatrix}$

$3\times 3 \quad 3\times 1$

$3\times 1 \quad \lambda = 3\times 1$

$\begin{bmatrix} \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \\ \end{bmatrix}$ $\quad \Rightarrow (A-\lambda I)X = 0$

homogen sistem enačb

1. $\det(A-\lambda I) = 0$
2. λ_1, λ_2
3. X

$$\Rightarrow A - \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \left|\begin{matrix} 5-\lambda & 2 & -3 \\ 4 & 5-\lambda & -4 \\ 6 & 4 & -4-\lambda \end{matrix}\right|$$

$$= (5-\lambda)(5-\lambda)(-4-\lambda) + \overset{-48}{2(-4)\cdot 6} + \overset{-48}{(-3)\cdot 4\cdot 4}$$
$$- (-3)\cdot 6\cdot(5-\lambda) - 2\cdot 4\cdot(-4-\lambda) - (5-\lambda)\cdot(-4)\cdot 4$$

$$= (5-\lambda)^2(-4-\lambda) - 48 - 48 + 18(5-\lambda) - 8(-4-\lambda) + 16(5-\lambda)$$

$$= (25-10\lambda+\lambda^2)(-4-\lambda) - 96 + 90 - 18\lambda + 32 + 8\lambda + 80 - 16\lambda$$

$= -100 + 40\lambda - 4\lambda^2 - 25\lambda + 10\lambda^2 - \lambda^3 + 106 - 26\lambda$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$$\begin{array}{c|cccc} & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline 1 & -1 & 5 & -6 & 0 \end{array}$$

$= (\lambda - 1)(-\lambda^2 + 5\lambda - 6) = 0$

$= (\lambda - 1)(\lambda - 3)(-\lambda + 2)$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 3$

$(A - \lambda I)x = 0$

$\lambda = 1 \Rightarrow x_1$

$$\begin{array}{c} 6 \\ \\ 4 \end{array}\left[\begin{array}{ccc:c} 4 & 2 & -3 & 0 \\ 4 & 4 & -4 & 0 \\ 6 & 4 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 4 & 2 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 4 & 2 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -0 & 0 \end{array}\right]$$

$2x_2 - x_3 = 0$ $\qquad$ $4x_1 + 2\alpha - 6\alpha = 0$

$x_3 = 2x_2$ $\qquad$ $x = \alpha$

$z' \quad y' = \alpha$

$$x_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda = 2$

$$\begin{array}{c} -4 \\ 3 \\ \\ \end{array}\left[\begin{array}{ccc:c} 3 & 2 & -3 & 0 \\ 4 & 3 & -4 & 0 \\ 6 & 4 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 3 & 2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = 0$

$3x + 2y - 3z = 0$

$3x - 3z = 0$

$3x = 3z$

$x = z = \alpha$

$$X_2 = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$$\left[\begin{array}{ccc:c} 2 & 2 & -3 & 0 \\ 4 & 2 & -4 & 0 \\ 6 & 4 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & 2 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$-2y + 2z = 0$ $\qquad$ $2x + 2\alpha - 3\alpha = 0$

$z = \alpha$ $\qquad$ $2x = \alpha$

$y = \alpha$ $\qquad$ $x = \alpha/2$

$$X_3 = \alpha \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$X_3' = 2\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

POTENČNE VRSTE

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| \qquad \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$\sum_{n=1}^{\infty} a_n \quad \text{funkc. vrsta: } \sum_{n=1}^{\infty} a_n(x) \quad \text{Konv. območje: } \{x;\ \text{je konv.}\}$$

$x = x_0$

konvergenčno območje potenčne vrste

$|x-x_0| < R$ $\qquad x_0 - R,\ x_0 + R$ je konv.

$\sum \frac{1}{n}$ – harmonična je div

$\sum \frac{1}{n^\alpha}$ je konv., če je $\alpha > 1$

$\sum \frac{p(n)}{q(n)}$ konv., če je razlika stopenj vsaj 2

Imamo splošno funkcijsko vrsto, najdi njeno konvergenčno območje.

$$\sum_{n=1}^{\infty} \frac{1}{n\, 3^n \sqrt{(x+2)^n}}$$

konvergenčno območje: Kvocientni kriterij, primerjalni kriterij, integralski kriterij

$x+2 > 0$

$x > -2$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)3^{n+1}\sqrt{(x+2)^{n+1}}}}{\frac{1}{n\,3^{n}\sqrt{(x+2)^n}}} = \frac{n\sqrt{(x+2)^n}}{(n+1)\,3\sqrt{(x+2)^{n+1}}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}\cdot\frac{1}{3}\sqrt{\frac{1}{(x+2)}}$$

$\frac{n}{n+1} \to 1$

$$\lim = \frac{1}{3\sqrt{x+2}} < 1 \text{ je konv.}$$

$$\frac{1}{3} < \sqrt{x+2}$$

$$\frac{1}{9} < x+2$$

$$\frac{-17}{9} < x \text{ konv.}$$

$$x < -\frac{17}{9}$$

je konv.

$$x > -\frac{17}{9}$$

$$\left(-\frac{17}{9}, \infty\right)$$

$$x = -\frac{17}{9}$$

$$\sum \frac{1}{n\, 3^n \sqrt{\left(\frac{1}{9}\right)^n}} = \sum \frac{1}{n} \text{ vrsta je divergentna}$$

1.b)

$$\sum_{n=1}^{\infty} \frac{n^n}{x^{n^n}} \qquad \lim \sqrt[n]{a_n} = q < 1$$

$$\lim \frac{n}{x^n}$$

POTENČNE VRSTE

Kaj bo konvergenčno območje vrste

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n^2 2^n} = \sum_{n=0}^{\infty} C_n (x-x_0)^n$$

$x_0 = 1$

$C_n = \frac{1}{n^2 2^n}$

$$R = \lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \frac{(n+1)^2 \cdot 2^{n+1}}{n^2 \cdot 2^n} = \lim_{n\to\infty} \frac{(n^2+2n+1) \cdot 2}{n^2} = 2$$

Vrsta je konvergentna, če je $[-1, 3]$

$x=3$ $\sum \frac{2^n}{n^2 2^n}$ konv

$x=-1$ $\frac{(-2)^n}{n^2 2^n}$ je konv.

za alternirajoča

$\sum (-1)^n a_n$ je konv.

1. $\lim_{n\to\infty} a_n = 0$

$\wedge$

2. $a_n \geq a_{n+1}$

$$\sum_{n=0}^{\infty} \frac{1}{3n+1} \left(\frac{2-x}{1+x}\right)^n$$

označi $y = \frac{2-x}{1+x}$ nariši graf

$y \in (a, b)$

$$= \sum \frac{1}{3n+1} y^n = \sum C_n (y-y_0)^n$$

$y_0 = 0$

$$R = \lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{3(n+1)+1}} = 1$$

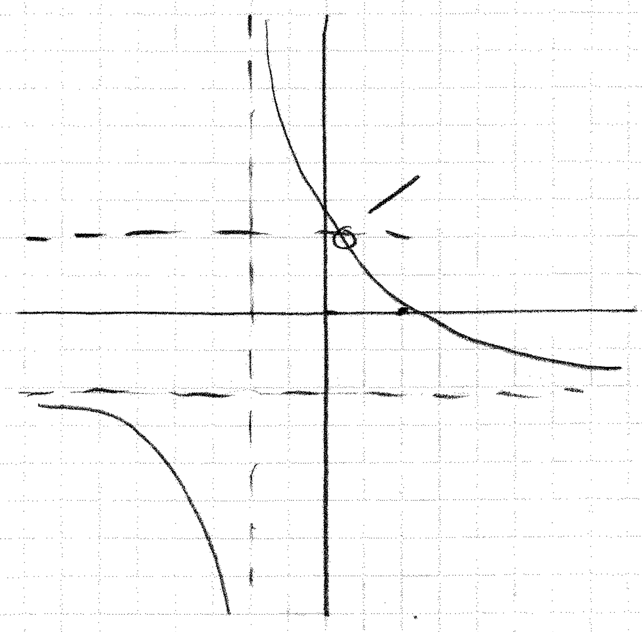
$-1 < y < 1$

$y=1$ $\sum \frac{1}{3n+1}$ div.

$y=-1$ $\sum \frac{(-1)^n}{3n+1}$ konv.

graf: $y = \frac{2-x}{1+x}$

n: $x = 2$
p: $x = 1$
a: $y = -1$



$\frac{2-x}{1-x} = 1$

$2 - x = 1 + x$
$1 = 2x$
$x = 1/2$

$\frac{1}{2} < x$ takrat je konvergenčno območje

$(+1/2, \infty)$

$A \cdot X = B \,/ A^{-1}$
$X = A^{-1} \cdot B$

$[A \,\vdots\, B] \ldots\ldots [I \,;\, BA^{-1}]$

Baza
$a_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $a_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $a_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $b_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} \quad \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$X \cdot A = B \,/^{T}$
$A^T \cdot X^T = B^T$
$X^T = \cancel{B^T} (A^T)^{-1} \cdot B$
$X = ((A^T)^{-1} \cdot B^T)^T$

$$\begin{bmatrix} A^T & \vdots & B^T \end{bmatrix} \ldots \begin{bmatrix} I & \vdots & A^{-1} B^T \end{bmatrix} \quad X^T$$

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$$\left[\begin{array}{ccc|ccc} 2 & 3 & 5 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 & -1 \\ 1 & 0 & 0 & 2 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & -1 \\ 2 & 3 & 5 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & -1 \\ 0 & 3 & 5 & -3 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & +1 & 1 & -1 \\ 0 & 0 & +1 & +6 & +4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & -11 & -7 & -1 \\ 0 & 0 & 1 & 6 & 4 & 0 \end{array}\right] \qquad X = \begin{bmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\Downarrow$$
$$X^T$$

Dan je $\vec{a} = (3,-1,2)$ in linearna preslikava:

$\vec{v} \longrightarrow \vec{a} \times \vec{v}$ Poišči matriko te linearne preslikave in izračunaj še lastne vrednosti matrike in vse lastne vektorje. $\lambda_1 = 0$

$\vec{v} = (x, y, z)$

$$\vec{a} \times \vec{v} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ x & y & z \end{vmatrix} = (-z-2y)\,i - j\,(3z-2x) + k\,(3y+x)$$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z-2y \\ -3z+2x \\ 3y+x \end{bmatrix} \qquad \text{za } \forall (x,y,z)$$

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$$

Lastni vektor:

$A \cdot x = \lambda \cdot x$

$A \cdot x = 0$

rezultat vekt. prod.

$\vec{a} \times \vec{v} = 0$

za $\vec{a} \parallel \vec{v}$

$$\begin{vmatrix} -\lambda & -2 & -1 \\ 2 & -\lambda & -3 \\ 1 & 3 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 6 - 6 - \lambda + 4\lambda + 9\lambda = 0$$
$$-\lambda^3 - 14\lambda = 0$$
$$\lambda^3 + 14\lambda = 0$$
$$\lambda(\lambda^2 + 14) = 0$$

$\lambda = 0 \qquad \lambda_{2,3} = \pm\sqrt{14}\,i$

$$\left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 2 & 0 & -3 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & -6 & -3 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right]$$

$-2x_2 - x_3 = 0$

$x_2 = \alpha$

$x_3 = -2\alpha$

$$\vec{x} = \begin{bmatrix} -3\alpha \\ \alpha \\ -2\alpha \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \qquad \begin{aligned} x_1 + 3\alpha &= 0 \\ x_1 &= -3\alpha \end{aligned}$$

Analiziraj sistem enačb glede na parameter a:

$$\begin{aligned} ax+2y-z&=-1\\ -2x-4y+2z&=2\\ x+y+z&=-2 \end{aligned}$$

če je sistem rešljiv poišči rešitev

$$\left[\begin{array}{ccc|c} a & 2 & -1 & -1\\ -2 & -4 & 2 & 2\\ 1 & 1 & 1 & -2 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2\\ -2 & -4 & 2 & 2\\ a & 2 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2\\ 0 & -2 & 4 & -2\\ 0 & 2-a & -1-a & -1+2a \end{array}\right]\begin{array}{l} \\ \cdot\, 2-a\\ \cdot\, 2 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2\\ 0 & -2 & 4 & -2\\ 0 & 0 & \underbrace{-2-2a+8-4a} & \begin{array}{c}-2+4a\\ -4+2a\end{array} \end{array}\right] 2-a$$

$6-6a=0$

$a=1$

sistem nima rešitev

za $a \neq 1$

$(6-6a)\cdot z = 6a-6$

$z=-1$

$-2y+4\cdot(-1)=-2$

$-2y=2$

$y=-1$

$x+y+z=-2$

$x-1-1=-2$

$x=0$

če $a=1$

$0\cdot z=0$

z je poljuben

$-2y+4z=-2$

$y=2z+1$

$x+2z+1+z=-2$

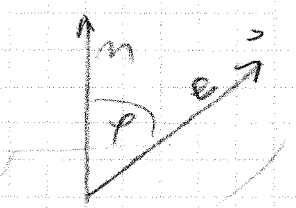
$x=-3z-3$

Izračunaj kot med premico in ravnino

$$\frac{x-1}{2}=\frac{y+2}{1}=\frac{z-3}{1} \qquad x+2y-z=9$$

$\vec{e}: (2,1,1)$

$\vec{n}: (1,2,-1)$



$$\cos\alpha = \frac{\vec{n}\cdot\vec{e}}{|\vec{n}|\cdot|\vec{e}|} = \frac{(1,2,-1)(2,1,1)}{\sqrt{6}\cdot\sqrt{6}} = \frac{2+2-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\alpha = \frac{\pi}{3}$$

$$\beta = \frac{\pi}{2}-\frac{\pi}{3} = \frac{\pi}{6}$$

TAYLORJEVA VRSTA

MAT II V
16. 4. 2008

$f(x) = \sin x$
Taylorjeva vrsta v okolici $x_0 = 0$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

$f(x)$ Taylorjeva vrsta za $f(x)$ v okolici x_0
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \qquad (x_0 - R, x_0 + R)$$
$$= \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \underset{\text{ostanek Taylorjeve vrste}}{R}$$
$n = N+1$
$$R_N = \frac{f^{(N+1)}(\xi)}{(N+1)!} (x - x_0)^{N+1} \qquad \exists\ x_0 < \xi < x$$

osnovne funkcije za Taylorjevo vrsto $x = 0$

konv. za vsak x:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots (-1)^{n-1} \frac{x^{2n+1}}{2n+1!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots (-1)^n \frac{x^{2n}}{2n!}$$

$|x| < 1$:
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - (-1)^{n+1} \frac{x^n}{n}$$
binomska formula:
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$f(x) = x^2 e^{2x}$ zapiši Taylorjevo vrsto $x = 0$ ..... $c_n x^n$

vrsta za e^{2x} $\quad y = 2x$
$$= x^2 \left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!}\right)$$
$$= x^2 \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots\right)$$
$$= x^2 + 2x^3 + \frac{4x^4}{2} + \frac{8x^5}{3 \cdot 2} + \ldots \frac{2^n}{n!} x^{n+2}, \ldots \text{ za } x \in \mathbb{R}$$
$$= x^2 + 2x^3 + 2x^4 + \frac{4x^5}{3} + \ldots \frac{2^{(n-2)}}{(n-2)!} x^n \ldots \quad n+2 = k$$
$$= \sum_{n=1}^{\infty} \frac{2n}{n!} x^{n+2}$$

$f(x) = \cos^2 x$ Taylorjeva vrsta $x_0 = 0$

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots (-1)^n \frac{x^{2n}}{2n!}\right)^2$$

$$= 1 - x^2 + \frac{1}{3}x^4 \qquad \left(\frac{1}{2!}\right)^2 + \frac{2}{4!} = \frac{1}{4} + \frac{2}{24} = \frac{8}{24} = \frac{1}{3}$$

raje na drug način $\cos^2 x - \sin^2 x = \cos 2x$

$$+ \qquad = 1$$

$$2\cos^2 x = 1 + \cos 2x$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \dots (-1)^n \frac{x^{2n}}{2n!}\right) \qquad \frac{\cancel{16}\, 8}{24 \cdot \cancel{2}} = 1/3$$

$$= 1 - x^2 + \frac{1}{3}x^4 + \dots (-1)^n \frac{2^{n-1}}{(2n)!} x^{2n}$$

$$f(x) = \ln(3 - 6x^3) \qquad x_0 = 0$$

$$= \ln 3(1 - 2x^3)$$

$$= \ln 3 + \ln(1 - 2x^3)$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-2x^3)^n}{n}(-1)^{n+1}$$

$$= \ln 3 - \sum_{n} \frac{2^n}{n} x^{3n} \qquad \text{če } |-2x^3| < 1$$

$$2x^3 < 1$$

$$x^3 < 1/2$$

$$|x| < 1/\sqrt[3]{2}$$

Razvij funkcijo

$$f(x) = \frac{x}{1+x^2} \quad v \quad x_0 = 0$$

$$= x \cdot \frac{1}{(1+x)} = x \cdot (1+x)^{-2}$$

$$= x \sum_{n=0}^{\infty} \binom{-2}{n} x^n = \sum_{n=0}^{\infty} \binom{-2}{n} x^{n+1} \qquad |x| < 1$$

$$\binom{-2}{0} = 1 \qquad \binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7 \cdot \cancel{6} \cdot 5\, \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2} \cdot 1\, \cancel{4 \cdot 3 \cdot 2 \cdot 1}} = 35$$

$$\binom{-2}{1} = 1 = -2 \qquad \binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1}$$

$$\binom{-2}{2} \quad \frac{(-2)\cdot(-3)}{2 \cdot 1} = 3$$

$$\binom{-2}{3} \quad \frac{(\cancel{-2})(-\cancel{3})(-4)}{\cancel{3} \cdot \cancel{2} \cdot 1} = -4$$

$$= x - 2x^2 + 3x^3 - 4x^4 + \dots$$

$$f(x)=\frac{3x+5}{x^2+3x+2} \qquad x_0=0$$

damo na parcialne ulomke:

$$\frac{3x+5}{(x+2)(x+1)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$
$$=\frac{A(x+2)+Bx+1}{(x+1)(x+2)}=3x+5$$

$(A+B)=3 \quad = A$
$2A+B=5$
$A=2$
$B=1$

$$=\frac{2}{(x+1)}+\frac{1}{(x+2)}$$
$$=2(1+x)^{-1}+(x+2)^{-1}$$
$$=2(1+x)^{-1}+(1+(1+x))^{-1}$$
$$=2(1+x)^{-1}+\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$
$$=2\sum_{n=0}^{\infty}\binom{-1}{n}x^n+\frac{1}{2}\sum_{n=0}^{\infty}\binom{-1}{n}\left(\frac{x}{2}\right)^n$$

$|x|<1 \qquad =(-1)^n \qquad |x|<2$

$$=\sum_{n=0}^{\infty}\binom{-1}{n}\left(2+\frac{1}{2^{n+1}}\right)x^n$$

$$\binom{-1}{3}=\frac{-1(-2)(-3)}{3\cdot 2\cdot 1}=-1$$
$$\binom{-1}{4}=\frac{(-1)(-2)(-3)(-4)}{4\cdot 3\cdot 2\cdot 1}=1$$

$$f(x)=\sum(-1)^n\left(2+\frac{1}{2^{n+1}}\right)x^n \quad \text{če} \quad |x|<1$$

razvij $f(x)$

$$f(x)=\int_0^x\frac{e^{-t^2}-1}{t}\,dt \text{ okoli } x_0=0$$

$$e^{-t^2}=\sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} \qquad e^x=\sum\frac{x^n}{n!} \quad x=-t^2$$

$$=1+\sum_{n=1}^{\infty}\frac{(-t^2)^n}{n!}-1$$

$$e^{-t^2}-1=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^{2n} \,/:t$$

$$\frac{e^{-t^2}-1}{t}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^{2n-1} \quad \int_0^x dt$$

$$\int_0^x\frac{e^{-t^2}-1}{t}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{x^{2n}}{2n} \quad \text{za } \forall x$$

$$\int_0^x t^{2n-1}\,dt=\left.\frac{t^{2n}}{2n}\right|_0^x=$$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2\cdot\sin(x^2)}$$

po Lopitalu:

$$\lim_{x\to 0}\frac{+2\cos x\sin x}{2x\sin(x^2)+x^2\cos(x^2)2x}$$

raje razvijmo v Taylorjevo vrsto:

$$= \lim_{x\to 0} \frac{1-\left(1-x^2+\frac{1}{3}x^4-\dots\right)}{x^2\cdot\left(x^2-\frac{x^6}{3!}-\dots\right)} = \qquad \sin x = x - \frac{x^3}{3!}$$

$$= \lim_{x\to 0} \frac{x^2-\frac{1}{3}x^4-\dots}{x^4-\frac{x^8}{3!}} = \lim_{x\to 0}\frac{\cancel{x^2}\left(1-\frac{1}{3}x^2+\dots\right)}{x^{4\,2}\left(1-\frac{x^4}{4!}+\dots\right)} = \infty$$

Izračunaj $\cos 1^\circ$ z natančnostjo $\varepsilon = 10^{-6}$ - toliko dec. natančno

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!} \qquad 360 = 2\pi$$

$$x = \frac{\pi}{180} \qquad 1^\circ = \frac{\pi}{180}$$

$$\text{Kdaj bo } \frac{\left(\frac{\pi}{180}\right)^{2n}}{2n!} = \frac{\frac{\pi^4}{180^4}}{24} \simeq \frac{1}{60^4\cdot 24} = \frac{1}{6^4\cdot 24\cdot 10^{-4}} < 10^{-6}$$

Razvij funkcijo $\operatorname{arctg} x$ $\quad x_0 = 0$

$$x = 1$$
$$\operatorname{arctg} 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$
$$= \sum_{n=0}^{\infty}\binom{-1}{n}x^{2n} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\operatorname{arctg} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

$$x^2 - 1$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\right)$$

MAT II
23. 4. 2008

Izračunaj vsoto vrste

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\dots\frac{x^n}{n}+\dots\dots = \int\frac{dx}{1-x}$$

$$= \int 1+x+x^2+x^3+\dots\dots+x^{n-1}+\dots\dots = \frac{a_0}{1-q} = \frac{1}{1-x} \qquad \Big/\int dx$$

geometrijska vrsta $q = x$ $\qquad |x|<1$

$$= \int\frac{dx}{1-x} = \frac{-dt}{t} = -\ln(1-x)+C$$

$x=0 \quad C=0$

vse potenčne vrste so enakomerno konvergentne - lahko odvajamo in integriramo.

Izračunaj vsoto številske vrste

$$\frac{1}{2}+\frac{3}{2\cdot 2}+\frac{5}{2\cdot 3}+\dots\frac{2n+1}{2(n+1)}+\dots$$

$$\sum \frac{2n+1}{2(n+1)} \qquad \lim a_n \neq 0 = 1$$

Izračunaj vsoto

$$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{8}+\dots\frac{2n-1}{2^n}+\dots \qquad \sum a_n(x) = \sum c_n x^n$$

$$x \quad 3x^2 \quad 5x^3 \qquad = \frac{2x}{(1-x)^2} - \frac{x}{(1-x)} \qquad (2n-1)x^n \qquad x=\frac{1}{2}$$

$$\Rightarrow \quad 2x+4x^2+6x^3+8x^4 \quad /\cdot x$$

$$2+4x+6x^2+8x^3 \quad /\int$$

$$\sum = \frac{1}{(\frac{1}{2})^2} - \frac{1/2}{1/2} = 3$$

Stevilna

$$2x+2x^2+2x^3+2x^4 = \frac{2x}{1-x} \qquad \frac{2-2x+2x}{(1-x)^2} \quad X \quad 1/2$$

$$2+4x+6x^2+8x^3+\dots + \quad = $$

$$2x+4x^2+6x^3+8x^4+\dots + \quad = \quad 2x \cancel{X}(1-x^2)$$

$$x+x^2+x^3+x^4 \quad = \frac{x}{(1-x^2)}$$

Poišči 1. 3. člene v Taylorjevi vrsti.

$f(x) = \arcsin x$

x v okolici x_0

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n \qquad /\int$$

$$\sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots$$

$$=\binom{-1/2}{1} = -\frac{1}{2}$$

FOURIERJEVA VRSTA

$$f(x) = \begin{cases} 0 \; ; & \text{če } \pi < x < 0 \\ x \; ; & \text{če } 0 \leq x < \pi \end{cases}$$

Razvij v Fourierjevo vrsto:

$$f(x) = a_0 + \sum a_n \cos(nx) + b_n \sin nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \left\{ \int_0^{\pi} x\,dx + \int_{-\pi}^{0} 0 \right\} = \frac{1}{2\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \underset{u=x}{x} \cdot \underbrace{\cos(nx)\,dx}_{dv} = \frac{1}{\pi}\left[x \cdot \frac{\sin nx}{n} \Big| - \int \frac{\sin nx}{n}\,dx\right]\Big|_0^{\pi}$$

$$\int u\,dv = uv - \int v\,du = \frac{1}{\pi}\left[x \cdot \sin\frac{nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\,\frac{(-1)^n - 1}{n^2}$$

$\cos(n\pi) = \pm 1$

če je n sodo je +, če ne −

$$= \frac{1}{\pi}\begin{cases} 0, & \text{če } n \text{ sodo} \\ \frac{-2}{n^2} & n \text{ liho} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^\pi \frac{x \sin nx}{n^2} = \frac{1}{\pi}\left[x \cdot \frac{\cos nx}{n} + \int \frac{\cos nx}{n}\,dx\right]\Big|_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-x \cdot \cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}\right]\Big|_0^\pi =$$

$$= \frac{1}{\pi}\left[\pi \cdot \frac{(-1)^{n+1}}{n}\right] = \frac{(-1)^{n+1}}{n}$$

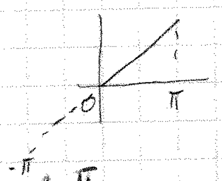
$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{sod} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

a) Razvij funkcijo $f(x) = x$ v Fourie-sinusno vrsto na intervalu 0 do π.

b) v Fourie-cos vrsto na intervalu $0 \to \pi$

Če je $f(x)$ soda, $b_n = 0$
če je $f(x)$ liha $a_n = 0$

a) $x = \sum b_n \sin nx\,dx$

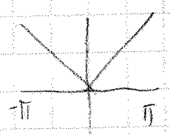


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x \cdot \sin nx}_{soda}\,dx = 2 \cdot \frac{1}{\pi}\int_0^\pi x \cdot \sin(nx)\,dx = 2 \cdot \frac{1}{\pi}\,\frac{(-1)^{n+1}}{n}$$

$$x = 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)\,dx$$

Fourierjeva vrsta za $f(x) = x$ na $(0, \pi)$

b) $x = a_0 + \sum a_n \cos nx$



$x = |x|$ v razširn

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = 2\,\frac{1}{\pi}\int_0^\pi x\,dx$$

$$= 2\,\frac{1}{2\pi}\,\frac{x^2}{2}\Big|_0^\pi$$

$$= \frac{\pi}{2}$$

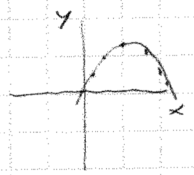
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\,dx = 2\,\frac{1}{\pi}\int_0^\pi x \cdot \cos nx\,dx$$

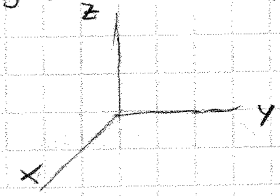
$$= \begin{cases} 0, & \text{če } n \text{ sodo} \\ -\frac{4}{\pi n^2} & \text{če } n \text{ liho} \end{cases}$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ liho} \frac{\cos nx}{n^2}$$

FUNKCIJE VEČ SPREMENLJIVK

graf: $y = f(x)$ graf $z = f(x,y)$ je ploskev v prostoru





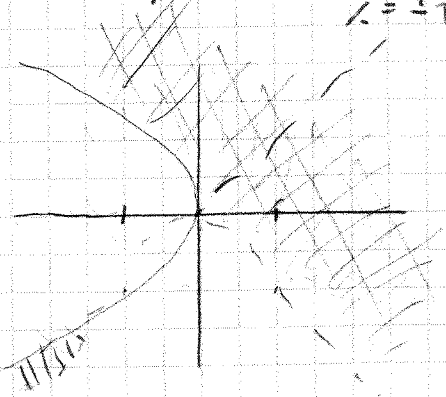
$z = 1 - x + y$
je ravnina

$z = x^2 + y^2$

Najši def območje funkcije

$z = \sqrt{x+y^2} + \sqrt{x^2+y}$

$x + y^2 \geq 0$
$x^2 + y \geq 0$ $x^2 + y = 0$ $= y = -x^2$ parabola
$x = \pm\sqrt{y}$



ali $T(0,-1)$ motrj
" $T(0,1)$ reši

$z = f(x,y)$ $z' = \frac{dz}{dx}$

MAT II-V
7.6.2008

$z_x = \frac{\partial z}{\partial x} = \lim\limits_{h\to 0} \frac{z(x+h,y) - z(x,y)}{h}$, pri tem $y =$ konst

1 $z = y \cdot \sin(x^2 y)$

$z_x = y \cdot \cos(x^2 y) \cdot 2xy$

$z_y = \sin(x^2 y) + y \cdot \cos(x^2 y) \cdot x^2$

$z_{xx} = -y^2 \cdot \sin(x^2 y) \cdot 2x \cdot y \cdot 2x + y^2 \cos(x^2 y) \cdot 2$

$z_{xy} = 2x \cdot 2y \cdot \cos(x^2 y) + 2x \cdot y^2 \cdot \sin(x^2 y) \cdot x^2$

2. $z = f(x) - e^{3x} + 3x + 3^2$

$z' = e^3 x \cdot 3 + 3$ - podano imamo to funk - njen odvod, katera je f

$z = \int (e^3 x \cdot 3 + 3)\, dx$

$z = e^{3x} + 3x + C$ $3 \to y$

$z = z(x,y) = e^{yx} + yx + y^2$

$z_x = ye^{x} + y$

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Poišči $z(x,y)$ za katero je $z_x = e^{yx} \cdot y + y$

$$z(x,y) = \int (y e^{yx} + y)\,dx$$
$$= \frac{y}{y} e^{yx} + yx + C(y)$$ rešitev ni unsto popolnoma poljubna funkcija y.

$$\int e^{yx} = \frac{1}{y} e^{yx}$$

graf $z = f(x,y)$ je ploskev v 3D

$$\lim_{T \to T_0} f(x,y)$$

EKSTREMI FUNKCIJE 2 NEODVISNIH SPR

1. Potreben pogoj za ekstrem

$$\left.\begin{array}{l} z_x = 0 \\ z_y = 0 \end{array}\right\}$$ reši sistem enačb

$$\underbrace{T_1(x, y_1),\ T_2}_{\text{stacionarne točke}}$$

2. Zadostni pogoj:

$$D(x,y) \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} \qquad z_{xy} = z_{yx}$$

Hessejeva det

če je $D(T_1) > 0$ – je ekstrem, če je $D < 0$ ni, če je $D = 0$ je pa vse odprto.

če $z_{xx} > 0$ je minimum
če $z_{yy} < 0$ je max.

$$z = xy \cdot (2 - x - y)$$

Poišči ekstreme te funkcije

$$\left.\begin{array}{l} z_x = y \cdot (2-x-y) - xy = 0 \\ z_y = x(2-x-y) - xy = 0 \end{array}\right\}$$ r. i sist. enačb

$$(2-x-y)(y-x) = 0$$

$2 - x - y = 0$ $\qquad$ $y = x$

$y = 2 - x$

$$x(2-2x) - x^2 = 0$$
$$2x - 2x^2 - x^2 = 0$$
$$x(2-3x) = 0$$
$x_1 = 0 \qquad x_2 = 2/3$

$$x \cdot 0 - x(2-x) = 0$$
$x_3 = 0$
$x_4 = 2$

stacionarne točke

st. točke
$T_3(0,2)$
$T_4(2,0)$

$T_1(0,0)$
$T_2(2/3, 2/3)$

2. VEZANI EKSTREM:

Poišči ekstrem funkcije : pri pogoju $x+y=-3$

$z = x^2 + y^2 + xy + x + y$

$F(x,y) = x^2 + y^2 - xy + x + y + \lambda(x + y + 3)$

$F_x = 2x - y + 1 + \lambda = 0$
$F_y = 2y + x + 1 + \lambda = 0$ } reši sistem enačb
$x + y = -3$

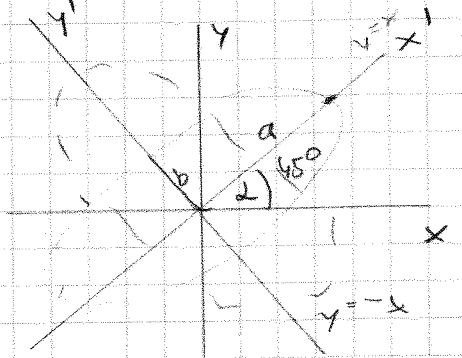
neznanke so x, y, λ

$3x - 3y = 0$
$+ \quad x + y = -3 \quad / \cdot 3$
$6x = -9$
$x = -9/6 = -3/2$

Poišči osi elipse

(5) $5x^2 + 8xy + 5y^2 = 9$ - pogoj

Iščemo ekstrem od funkcije razdalje točke od izhodišča



$d = \sqrt{x^2 + y^2}$

$F(x,y) = \sqrt{x^2 + y^2} + \lambda(5x^2 + 8xy + 5y^2 - 9)$

$F_x = \frac{1 \cdot 2x}{2\sqrt{x^2+y^2}} + \lambda(10x + 8y) = 0$ (1)

$F_y = \frac{1 \cdot 2y}{2\sqrt{x^2+y^2}} + \lambda(8x + 10y) = 0$ (2)

1. : 2.

$\lambda(10x + 8y) = -\frac{x}{\sqrt{x^2+y^2}}$
$\lambda(8x + 10y) = -\frac{y}{\sqrt{x^2+y^2}}$
$\quad = \frac{10x + 8y}{8x + 10y} = \frac{x}{y}$

$\Rightarrow \cancel{10xy} + 8y^2 = 8x^2 + \cancel{10xy}$
$y^2 = x^2$
$y = \pm x$

1. $y = x$
$5x^2 + 8x^2 + 5x^2 = 9$
$18x^2 = 9$
$x^2 = 1/2$
$x_1 = \pm\frac{1}{\sqrt{2}}$

2. $y = -x$
$5x^2 - 8x^2 + 5x^2 = 9$
$2x^2 = 9$
$x^2 = 9/2$
$x = \pm 3/\sqrt{2}$

$T_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$T_2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$b = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$T_3\left(3/\sqrt{2}, -3/\sqrt{2}\right)$
$T_4\left(-3/\sqrt{2}, 3/\sqrt{2}\right)$

$a = \sqrt{\frac{9}{2} + \frac{9}{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

$z_{xx} = -y - y = -2y \qquad z_{xy} = (2-x-y) \cdot y - x$ 1.

$= 2 - 2x - 2y$

$z_{yy} = -x - x = -2x$

$D = (-2y)\cdot(-2x) - (2 - 2x - 2y)^2$

kar vstavimo številke

$D(T_1) = -4 < 0$ ni ekst.

$D(T_2) = -4/3 \cdot (-4/3) - (2 - 4/3 - 4/3)^2 =$

$= 16/9 - 4/9 = 12/9 +$ max.

$D(T_3) = -(\ )^2$ ni ekstrema

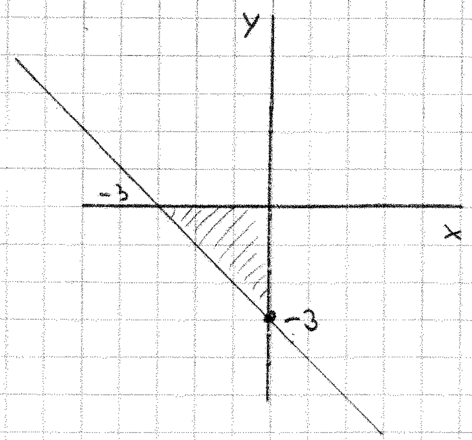
$D(T_4) = -(\ldots)^2$ ni ekstrema.

Poišči maximum in minimum na trikotniku

$\{(x,y)\ ;\ x \leq 0$ in $y \leq 0$ in $x + y \geq -3\}$

$y = -x - 3$

$z = x^2 + y^2 - xy + x + y$



1. $z_x = 2x - y + 1 = 0$

$z_y = 2y - x + 1 = 0$

$2x - y + 1 - 2y + x - 1 = 0$

$3x - 3y = 0$

$y = x$

$2x - x + 1 = 0$

$x + 1 = 0$

$x = -1$

$y = -1$

$\cancel{z_{xx}} = z = 1 + 1 - 1 - 1 - 1 = -1$ minimum.

$z(0) = 0$

$z(-3,0) = 6$

2. $y = 0$

$z = x^2 + x$

$z' = 2x + 1 = 0$

$x = -1/2$

$z = 1/4 - \frac{1}{2} = -1/4$

3. $x = 0$

$x \to y \quad = -1/4$

4. $y = -x - 3$

$z = x^2 + (-x-3)^2 + x(-x-3) + x + (-x-3)$

$= x^2 + x^2 + 6x + 9 + x^2 + 3x + x - x - 3$

$z = 3x^2 + 9x + 6 \quad$ za $\quad -3 \leq x \leq 0$

$z' = 6x + 9 = 0$

$x = -3/2$

$z = \frac{3 \cdot 9}{4} - \frac{9 \cdot 3}{2} + 6 =$

$= \frac{27}{4} - \frac{27}{2} + 6 =$

$= \frac{27 - 54 + 24}{4} = -3/4$

5. $z(0,0) = 0$

$z(-3,0) = 6$

$z(0,-3) = 6$

max $= 6$

min $= -1$

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DIFERENCIALNA ENAČBA

$y = 2xy'$

$y = f(x)$

ali je $y = x$ rešitev?

$x = 2x \cdot 1$ ni enako za vsak x ni rešitev

ali $y = x^2$ rešitev

$x^2 = 2x \cdot 2x$ ni

ali je $y = \sqrt{x}$ rešitev

$\sqrt{x} = 2x \cdot \frac{1}{2\sqrt{x}}$

$\sqrt{x} = \sqrt{x}$ je rešitev.

Dif. enačbe:

1. Dif. enačba z ločljivimi spremenljivkami
2. Linearna dif. enačba 1 reda

1.

$y' = f(x)\, g(y)$

$\frac{dy}{dx} = f(x) \cdot g(y)$

$\int \frac{dy}{g(y)} = \int f(x) \cdot dx$

$y' = y \cdot tg\, x$

$\frac{dy}{dx} = y \cdot tg\, x$

$\int \frac{dy}{y} = \int tg\, x \cdot dx$ $\qquad tg\, x = \frac{\sin x}{\cos x}$, $\cos x = t$, $dt = -\sin dx$

$\ln y = \int \frac{-dt}{t}$

$\ln y = -\ln t$

$\ln y = \ln \frac{1}{\cos x} + \ln C \quad /e$

$y = \frac{C}{\cos x}$

$y' = e^{x+y}$

$\frac{dy}{dx} = e^x \cdot e^y$

$\frac{dy}{e^y} = e^x \cdot dx$

$\int e^{-y}\, dy = \int e^x\, dx$

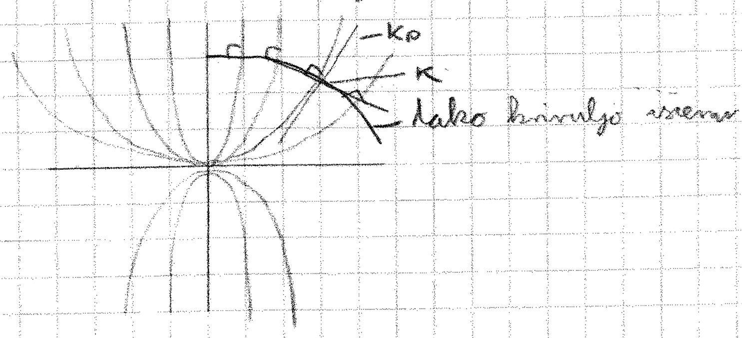
$-e^{-y} = e^x + e^c$

$e^{-y} = -e^x + e^c$

$-y = ~~\ln x + c~~$

$y = -\ln(-e^x - C)$

Poišči ortogonalne trajektorije k družini krivulj $y = Cx^2$



Zapiši dif. enačbo katere rešitve so $y = Cx^2$

$y' = -\frac{1}{k_T}$ je dif. enačba za ortog. traj.

$k_T = y_T$
$k_\sigma = -1/k_T$

$$\left.\begin{aligned} &F(x, y, y') = 0 \\ &y' = C \cdot 2x \\ &y = 2x^2 \end{aligned}\right\}$$ zveza, ki velja za vsak C

$\frac{y}{y'} = \frac{x^2}{2x}$

$\frac{y'}{y} = \frac{2}{x}$

$y' = \frac{2y}{x}$ ← za parabole

$\frac{dy}{dx} = \frac{2y}{x}$

$\frac{dy}{y} = \frac{2}{x} \cdot dx$

$y' = -\frac{1}{k_T}$

$y' = \frac{-x}{2y}$

$\frac{dy}{dx} = -\frac{x}{2y}$

$\int 2y\, dy = -\int x\, dx$

$2\frac{y^2}{2} = -\frac{x^2}{2}\, dx + C$

$y^2 + \frac{x^2}{2} = C$

2. Lin dif. enačba 1 reda

$a(x)y' + b(x)y = c(x)$

1. Homog. enačba
$ay' + by = 0$
$\vdots$
$y = f(x, C)$

2. Metoda variacije konstante
$y = f(x, C(x))$

Primer:

$y' - 2y = (e^x - x)$

$y' - 2y = 0$

$\frac{dy}{dx} = 2y$

$\int \frac{dy}{y} = \int 2\, dx$

$\ln y = 2x + \ln C$

$y = e^{2x} \cdot C$

$y_h = C \cdot e^{2x}$

$y = C(x) \cdot e^{2x}$

$C'(x) \cdot e^{2x} + \cancel{C(x) \cdot 2e^{2x}} - \cancel{2C(x)e^{2x}} = e^x - x$

se nujno krajša

$C'(x) = (e^{x} - x)e^{-2x}$

$C(x) = \int \frac{e^x - x}{e^{2x}}\, dx$

$\int v\, du$

per partes

$C(x) = \int e^{-x} - xe^{-2x}\, dx$

$u = x$
$dv = e^{-2x} dx$

$v = -\frac{1}{2}e^{-2x}$
$du = 1$

$= -e^{-x} - (x \cdot \frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\, dx)$

$C(x) = -e^{-x} + \frac{x}{2} e^{-2x} + \frac{e^{-2x}}{4} + C$

$y = \left(-e^{-x} + \frac{x}{2} e^{-2x} + \frac{e^{-2x}}{4} + C\right) \cdot e^{2x}$

$y = -e^{2x} + \frac{x}{2} + \frac{1}{4} + Ce^{2x}$

$xy' - 2y = 2x^4$

MAT II-V
14. 5. 2008

1. pripadajoča homogena enačba

$xy' - 2y = 0$

$x\frac{dy}{dx} = 2y$

$\frac{dy}{y} = 2\frac{dx}{x}$

$\ln y = 2\ln x + \ln C$

$y_h = Ce^{x^2}$

2. metoda variacije konstante:

$y_p = C(x) \cdot x^2$ partikularna rešitev

$x\left(C'(x) \cdot x^2 + C(x) \cdot 2x\right) - 2C(x)x^2 = 2x^4$

$x^3 C'(x) + \cancel{2C(x) \cdot 2x^2} - \cancel{2C(x)x^2} = 2x^4$

se krajša

$C'(x) = 2x$

$\int C'(x)dx = \int 2x\,dx$

$C(x) = x^2$ $\qquad$ $y_p = x^4$

splošna rešitev

$y = C \cdot x^2 + x^4$

$y' + \frac{y}{x} = 2\ln(x+1)$

1. hom.

$y' + \frac{y}{x} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$\frac{dy}{y} = -\frac{dx}{x}$

$\ln y = -\ln x + \ln C$

$\ln y = \ln\frac{1}{x} + \ln C$

$y_h = C \cdot \frac{1}{x}$

2. metoda variacije konst.

$y_p = C(x) \cdot \frac{1}{x}$

$C'(x) \cdot \frac{1}{x} - \cancel{C(x) \cdot \frac{1}{x^2}} + \cancel{C(x)\frac{1}{x^2}} = 2\ln x + 1$

$C'(x) \cdot \frac{1}{x} = 2x\ln x + 1$

$C'(x) = \cancel{2x(\ln x + 1)}$

$\int C(x) = 2\int x\ln x + \int x$

$C(x) = 2 \cdot \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + \frac{x^2}{2}$

$= x^2 \cdot \ln x - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}}$

$x \cdot \ln x$

$du = x\,dx$

$v = \ln x\,dx$

$\ln(x+1)$

$\ln x \quad + 1$

$u = \frac{x^2}{2}$ $\qquad$ $u \cdot v - \int u\,dv$

$x \cdot \ln x$

$dv = \frac{1}{x}dx$ $\quad \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}dv$ $\quad 2\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)$

splošna rešitev

$$y = \frac{C}{x} \cancel{+ x^2 \cdot \ln x}$$

$$y_0 = x \cdot \ln x$$

$$y = \frac{C}{x} + x \ln x$$

Bernoullijeva enačba

$a(x)y' + b(x)y = c(x)y^n$ rešimo z vpeljavo nove konst.

$y^{1-n}(x) = z(x)$

$$xy' - 4y = 2x^2\sqrt{y}$$

$\sqrt{y} = z$ - nova spr.

$$x \cdot 2zz' - 4z^2 = 2x^2 \cdot z \;/ : z$$

$$\frac{1}{2\sqrt{y}} = \frac{dz}{dx} \quad \text{ali} \quad y = z^2 \;/ \cdot \frac{d}{dx}$$

$$y' = \frac{2z\,dz}{dx}$$

$$2xz' - 4z = 2x^2$$

lin dif enačba 1 reda

1. $2xz' - 4z = 0$

$$2x\frac{dz}{dx} = 4z$$

$$\frac{dz}{z} = 2\frac{dx}{x}$$

$$\ln z = \ln x^2 + \ln C$$

$$z_h = C \cdot x^2$$

2. partik.

$$z = C(x) \cdot x^2$$

$$2x(C'(x) \cdot x^2 + \cancel{C(x) \cdot 2x}) \cancel{- 4Cx^2} = 2x^2$$

$$2x^3 C'(x) = 2x^2$$

$$C'(x) = \frac{1}{x}$$

$$C(x) = \ln x$$

$$z_p = \ln x \cdot x^2$$

$$z = Cx^2 + x^2 \cdot \ln x$$

$\sqrt{y} = z$ $\quad y = (Cx^2 + x^2 \ln x)^2$

$y = z^2$ $\quad = C^2x^4 + 2Cx^4 \ln x + x^4 \ln^2 x$

HOMOGENA DIF. ENAČBA

$y' = f\left(\frac{y}{x}\right)$ nova neznanka $\frac{y}{x} = U(x)$

$xy' - y = x \operatorname{tg} \frac{y}{x}$

$y = x \cdot U \quad /'$

$y' = U + xU'$

$xy' = y + x \operatorname{tg} \frac{y}{x}$

$y' = \frac{y}{x} + \operatorname{tg} \frac{y}{x}$

$\cancel{U} + xU' = \cancel{U} + \operatorname{tg} U$

$\frac{du}{dx} = \frac{\operatorname{tg} u}{x}$

$\int \frac{du}{\operatorname{tg} u} = \int \frac{dx}{x}$

$\Rightarrow \frac{\cos u \, du}{\sin u} = \ln x$

$\sin u = z$

$\cos u = dz$

$\frac{dz}{z} = \ln x$

$U = \frac{y}{x}$

$\ln z = \ln x$

$\ln \sin u = \ln x + \ln C$

$\sin u = C \cdot x$

$\sin \frac{y}{x} = C \cdot x$

$\frac{y}{x} = \arcsin(Cx)$

$y = x \cdot \arcsin(Cx)$

EKSAKTNA DE

$du = (2 - 9xy^2)x\,dx + \underbrace{(4y^2 - 6x^3)y\,dy}_{2} = 0$

$u = C$

$du(x,y)$

totalni diferencial

$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$(2 - 9xy)x = \frac{\partial u}{\partial x} \quad - y$ je konst

$u = \int 2x - 9x^2y^2$

$= x^2 - 9y^2\frac{x^3}{3} + C(y) = C \Rightarrow x^2 - 3y^2x^3 + y^4 = C$

$\frac{\partial u}{\partial y} = -3x^3 \cdot 2y + C'(y) = (4y^2 - 6x^3)y$

$\Rightarrow -\cancel{6x^3y} + C'(y) = -\cancel{6x^3y} + 4y^3$

$C'(y) = 4y^3$

$C(y) = \int 4y^3 dy$

$C = y^4$

LIN DIF. ENAČBA S KONST. KOEFICIENTI

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y =$$

splošna rešitev $y = y_h + y_p$

y_h – reši karakteristično enačbo

$$y''' + y'' - 2y' = 0$$

karakteristična enačba : $\lambda, \ldots$

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$
$$\lambda(\lambda^2 + \lambda - 2) = 0$$
$$\lambda(\lambda + 2)(\lambda - 1) = 0 \qquad \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = -2$$

$$y = C_1 e^{-2x} + C_2 e^{x} + C_3 \qquad y = C_1 \cdot e^{\lambda x} + \dots$$

$$y''' - 3y' + 2y = 0$$

$$\lambda^3 - 3\lambda + 2 = 0$$

	1	0	-3	2
		1	1	-2
1	1	1	-2	0

$$(\lambda - 1)(\lambda^2 + \lambda - 2)$$
$$(\lambda - 1)(\lambda - 1)(\lambda + 2)$$

$$\lambda_{1,2} = 1 \qquad \lambda_3 = -2$$

$$y = C_1 e^{-2x} + C_2 e^{x} + C_3 e^{x} \cdot x$$

$$y'' + 2y' + 10y = 0$$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$\lambda_{1,2} = -1 \pm i3$$

$$y = C_1 e^{(-1+3i)x} + C_2 e^{(-1-3i)x}$$

$$y = C_1 e^{-x} \cdot \cos 3x + C_2 e^{-x} \sin 3x$$

nehomog. $y = y_h + y_p$

$$y'' - 2y' - 3y = e^{4x} \qquad y_p = C e^{4x}$$

$$\lambda^2 - 2\lambda - 3 = 0 \qquad y' = C e^{4x} \cdot 4$$
$$(\lambda - 3)(\lambda + 1) = 0 \qquad y'' = C e^{4x} \cdot 16$$
$$\lambda_1 = 3 \quad \lambda_2 = -1$$

$\check{y}_p$ – nastavek ali metoda variacije konst.

$$y_h = C_1 e^{-x} + C_2 e^{3x}$$

$$(16C - 2 \cdot C \cdot 4 - 3C) e^{4x} = e^{4x}$$
$$(5C) e^{4x} = e^{4x}$$

$$C = \frac{1}{5}$$

$$y = C_1 e^{-x} + C_2 e^{3x} + 1/5\, e^{4x}$$

desna stran f(x)	nastavek y_p
$-e^{ax}$	$\cdot A \cdot e^{ax}$
$-p_n(x)$ polinom n-te stopnje	$-A_n x^n + A_{n-1} x^{n-1}$
$-\cos bx$, $\sin bx$	$A \cdot \cos(bx) + B \cdot \sin(bx)$
$e^{3x} + \sin x$	$Ae^{3x} + B \sin 3x + C \cos 3x$

$y_p = Ax^2 + Bx + C$

če je na D npr. $= x^2$ moramo to gledat kot polinom

$y_p = Ax^2 + Bx + C$

$$y'' - 2y' - 3y = x \cdot e^{2x}$$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda+1)(\lambda-3) = 0$

$\lambda_1 = -1 \quad \lambda_2 = 3$

$y_h = C_1 e^{-x} + C_2 e^{3x}$

$y_p = (Ax+B)e^{2x}$

$y' = Ae^{2x} + (Ax+B) \cdot e^{2x} \cdot 2$

$y'' = 2Ae^{2x} + A2e^{2x} + (Ax+B)4 \cdot e^{2x}$

$$4Ae^{2x} + \cancel{4(Ax+B)e^{2x}} - 2Ae^{2x} - \cancel{4(Ax+B)e^{2x}} - 3(Ax+B)e^{2x} = xe^{2x}$$

$$e^{2x}[(\underbrace{-3A}_{1})x + (\underbrace{2A-3B}_{0})] = xe^{2x}$$

$A = -1/3 \qquad B = -\frac{2}{9}$

$y_p = (-\frac{x}{3} - \frac{2}{9})e^{2x}$

$y = C_1 e^{-x} + C_2 e^{3x} + (-\frac{x}{3} - \frac{2}{9})e^{2x}$

včasih se nekaj manjka

nov primer:

$y'' - 2y' - 3y = xe^{-x}$

$y_h = C_1 e^{-x} + C_2 e^{3x}$

$y_p = (Ax+B)e^{-x}$

$y' = Ae^{-x} - (Ax+B)e^{-x}$

$y'' = -Ae^{-x} - Ae^{-x} + (Ax+B)e^{-x}$

$$-2Ae^{-x} + \cancel{(Ax+B)e^{-x}} - 2Ae^{-x} + \cancel{2(Ax+B)e^{-x}} - \cancel{3(Ax+B)e^{-x}} = xe^{-x}$$

$$-4A\cancel{e^{-x}} = x\cancel{e^{-x}}$$

$-4A = x$

ni rešitve predpostavka je napačna

vprašamo se ali je $\lambda = a$ ničla karakt. enačbe

v našem primeru da. Le da dodamo x in x

$y_p = (Ax+B)e^{-x} \cdot x$

$= (Ax^2 + Bx)e^{-x}$

$$y' = (2Ax+B)e^{-x} - (Ax^2+Bx)e^{-x}$$

$$y'' = 2Ae^{-x} \underbrace{- (2Ax+B)e^{-x} - (2Ax+B)e^{-x}}_{2} + (Ax^2+Bx)e^{-x}$$

$$\left.\begin{aligned} &2Ae^{-x} - 2(2Ax+B)e^{-x} + \cancel{(Ax^2+Bx)}e^{-x} - \\ &-2(2Ax+B)e^{-x} + 2\cancel{(Ax^2+Bx)}e^{-x} \\ &+ 3\cancel{(Ax^2+Bx)}e^{-x} \end{aligned}\right\} = xe^{-x}$$

$$\cancel{e^{-x}}\,[\,(\underbrace{-8A}_{1})x + (\underbrace{2A-4B}_{0})\,] = x\cancel{e^{-x}}$$

$$A = -\tfrac{1}{8} \qquad B = -\tfrac{1}{16}$$

$$y_p = \left(-\frac{x^2}{8} - \frac{x}{16}\right)e^{-x}$$

$$y = y_h + y_p = C_1e^{-x} + C_2e^{3x} - \left(\frac{x^2}{8} + \frac{x}{16}\right)e^{-x}$$

nastavek

$$y_p = \text{nastavek} \cdot x^{\text{stopnja ničle}}$$

$$y''' - 3y' - 2y = 9e^{2x}$$

začetni pogoji

$y(0) = 0$

$y'(0) = -3$

$y''(0) = 3$

splošna rešitev:

$$y = f(x, C_1, C_2, C_3)$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$$(\lambda+1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

	1	0	-3	-2
		-1	1	2
-1	1	-1	-2	0

$$\lambda_{1,2} = -1 \qquad \lambda_3 = 2$$

$$y_p = Ae^{2x} \cdot x$$

$$y' = Ae^{2x} \cdot 2x + Ae^{2x}$$
$$= Ae^{2x}(2x+1)$$

$$y'' = \cancel{4A}\; 2Ae^{2x}(2x+1) + Ae^{2x} \cdot 2$$
$$= Ae^{2x}(4x+2+2)$$

$$y''' = Ae^{2x}(8x+8+4) = Ae^{2x}(8x+12)$$

$$Ae^{2x}(8x+12) - Ae^{2x}(6x+3) - 2Ae^{2x}x = 9e^{2x}$$

$$A\cancel{e^{2x}}(\cancel{8x} + 12 - \cancel{6x} + 3 - \cancel{2x}) = 9\cancel{e^{2x}}$$

$$9A = 9$$

$$A = 1$$

$$y = C_1 e^{-x} + C_2 e^{-x} \cdot x + C_3 e^{2x} + e^{2x} \cdot x$$

1. $C_1 + C_3 = 0$
2. $-C_1 + C_2 + 2C_3 = 2 \quad / 2$
3. $C_1 - 2C_2 + 4C_3 = -1 \quad +$

$$y' = -C_1 e^{-x} - C_2 e^{-x} \cdot x + C_2 e^{-x} + 2C_3 e^{2x} + 2e^{2x} \cdot x + e^{2x}$$

$$y'(0) = -C_1 + C_2 + 2C_3 + 1 = 3$$

$$y'' = C_1 e^{-x} + C_2 (e^{-x} \cdot x - e^{-x} - e^{-x}) + 4C_3 e^{2x} + e^{2x} 4x + e^{2x} \cdot 2 + 2e^{2x}$$

$$y''(0) = C_1 - 2C_2 + 4C_3 + 2 + 2 + = 3$$

$$\left.\begin{array}{l} -C_1 + 8C_3 + 6 = -3 \\ C_1 + C_3 = 0 \end{array}\right\} + \Rightarrow 9C_3 = -9 \quad C_3 = -1 \quad C_1 = 1 \quad C_2 = 4$$

$$y = e^{-x} - e^{-x} \cdot x - e^{2x} + e^{2x} \cdot x$$

EULERJEVA DIF ENAČBA

$$x^2 y'' - 3x y' + 5y = 0 \quad \text{poišči splošno rešitev}$$

3 načini reš.

1. $e^t = x \qquad y' \to \dot{y}$
2. $\lambda(\lambda-1) - a\lambda + b = 0 \iff r^2 + (-3-1)\lambda + 5 = 0$
3. $y = x^r$

$$r^2 - 4r + 5 = 0$$
$$(r+5)(r-1)$$
$$r = 5 \quad r = 1$$

$$r_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$y = \overset{p}{x^2} \cdot \cos(\underset{q}{1} \ln x) + C_2 x^2 \sin(1 \ln x)$$

SISTEM DE

$x(t)$ $\quad \dot{x} = 3x + 2y + 4e^{5t}$

$y(t)$ $\quad \dot{y} = x + 2y \quad / \cdot \quad x = \dot{y} - 2y$

Iščemo ti 2 funkciji.

SISTEM DIF ENAČB $\Updownarrow$ ENA DIF. ENAČBA

$$\ddot{y} = \dot{x} + 2\dot{y}$$
$$\ddot{y} = 3x + 2y + 4e^{5t} + 2\dot{y}$$
$$\ddot{y} = \underline{3\dot{y}} - \underline{6y} + \underline{2y} + 4e^{5t} + \underline{2\dot{y}}$$
$$\ddot{y} - 5\dot{y} + 4y = 4e^{5t}$$

karakt. enačba

$$\lambda^2 - 5\lambda + 4 = 0$$
$$(\lambda - 1)(\lambda - 4) = 0$$
$$\lambda_1 = 4 \qquad \lambda_2 = 1$$

$$y = C_1 e^{t} + C_2 e^{4t}$$

$y^p =$

$y_p = Ae^{5t}$

$\dot{y}' = 5Ae^{5t}$

$y'' = 25Ae^{5t}$

$$\cancel{25Ae^{5t}} - \cancel{25Ae^{5t}} + 4Ae^{5t} = 4e^{5t}$$

$$A = 1$$

$y = C_1 e^{t} + C_2 e^{4t} + e^{5t}$ $\qquad x = \dot{y} - 2y$

$x = C_1 e^{t} + 4C_2 e^{4t} + 5e^{5t} - 2C_1 e^{t} - 2C_2 e^{4t} - 2e^{5t}$

$x = -C_1 e^{t} + 2C_2 e^{4t} + 3e^{5t}$

Imamo enačbo, poišči partikularno rešitev

$y'''' + y''' + y'' + y' = f(x)$?

$f_1(x) = e^{2x}$ $\qquad y_{p1} =$

$f_2(x) = e^{-x} \cdot x$

$f_3(x) = \cos 2x$

$f_4(x) = \cos x$

$f_5(x) = 1 = e^0$ $\qquad y_{p5} =$

$\lambda^4 + \lambda^3 + \lambda^2 + \lambda = 0$

$\lambda(\lambda^3 + \lambda^2 + \lambda + 1) = 0$

$\lambda(\lambda+1)(\lambda^2+1) = 0$

$\lambda(\lambda+1)(\lambda^2+1) = 0$

	1	1	1	1
		-1	0	-1
-1	1	0	1	0

$\lambda_1 = 0$

$\lambda_2 = -1$

$\lambda_{3,4} = \pm i$

$y_{p1} = C \cdot e^{2x}$ ni 2 ničla karakt enačbe

$y_{p2} = e^{-x}(ax+b) \cdot x$

$y_{p3} = A \cdot \cos 2x + B \cdot \sin 2x$ $\qquad$ ali je $\lambda = a + bi$

$y_{p4} = ((Ax+B)\cos x + (Ax+B)\sin x) x$ ali je $\lambda = i$ ničla da

$y_{p5} = C \cdot x$ $\qquad$ ali je $\lambda = 0$ ničla

Dif enačba

$x^2 y'' = (y')^2$ $\qquad y' = u$

$x^2 u' = u^2$ $\qquad y'' = u'$

$x^2 \frac{du}{dx} = u^2$

$\frac{du}{u^2} = \frac{dx}{x^2}$

$-\frac{1}{u} = -\frac{1}{x} + D$

$\frac{1}{u} = \left(\frac{1}{x} + C\right)^{-1}$

$y' = u$

$y = \int \frac{x}{1-Cx} dx$

$u = \int \left(-\frac{1}{C} + \frac{1/C}{1-Cx}\right) dx$

$= -\frac{x}{C} - \frac{1}{C^2} \ln|1-Cx| + D$

$= -\cancel{D}x - C^2 \ln\left(1 - \frac{x}{C}\right)$

$y'' + 4y = 2\,tg\,x$ $\qquad y_p = C_1(x)\cdot\cos 2x + C_2(x)\cdot\sin 2x$

neznanke so: $C_1'(x)$, $C_2'(x)$

$y'' + 4y = 0$ $\qquad$ sestavimo linearen sistem enačb:

$\lambda^2 + 4 = 0$

$\lambda_{1,2} = \pm 2i$

$\cos 2x\, C_1'(x) + \sin 2x\, C_2'(x) = 0 \quad /\cdot \sin 2x$

$-2\sin 2x\, C_1'(x) + 2\cos 2x\, C_2'(x) = 2\,tg\,x \quad /\frac{\cos 2x}{2} \quad +$

$-4\cos 2x\, C_1'(x) - 4\sin 2x\, C_2(x) =$

$y = C_1\cos 2x + C_2\sin x$

1. $C_2'(x)\,[\sin^2 2x + \underbrace{\cos^2 2x]}_{=1} = tg\,x\cdot\cos 2x$

$C_2(x) = \int tg\,x\cdot\cos 2x\,dx = \int \frac{\sin x\,dt}{\cos x}(\cos^2 x - \sin^2 x)\,dx$ $\qquad \cos x = t$, $-\sin x\,dx = dt$, $dx = -\frac{dt}{t}$

$= \frac{-dt}{t}(t^2 - 1 + t^2) =$

$= \int\frac{1}{t} - \int 2t = \ln t - t^2$

$C_2(x) = \ln\cos x - \cos^2 x$

2. $/\cos 2x$

$/\sin 2x/2$

$C_1'(x) = tg\cdot\sin 2x$

$C_1(x) = \int tg\cdot\sin 2x$

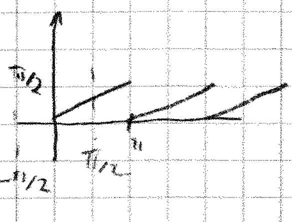
$= \int\frac{\sin x}{\cos x}\cdot 2\sin x\cancel{\cos x}\,dx = \int 1 - \cos 2x\,dx$

$C_1(x) = x - \frac{\sin 2x}{2}$

$y = C_1\cos 2x + C_2\sin 2x + (x - \frac{1}{2}\sin 2x)\cos 2x + (\ln\cos x - \cos^2 x)\sin 2x$

Funkcijo $f(x) = x/2$ na $[0, \pi)$ razvij s periodo π.

$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cdot\cos\frac{n\pi x}{T}\,dx$



$2T = \pi$

$T = \pi/2$

$a_0 = \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = \frac{1}{2T}\int_0^{2T} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi}\frac{x}{2}\,dx = \frac{1}{\pi}\frac{x^2}{4}\Big|_0^{\pi} = \frac{\pi}{4}$

$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos x\frac{n\pi x}{\pi/2}\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\cdot\cos 2nx\,dx =$

$u = \frac{x}{2}$ $\qquad du = \frac{1}{2}dx$ $\qquad = \frac{2}{\pi}\Big[\overbrace{\frac{x}{2}\frac{\sin 2nx}{2n}\Big|_0^{\pi}}^{0} - \frac{1}{2}\int_0^{\pi}\frac{\sin 2nx}{2n}\Big]$

$dv = \cos 2nx$ $\qquad v = \frac{\sin 2nx}{2x}$ $\qquad = \frac{2}{\pi}\frac{\cos 2nx}{4n^2}\Big|_0^{\pi} = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\cdot \sin 2nx$$

$$= \frac{2}{\pi}\left[-\frac{x}{2}\,\frac{\cos 2nx}{2n}\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \frac{\cos 2nx}{2n}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\cdot\frac{1}{2n} + \underbrace{\frac{1}{2}\,\frac{\sin 2nx}{4n^2}\Big|_0^{\pi}}_{0}\right]$$

$$= -\frac{1}{2n}$$

$$\frac{x}{2} = \frac{\pi}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\cdot \sin(2nx)$$

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VELIKO SREČE NA IZPITIH!