MATEMATIKA III
odgovori na ustna vprašanja

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Izvajalec Gregor Dolinar

Avtor dokumenta Dominik Peruško

UREJANJE DOKUMENTA			
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OPOMBE

- kvadratne krivulje so mišljene koordinatne
- pri naravnem parametru dx, bi moral biti dt
- pri ploskvah v prostoru, r(u,v) zares opisuje ploskev če ru x rv NI enako 0
- napaka je pri vprasanju trojni integral, ker je preveckrat napisano dx dy

POPRAVKI

Dokument še ni doživel revizije.

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Operator nabla

Gradient skalarnega polja

Smerni odvod skalarnega polja

Potencial - potencialna funkcija vektorskega polja

Divergenca vektorskega polja

Rotor vektorskega polja

Laplaceov operator

Krivuljni integral 1. vrste

Krivuljni integral 2. vrste

Greenova formula

Neodvisnost krivuljnega integrala od poti

Ploskovni integral 1. vrste

Ploskovni integral 2. vrste

Gaussov izrek

Stokesov izrek

Zveza med Stokesovim izrekom in Greenovo formulo

Analitična funkcija

Cauchy - Riemannovi enačbi

Harmonična funkcija

Konstrukcija funkcije $f(z)=e^z$

Funkcije $\cos z, \sin z, \cosh z, \sinh z, \log z$

Integracija v kompleksni ravnini ne nujno analitičnih funkcij

Integral funkcije $\frac{1}{(z-z_0)^n}$

Cauchy-jeva integralska formula

Integracijska formula za $f^{(n)}(z_0)$

Laurentova vrsta

Residuum, singularnosti vrste

Izrek o residuih

Računanje realnih integralov s pomočjo integracije v kompleksnem

Konformne preslikave

Lomljena linearna transformacija

Ločna dolžina krivulje v prostoru

krivulja: $\vec{r}(t) = (x(t), y(t), z(t))$
(parametrični zapis)

vsota vseh dolžin je:

$$S_m = \sqrt{(x(t_1)-x(t_0))^2+(y(t_1)-y(t_0))^2+(z(t_1)-z(t_0))^2} + \ldots + \sqrt{(x(t_m)-x(t_{m-1}))^2+(y(t_m)-y(t_{m-1}))^2+(z(t_m)-z(t_{m-1}))^2} =$$

$$= \sum_{i=1}^{m} \sqrt{(x(t_i)-x(t_{i-1}))^2+(y(t_i)-y(t_{i-1}))^2+(z(t_i)-z(t_{i-1}))^2} = *$$

Lagrange:

$$f'(\xi_1) = \frac{f(t_i)-f(t_{i-1})}{t_i - t_{i-1}}$$

t_i in odvod, samo ξ_i so različni

$$* = \sum_{i=1}^{m} \sqrt{\dot{x}(\xi_i)^2(t_i-t_{i-1})^2 + \dot{y}(\xi_i')^2(t_i-t_{i-1})^2 + \dot{z}(\xi_i'')^2(t_i-t_{i-1})^2} =$$

$$= \sum_{i=1}^{m} \sqrt{\dot{x}(\xi_i)^2+\dot{y}(\xi_i')^2+\dot{z}(\xi_i'')^2}\,\underbrace{(t_i-t_{i-1})}_{\Delta t} \xrightarrow[m\to\infty]{\Delta t \to 0} \lim_{\substack{m\to\infty \\ \Delta t\to 0}} S_m = \int_\alpha^\beta \sqrt{\dot{x}(t)^2+\dot{y}(t)^2+\dot{z}(t)^2}\,dt = l$$

Če krivulja podana eksplicitno:

$x = x$
$y = y(x)$
$z = z(x)$

$$l = \int_{x_0}^{x_1} \sqrt{1+y'(x)^2+z'(x)^2}\,dx$$

Naravni parameter: zg. meja ni fiksirana

$$s(t) = \int_{t_0}^{t} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dx$$

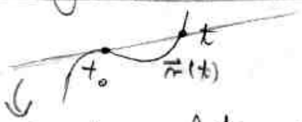
$$\dot{s} = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \Rightarrow \dot{s}^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$$

$s(t)$ injektivna fja parametra t, torej lahko izrazimo x, y in z s s

$x = x(s)$
$y = y(s)$
$z = z(s)$

s – naravni parameter, meri dolžino krivulje

Tangenta na krivuljo



Tangenta – tista premica, ki se v dani točki najbolje prilega krivulji

smerni vektor te premice: $\dfrac{(x(t), y(t), z(t)) - (x(t_0), y(t_0), z(t_0))}{t - t_0}$

$$\lim_{t\to t_0} \frac{(x(t), y(t), z(t)) - (x(t_0), y(t_0), z(t_0))}{t - t_0} = (\dot{x}(t_0), \dot{y}(t_0), \dot{z}(t_0)) \overset{\text{ne}}{=} \dot{\vec{r}}(t_0)$$

- da v limiti smerni vektor ne postal $(0,0,0)$

Smerni vektor tangente na krivuljo

Tangenta je potem:

$$\vec{Tan}(\lambda) = \vec{r}(t_0) + \lambda\,\dot{\vec{r}}(t_0)$$

Ploskve v prostoru

načini zapisa ploskve:

1) eksplicitno

$z = z(x,y)$

 problem!

2) implicitno

$f(x,y,z) = 0$

3) parametrično

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$

4) vektorsko

$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Zares opisuje ploskev, če $\vec{r}_u \times \vec{r}_v = 0$

Kvadratne krivulje:

$\vec{r}(u,v)$ krivulja, če v fiksiramo na v_0, dobimo $\vec{r}(u,v_0)$, kar je vekt. zapis krivulje. Podobno za u, dobimo $\vec{r}(u_0,v)$

$\vec{r}(u,v_0)$ in $\vec{r}(u_0,v)$ ležijo na ploskvi $\vec{r}(u,v)$

Normala na ploskev

točka je na ploskvi

normala - ravnina, ki se v dani točki $T = \vec{r}(u_0,v_0)$ najbolje prilega ploskvi $\vec{r}(u,v)$.

$r_1(t) = \vec{r}(u(t),v(t))$ poljubna krivulja na ploskvi. Njen smerni vektor je:

$\dot{\vec{r}}_1 = \vec{r}_u \cdot \dot{u} + \vec{r}_v \cdot \dot{v}$ torej $\dot{\vec{r}}_1$ linearna kombinacija $\vec{r}_u$ in $\vec{r}_v$ - leži na ravnini, ki jo določata.

Ker $\dot{\vec{r}}_1$ poljubna, torej tangenta vsake krivulje leži na tej ravnini, ki jo imenujemo tangentna ravnina na ploskev.

Njena normala je $\vec{r}_u \times \vec{r}_v$.

Če pl. podana eksplicitno $z = z(x,y)$

$\vec{r} = (x,y) = \vec{r}(x,y,z(x,y))$

$\vec{r}_x \times \vec{r}_y = (1,0,z_x) \times (0,1,z_y)$ $z_x = p$, $z_y = q$

$\vec{n} = (1,0,p) \times (0,1,q) = (-p,-q,1)$

Če pl. podana implicitno $F(x,y,z) = 0$

vzamemo $z = z(x,y) \Longrightarrow F(x,y,z(x,y)) = 0$

odvajamo po x in y

$F_x + F_z \cdot z_x = 0 \longrightarrow p = -\frac{F_x}{F_z}$

$F_y + F_y \cdot z_y = 0 \longrightarrow q = -\frac{F_y}{F_z}$

$\vec{n} = (-p,-q,-1) = \left(\frac{F_x}{F_z}, \frac{F_y}{F_z}, 1\right) \xrightarrow{\text{zamenjamo le normo}} \vec{n}_1 = (F_x, F_y, F_z)$

dolžina tangente:

$$|\vec{r}_u \times \vec{r}_v|^2 = (\vec{r}_u \times \vec{r}_v)\cdot(\vec{r}_u \times \vec{r}_v) = \ldots = \overbrace{(\vec{r}_u \cdot \vec{r}_u)}^{E}\overbrace{(\vec{r}_v \cdot \vec{r}_v)}^{G} - \overbrace{(\vec{r}_u \cdot \vec{r}_v)^2}^{F} =$$
$$= EG - F^2$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{EG-F^2}$$

Integrali odvisni od parametra

$F(y) = \int_a^b f(x,y)\,dx$ je integral s parametrom y, integral je fja y.

Če f zvezna fja, je zvezna tudi $F(y)$

Dok: $F(y+h) - F(y) = \int_a^b f(x,y+h)\,dx - \int_a^b f(x,y)\,dx = \int_a^b (f(x,y+h) - f(x,y))\,dx \xrightarrow{h\to 0} 0$

Če f in $\frac{\partial f}{\partial y}$ zvezni, je $F(y)$ odvedljiva in velja $F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx$

Dok: $F'(y) = \lim_{h\to 0} \frac{F(y+h)-F(y)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_a^b (f(x,y+h) - f(x,y))\,dx\right) =$

$= \lim_{h\to 0} \int_a^b \frac{f(x,y+h)-f(x,y)}{h}\,dx = \int_a^b f_y(x,y)\,dx$

Odvod integrala s parametrom

(Glej še prejšnje vprašanje)

Naj bosta f in $\frac{\partial f}{\partial y}$ zvezni, $u(y)$ in $v(y)$ odvedljivi. potem je

$F(y) = \int_{u(y)}^{v(y)} f(x,y)\,dx$ odvedljiva fja in velja

$$F'(y) = \int_{u(y)}^{v(y)} \frac{\partial f(x,y)}{\partial y}\,dx + f(v(y),y)\,v'(y) - f(u(y),y)\,u'(y)$$

Dokaz:

$F(y) = G(y, u(y), v(y))$

$F' = G_y + G_u \frac{du}{dy} + G_v \frac{dv}{dy}$

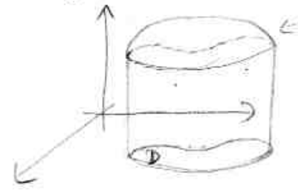
$F(y) = \int_{u(y)}^{v} f(x,y)\,dx = \int_{u(y)}^{a} f(x,y)\,dx + \int_a^{v(y)} f(x,y)\,dx = \int_a^{v(y)} f(x,y)\,dx - \int_a^{u(y)} f(x,y)\,dx$

$G(y,u,v) = \int_a^{v} f\,dx - \int_a^{u} f\,dx$

$G_u = -f(u,y) \quad G_v = f(v,y) \quad G_y = \int_u^v f_y\,dx$?

$F'(y) =$ glej zgoraj

Dvojni integrali



$f(x,y)$ volumen območja med f in D?

D razdelimo na podpravcele D_i, M_i – maksimalna vrednost f na D_i, m_i min. vrednost na D_i, w_i ploščina D_i

$$\sum_{i=1}^{m} w_i m_i \le \sum_{i=1}^{m} f(x_i,y_i)\, w_i \le \sum_{i=1}^{m} w_i M_i$$

Ker f zvezna, sta limiti $\lim\limits_{\substack{m\to\infty\\ w_i\to 0}} \sum_{i=1}^{m} w_i m_i$ in $\lim\limits_{\substack{m\to\infty\\ w_i\to 0}} \sum_{i=1}^{m} w_i M_i$ enaki. Dvojni integral je:

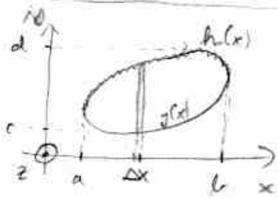
$$\iint_D f(x,y)\,dx\,dy = \lim_{\substack{m\to\infty\\ w_i\to 0}} \sum_{i=1}^{m} w_i\, f(x_i,y_i)$$

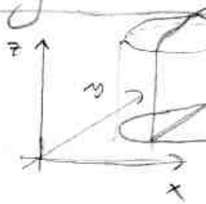
velja: $\iint_D dx\,dy = pl(D) \quad \left(\iint_D 1\,dx\,dy\right)$

$$\iint_D (f(x,y)+g(x,y))\,dx\,dy = \iint_D f(x,y)\,dx\,dy + \iint_D g(x,y)\,dx\,dy$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy \qquad D_1 \cup D_2 = D \text{ in } D_1 \cap D_2 = 0$$

Prevedba dvojnega integrala na dvakratnega



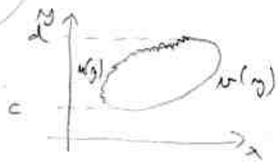


volumen območja

pl. ene rezine

$$\iint_D f(x,y)\,dx\,dy = \lim_{\substack{m\to\infty\\ w_i\to 0}} \sum_{i=1}^{m} f(x_i,y_i)\,\Delta x_i\,(h(x_i)-g(x_i)) = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right)dx$$

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right)dx = \int_c^d \left(\int_{u(y)}^{v(y)} f(x,y)\,dx\right)dy$$



Zamenjava vrstnega reda integriranja

Integral s parametrom $F(y)=\int_a^b f(x,y)\,dx$ je zvezna fja, torej lahko integriramo po y: $\int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$ – dvakratni integral

Podobno $G(x) = \int_c^d f(x,y)\,dy$ in $\int_a^b G(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$

Če $f(x,y)$ zvezna fja na $x\in[a,b]$ in $y\in[c,d]$ velja

$$\int_c^d \left(\int_a^b f(x,y)\,dx\right)dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

Če $\int_a^\infty f(x,y)\,dx$ enak. konv. na intervalu velja:

$$\int_c^d \left(\int_a^\infty f(x,y)\,dx\right)dy = \int_a^\infty \left(\int_c^d f(x,y)\,dy\right)dx$$

(Enakomerna konvergentnost:
če za vsak $\varepsilon>0$, obstaja tak $M>0$, da:
$\left|\int_M^\infty f(x,y)\,dx\right| < \varepsilon$ za vsak $y\in[c,d]$)

glej še vprašanje zgoraj in enak. konv. v zapiskih

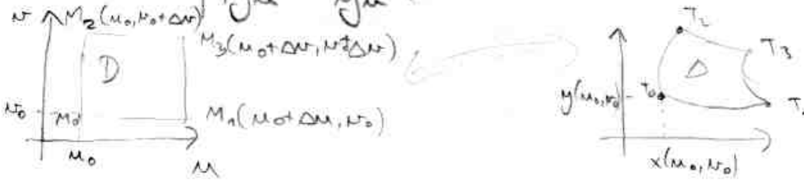
Uvedba novih spremenljivk v dvojni integral

Območje D v ravnini xy bi radi opisali s spremenljivkama u in v, torej:

$x = x(u,v)$
$y = y(u,v)$

Območje v ravnini uv mora biti bijektivna preslikava območja v xy.
Pogoj za to

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$$



$\Delta x \Delta y = J \, \Delta u \Delta v$
(glej izpeljavo v zapiskih)

$$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(x(u,v),y(u,v))\,|J|\,du\,dv$$

Pri polarnih koord. $x = r\cos\varphi$ in $y = r\sin\varphi$ je $J = r$

Posplošeni dvojni integral

Če $f: D \to \mathbb{R}$ $D \subset \mathbb{R}^2$ je v točki $a \in D$ neomejena. Izrežemo krog s središčem v a in polmerom ε in izračunamo $\iint_{D-K_a(\varepsilon)} f(x,y)\,dx\,dy$. Potem je posplošeni integral limita

$$\iint_D f(x,y)\,dx\,dy = \lim_{\varepsilon \to 0} \iint_{D-K_a(\varepsilon)} f(x,y)\,dx\,dy$$, če ta obstaja

Če D neomejeno območje, računamo integral na omejenem podobmočju D_r, za katerega velja $\lim_{r\to\infty} D_r = D$. Posplošeni integral je potem

$$\iint_D f(x,y)\,dx\,dy = \lim_{r\to\infty} \iint_{D_r} f(x,y)\,dx\,dy$$, če ta limita obstaja

Uporaba dvojnega integrala v geometriji

1. Računanje prostornine

$f(x,y) \geq 0$ na območju D. Potem je $\iint_D f(x,y)\,dx\,dy$ prostornina pod $f(x,y)$ na D

Če imamo dve ploskvi $f_1(x,y) \leq f_2(x,y)$ je prostornina med njima na D enaka

$$\iint_D f_2(x,y)\,dx\,dy - \iint_D f_2(x,y)\,dx\,dy = \iint_D (f_2(x,y) - f_1(x,y))\,dx\,dy$$

2. Računanje ploščine v 2D

Ploščina D je $\iint dx\,dy$

3. Računanje površine območja ⑥

$f(x,y)$ neka ploskev, zanima nas njena površina.

σ_k naj bo mali košček te površine, ω_k njegova projekcija na xy

$$pl\,\omega_k = \cos\varphi \; pl\,\sigma_k$$

$$\cos\varphi = \frac{\vec{n}\cdot\vec{k}}{|\vec{n}||\vec{k}|} = \frac{1}{\sqrt{p^2+q^2+1}}$$

$$pl\,\sigma_k = \sqrt{p^2+q^2+1}\; pl\,\omega_k \Rightarrow \sum_{k=1}^{n} pl\,\sigma_k = \sqrt{p^2+q^2+1}\sum_{k=1}^{n}\omega_k \text{ in } \lim_{\substack{n\to\infty\\ \omega_k\to 0\\ \sigma_k\to 0}}$$

$$\text{površina} = \iint_D \sqrt{p^2+q^2+1}\,dx\,dy$$

Če ploskev podana param. $\vec{r} = \vec{r}(u,v)$

$\sqrt{1+p^2+q^2} = \sqrt{EG-F^2}$, torej površina $= \iint_D \sqrt{EG-F^2}\,du\,dv$

Trojni integral

$f: V \to \mathbb{R}$, $V \subset \mathbb{R}^3$

V razdelimo na podobmočja V_k, v vsakem izračunamo $f(x_k,y_k,z_k)$

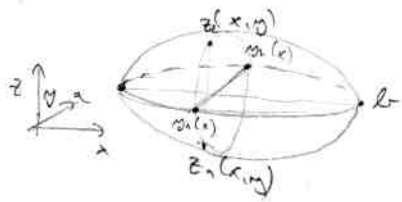
Integralska vsota: $\sum_{k=1}^{n} f(x_k,y_k,z_k)\,\mathrm{vol}\,V_k$, ko stvar limitiramo:

$$\lim_{\substack{n\to\infty\\ \mathrm{vol}V_k\to 0}} \sum_{k=1}^{n} f(x_k,y_k,z_k) = \iiint_V f(x,y,z)\,dx\,dy\,dz \quad \text{– trojni integral}$$

Če $f(x,y,z)=1 \Rightarrow \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V dx\,dy\,dz$ = prostornina V

Izračunamo ga tako, da ga prevedemo na trikratnega

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dx\,dy\,dz$$



Uvedba novih koordinat v trojni int.

Da lahko uvedemo nove spremenljivke, mora biti preslikava med območjema bijektivna. Preslikava
$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$
je bijektivna, če velja:

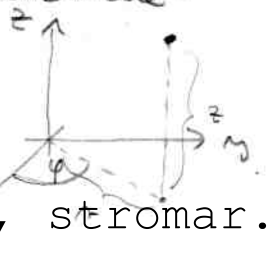
$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \neq 0$$

Velja:

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(x(u,v,w),y(u,v,w),z(u,v,w))\,|J(u,v,w)|\,du\,dv\,dw$$

Cilindrične koordinate:

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$



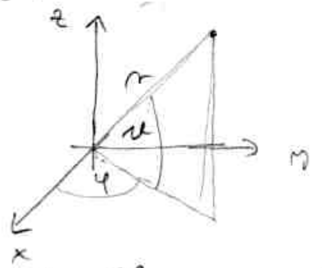
$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos\varphi^2 + r\sin^2\varphi = r$$

Sferične koordinate:

$x = r\cos\varphi\cos\vartheta$

$y = r\sin\varphi\cos\vartheta$

$z = r\sin\vartheta$



$$J = \begin{vmatrix} \cos\varphi\cos\vartheta & -r\sin\varphi\cos\vartheta & -r\cos\varphi\sin\vartheta \\ \sin\varphi\cos\vartheta & r\cos\varphi\cos\vartheta & -r\sin\varphi\sin\vartheta \\ \sin\vartheta & 0 & r\cos\vartheta \end{vmatrix} =$$

$$= \sin\vartheta\left(r^2\sin^2\varphi\sin\vartheta\cos\vartheta + r^2\cos^2\varphi\cos\vartheta\sin\vartheta\right) + r\cos\vartheta\left(r\cos^2\varphi\cos^2\vartheta + r\sin^2\varphi\cos^2\vartheta\right) =$$

$$= r^2\sin^2\vartheta\cos\vartheta + r^2\cos^3\vartheta = \underline{\underline{r^2\cos\vartheta}}$$

Uporaba trojnega integrala v mehaniki

1) Težišče

$$\frac{m_1}{m_2} = \frac{|T_2T|}{|T_1T|}$$

$$\frac{m_1}{m_2} + \frac{m_2}{m_2} = \left|\frac{T_2T}{T_1T}\right| + \left|\frac{T_1T}{T_1T}\right| = \left|\frac{T_1T_2}{T_1T}\right| \Longrightarrow \frac{m_2}{m_1+m_2} = \left|\frac{T_1T}{T_1T_2}\right| \text{ in } \frac{m_1}{m_1+m_2} = \left|\frac{T_2T}{T_1T_2}\right|$$

$\vec{r}_1 = (x_1, y_1, z_1)$, $\vec{r}_2 = (x_2, y_2, z_2)$, $\vec{r}_T = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1) = \vec{r}_1 + \frac{|TT_1|}{|T_1T_2|}(\vec{r}_2 - \vec{r}_1) = \vec{r}_1 + \frac{m_2}{m_1+m_2}(\vec{r}_2 - \vec{r}_1)$

$$x_T = x_1 + \frac{m_2}{m_1+m_2}(x_2 - x_1) = \frac{x_1m_1 + x_1m_2 + x_2m_2 - x_1m_2}{m_1+m_2} = \frac{x_1m_1 + x_2m_2}{m_1+m_2}$$

$$y_T = \frac{y_1m_1 + y_2m_2}{m_1+m_2}$$

$$z_T = \frac{z_1m_1 + z_2m_2}{m_1+m_2}$$

Za n točk:

$$x_T = \frac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k}, \quad y_T = \frac{\sum_{k=1}^{n} y_k m_k}{\sum_{k=1}^{n} m_k}, \quad z_T = \frac{\sum_{k=1}^{n} z_k m_k}{\sum_{k=1}^{n} m_k}$$

Če imamo dano telo z gostoto $\rho(x,y,z)$ (dif. maso v neki točki je $\rho\,dx\,dy\,dz$):

$$x_T = \frac{\iiint_V x\rho(x,y,z)\,dx\,dy\,dz}{\iiint_V \rho(x,y,z)\,dx\,dy\,dz}$$ podobno za y_T in z_T

2) Vztrajnostni moment

točkasta masa m, ki se vrti okoli osi s kotno hitrostjo ω.

$$\vec{v} = \omega\vec{r}$$

$$W_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2r^2$$

majhen delček telesa z maso m ima energijo $\frac{1}{2}\omega^2r^2dm$

$$W_k = \iiint_V \frac{1}{2}\omega^2r^2dm = \frac{1}{2}\omega^2\iiint_V r^2dm$$

$$dm = \rho(x,y,z)\,dx\,dy\,dz$$

$$\iiint_V r^2\rho(x,y,z)\,dx\,dy\,dz$$ – vztrajnostni moment J

Če vrtimo okoli z: $r^2 = x^2 + y^2$

$$J_z = \iiint_V \rho(x,y,z)(x^2+y^2)\,dx\,dy\,dz$$

podobno J_x in J_y.

1

Operator nabla

$f: V \to \mathbb{R}$, $V \subset \mathbb{R}^2$

$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ - Lahko zapišemo z operatorjem nabla

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Longrightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \operatorname{grad} f$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ Δ - Laplaceov operator

Δ lahko zapišemo z operatorjem ∇: $\Delta = \nabla \cdot \nabla$

$\vec{v}: V \to \mathbb{R}^3$, $V \subset \mathbb{R}^3$

$$\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (v_1, v_2, v_3) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \operatorname{div} \vec{v}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \operatorname{rot} \vec{v}$$

Gradient skalarnega polja

$f: V \to \mathbb{R}$, $V \subset \mathbb{R}^3$

$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \nabla f$

Uporaba:

- smerni odvod

kako se v dani smeri $\vec{b}$ spreminja fja
točki T $|\vec{b}| = 1$ - $\vec{b}$ enotski vektor

$T(x,y,z) \xrightarrow{\vec{b}}$

$$\frac{df}{d\vec{b}} = \operatorname{grad} f \cdot \vec{b}$$ kot med grad f in $\vec{b}$

$$\left|\frac{df}{d\vec{b}}\right| = \underbrace{|\operatorname{grad} f|}_{\text{konst}} \cdot \underbrace{|\vec{b}|}_{1} \underbrace{\cos\varphi}_{\text{največ, ko } \varphi = 0}$$, torej grad f in $\vec{b}$ kažeta v isto smer

Gradient kaže v smer največjega spreminjanja polja

- imejmo ploskev $f(x,y,z) = 0$, $\vec{r}(t)$ neka krivulja, ki leži na njej, torej $f(x(t), y(t), z(t)) = 0$

odvajamo na t

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = 0$$

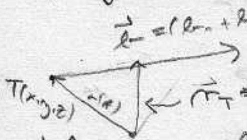
$\operatorname{grad} f \cdot \dot{x} = 0$ → torej grad f pravokoten na tangento poljubne krivulje na ploskvi; torej $\operatorname{grad} f = \vec{n}$

- obstajajo vektorska polja $\vec{v}$, za katera obstaja skalarno polje u, da
$\vec{v} = \operatorname{grad} u$ $\quad$ $\vec{v}$ - potencialno polje
u - potencial

Smerni odvod skalarnega polja

glej gradient skal. polja - uporaba

Izpeljava formule



$\leftarrow \vec{r}_T + \Delta\vec{b} = (x + \Delta b_1, y + \Delta b_2, z + \Delta b_3)$

$$\frac{df}{d\vec{b}} = \lim_{\Delta \to 0} \frac{f(x+\Delta b_1, y+\Delta b_2, z+\Delta b_3) - f(x,y,z)}{\Delta}$$

$$\frac{df}{d\vec{b}} = \left(\frac{\partial f}{\partial x}\frac{dx}{d\Delta}, \frac{\partial f}{\partial y}\frac{dy}{d\Delta}, \frac{\partial f}{\partial z}\frac{dz}{d\Delta}\right) = \left(\frac{\partial f}{\partial x} b_1, \frac{\partial f}{\partial y} b_2, \frac{\partial f}{\partial z}, b_3\right) = \operatorname{grad} f \cdot \vec{b}$$

Potencial

glej alinejo na vrhu strani

vekt. polje $\vec{v}$ je potencialno, če velja $\operatorname{rot} \vec{v} = (0,0,0)$

glej še neodvisnost krivuljnega integrala od poti

Divergenca vekt. polja

$\vec{v} : V \to \mathbb{R}^3 \quad V \subset \mathbb{R}^3 \qquad \vec{v} = (v_1, v_2, v_3)$

$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ - skalarno polje

$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}, \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \Delta f$

Če za vekt. polje v v vsaki točki velja $\operatorname{div} \vec{v} = 0$ - je polje solenoidno

Rotor vekt. polja

$\vec{v} = V \to \mathbb{R}^3 \quad V \subset \mathbb{R}^3$

$$\operatorname{rot} \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \nabla \times \vec{v}$$

$\vec{v}$ je potencialno, če velja $\operatorname{rot} \vec{v} = 0$

Laplaceov operator

$f: V \to \mathbb{R} \quad V \subset \mathbb{R}^3$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta = \nabla \cdot \nabla$$

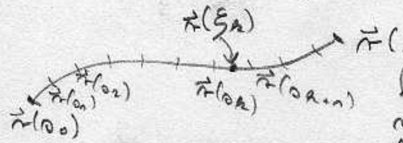
$\Delta f = \operatorname{div}(\operatorname{grad} f)$ (glej divergenco vekt. polja

(Transformacija Laplaceove D.E. 24)

Krivuljni integral 1. vrste

$\vec{r} = \vec{r}(s)$ krivulja v par. obliki z naravnim par. s

$f: V \to \mathbb{R}$, $V \subset \mathbb{R}^3$, V vsebuje $\vec{r}(s)$



$$\lim_{\substack{m \to \infty \\ \Delta s_k}} \sum_{k=1}^{m} f(\vec{r}(\xi_k)) \cdot \underbrace{(s_{k+1} - s_k)}_{\Delta s_k \text{ - dolžina krivulje od } \vec{r}(s_k) \text{ do } \vec{r}(s_{k+1})} = \int_C f\,ds$$

$$\int_C f(x,y,z)\,ds = \int_{a=s_0}^{b=s_{m+1}} f(x(s), y(s), z(s))\,ds$$

$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$ če $C_1 \cup C_2 = C$ in $C_1 \cap C_2 = 0$

$\int_C (f+g)\,ds = \int_C f\,ds + \int_C g\,ds$

$\int_C k \cdot f\,ds = k\int_C f\,ds$ — k konst.

Če krivulja ni dana z s:

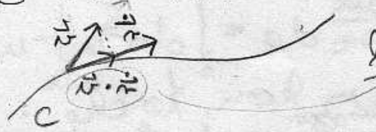
$$\int_C f(x(s), y(s), z(s))\,ds = \int_C f(x(t), y(t), z(t)) \frac{ds}{dt}\,dt =$$

$$= \int_C f(x(t), y(t), z(t)) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt$$

C ravno odvedljiva na t!

Krivuljni integral 2. vrste

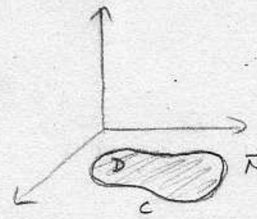
$\vec{v}: C \to \mathbb{R}^3 \quad C \subset \mathbb{R}^3$



ker Iščemo samo del $\vec{v}$, ki gleda v isto smer kot krivulja ($\vec{v} \cdot \dot{\vec{r}}$ je proj $\vec{v}$ na tangento)

$$\int_C \vec{v}\,d\vec{r} = \int_C \vec{v} \frac{d\vec{r}}{dt}\,dt = \int_C \vec{v}\,\dot{\vec{r}}\,dt = \int_a^b (v_1, v_2, v_3)(\dot{x}, \dot{y}, \dot{z})\,dt = \int_a^b (v_1\dot{x} + v_2\dot{y} + v_3\dot{z})\,dt$$

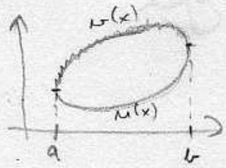
Greenova formula



$$\oint_C f\,dx + g\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

f, g parcialno zvezno odvedljivi fji, C meja D

Dokaz:



$$\iint_D \frac{\partial f}{\partial y}\,dx\,dy = \int_a^b dx \int_{u(x)}^{v(x)} \frac{\partial f}{\partial y}\,dy = \int_a^b f(x,y)\Big|_{u(x)}^{v(x)} =$$

$$= \int_a^b \left(f(x,v(x)) - f(x,u(x))\right)dx = -\int_a^b f(x,u(x))\,dx - \int_b^a f(x,v(x))\,dx =$$

$$= -\int_{C_1} f\,dx - \int_{C_2} f\,dx = -\oint_C f\,dx$$

$$\iint_D \frac{\partial g}{\partial x}\,dx\,dy = \oint_C g\,dy$$

$$\iint_D \frac{\partial f}{\partial y}\,dx\,dy = -\oint_C f\,dx$$

$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \oint_C f\,dx + g\,dy$$

Isto velja za bolj zapletena območja (razdelimo na preprosta podobmočja - integrali znotraj se odštejemo, ostane integral po zunanjem robu)

(Izražanje Laplaca z Greenom?)
(V porabla ?)

Neodvisnost krivuljnega integrala od poti

$\int_C \vec{v}\,d\vec{r}$ neodvisen od poti, če za vsako pot od P do Q dobimo enako vrednost.

$v_1dx + v_2dy + v_3dz$ je eksakten, če obstaja skalarno polje u, da

$$du = v_1dx + v_2dy + v_3dz$$
$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$\Longrightarrow$ $v_1 = \frac{\partial u}{\partial x}$, $v_2 = \frac{\partial u}{\partial y}$, $v_3 = \frac{\partial u}{\partial z}$ oz. $\vec{v} = \text{grad}\, u$

Če to velja za $\vec{v}$, potem je $\int_C \vec{v}\,d\vec{r}$ neodvisen od poti

Dok.: naj velja $\vec{v} = \text{grad}\, u$

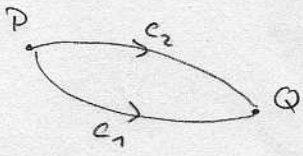
$$\int_C \vec{v}\cdot d\vec{r} = \int v_1dx + v_2dy + v_3dz = \int \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \int_C du = u\Big|_{T_k} - u\Big|_{T_z}$$

Torej $\int_C \vec{v}\cdot d\vec{r}$ ~~je~~ odvisen samo od zač. in kon. točke

Če je D enostavno povezano = nima lukenj (sklenjeno krivuljo se da skrčiti v točko), je vsako vekt. polje $\vec{v}$, za katerega velja rot $\vec{v} = 0$ eksaktno.

Če je $\int \vec{v}\cdot d\vec{r}$ neodvisen od poti, je $\oint \vec{v}\cdot d\vec{r} = 0$

Dok.:



$$\int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_2} \vec{v}\cdot d\vec{r}$$

$$\int_{C_1} \vec{v}\cdot d\vec{r} - \int_{C_2} \vec{v}\cdot d\vec{r} = 0$$

$$\int_{C_1} \vec{v}\cdot d\vec{r} + \int_{-C_2} \vec{v}\cdot d\vec{r} = 0 \qquad \oint_C \vec{v}\cdot d\vec{r}$$

Ploskovni integral 1. vrste

S gladka, orientirana ploskev (2. strani - iz ene na drugo samo čez rob). Razdelimo jo na manjše parcele S_k in na vsaki parceli izberemo točko $T_k(x_k, y_k, z_k)$

int. vsota:

$\sum_{k=1} u(x_k, y_k, z_k)\, pl\, S_k$

$$\iint_S u\, dS = \lim_{\substack{n \to \infty \\ pl S_k \to 0}} \sum_{k=1}^{n} u(x_k, y_k, z_k)\, pl\, S_k$$

$$pl\, S = \iint_D \sqrt{EG - F^2}\, du\, dv \quad , \quad E = \vec{r}_u \cdot \vec{r}_u,\ G = \vec{r}_v \cdot \vec{r}_v,\ F = \vec{r}_u \cdot \vec{r}_v$$

$\vec{r}$ - parametr. ploskev $\vec{r} = \vec{r}(u, v)$

$$\iint_S f\, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{EG - F^2}\, du\, dv$$

D območje, po katerem tečeta u in v

Ploskovni integral 2. vrste

S gladka, orientirana ploskev, $\vec{v}$ vekt. polje, definirano na njej

pl. int. 2. vrste:

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_S \vec{v} \cdot \vec{\nu}\, dS \quad , \text{kjer } \vec{\nu} \text{ enotska normala na } S \text{ v dani točki}$$

Če S podana v param. obliki $\vec{r} = \vec{r}(u, v)$ je $\vec{n} = \vec{r}_u \times \vec{r}_v$ in

$$\vec{\nu} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{r}_u \times \vec{r}_v}{\sqrt{EG - F^2}}$$

$$\iint_S \vec{v} \cdot \vec{\nu}\, dS = \iint_D \vec{v}\, \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\sqrt{EG - F^2}}} \cancel{\sqrt{EG - F^2}}\, du\, dv = \iint_D \vec{v} \cdot (\vec{r}_u \times \vec{r}_v) = \iint_D \begin{vmatrix} v_1 & v_2 & v_3 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\, dv$$

Če S dana eksplicitno $z = z(x,y)$, je $\vec{n} = (p, q, -1)$, kjer $p = z_x$ in $q = z_y$ in

$$\vec{\nu} = \frac{(p, q, -1)}{\sqrt{p^2 + q^2 + 1}}$$

$$\iint_S \vec{v} \cdot d\vec{S} = \iint_D \vec{v}\, (p, q, -1)\, dx\, dy$$

Gaussov izrek

S gladka zaključena ploskev, ki omejuje območje V

$$\boxed{\iiint_V \operatorname{div} \vec{v}\, dx\, dy\, dz = \oiint_S \vec{v}\, d\vec{S}}$$

Če $\vec{v}$ pot. polje, torej $\vec{v} = \operatorname{grad} u$:

$$\operatorname{div} \operatorname{grad} f = \operatorname{div}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \Delta f \quad \text{in} \quad \operatorname{grad} f \cdot \vec{\nu} = \frac{df}{d\vec{\nu}}$$

$$\iiint_V \operatorname{div} \operatorname{grad} f\, dx\, dy\, dz = \iiint_V \Delta f\, dx\, dy\, dz$$

$$\oiint_S \operatorname{grad} f\, d\vec{S} = \oiint_S \frac{df}{d\vec{\nu}}\, dS \quad \text{in}$$

$$\iiint_V \Delta f\, dx\, dy\, dz = \oiint_S \frac{df}{d\vec{\nu}}\, dS$$

Stokesov izrek

C gladka, sklenjena krivulja, ki omejuje S, $\vec{v}$ odvedljivo vektorsko polje na S.

$$\iint_S \mathrm{rot}\,\vec{v} \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r}$$

Dokaz: S projiciramo na xy in uporabimo Greenovo formulo

$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C \cancel{f}\, f\,dx + g\,dy$$

Potem projiciramo še na xz in zy in rezultate seštejemo

Zveza med Stokesovim izrekom in Greenovo formulo

Greenova formula $\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C f\,dx + g\,dy$ je poseben primer Stokesove $\iint_S \mathrm{rot}\,\vec{v}\, d\vec{S} = \oint_C \vec{v} \cdot d\vec{r}$

$$\vec{v} = (f(x,y), g(x,y), 0)$$

$$\mathrm{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x,y) & g(x,y) & 0 \end{vmatrix} = \left(0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

S območje v ravnini $\Rightarrow d\vec{S} = \vec{N}\,dS = (0,0,1)\,dS$

$$\iint_S \mathrm{rot}\,\vec{v}\, d\vec{S} = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C \cancel{f}\, f\,dx + g\,dy$$

Analitična funkcija

$f(z)$ kompleksna funkcija

Odvedljiva v točki z_0, če obstaja $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$f'(z_0) =$

f je analitična v z_0, če je odvedljiva v vsaki točki okolice z_0.

Če analitična, avtomatično zvezna, obratno ni nujno

Velja še, če $f(z) = u(x,y) + i\,v(x,y)$ analitična:

$\Delta u = 0$ in $\Delta v = 0$.

Dokaz: glej C-R: $u_x = v_y \quad / \frac{\partial}{\partial x}$

$u_y = -v_x \quad / \frac{\partial}{\partial y}$

$u_{xx} = v_{yx}$

$u_{yy} = -v_{xy}$ $)+ \longrightarrow u_{xx} + u_{yy} = 0 \qquad \Delta u = 0$

Podobno izpeljemo $\Delta v = 0$

Glej še: Cauchy-Riemann, Cauchyev izrek (na koncu)

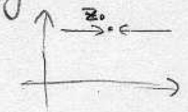
Cauchy-Riemannovi enačbi

$f(z)=u(x,y)+i v(x,y)$ naj bo analitična fja kompl. sprem.
Potem velja:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Velja tudi obratno, torej, če velja C-R. pogoj, fja analitična

Izpeljava:



odvod v horizontalni smeri, torej $\Delta y=0$:

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta x\to 0}\frac{f(z_0+\Delta x)-f(z_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{u(x_0+\Delta x,y_0)+iv(x_0+\Delta x,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}=$$
$$=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$$

odvod v vertikalni smeri, torej $\Delta x=0$

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta y\to 0}\frac{f(z_0+\Delta y)-f(z_0)}{\Delta y\cdot i}=\lim_{\Delta y\to 0}\frac{u(x_0,y_0+\Delta y)+iv(x_0,y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta y\cdot i}=$$
$$=\frac{1}{i}\lim_{\Delta y\to 0}\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}+\lim_{\Delta y\to 0}\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{\Delta y}=\frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0)$$

ker $\Delta z=\Delta x+i\Delta y$, $\Delta x=0$

Odvoda morata biti enaka, torej

$$u_x+iv_x=\frac{1}{i}u_y+v_y$$
$$u_x+iv_x=v_y-iu_y \implies \begin{matrix}u_x=v_y\\ u_y=-v_x\end{matrix}$$

Harmonična funkcija

Glej analitična fja

Fja realnih spremenljivk u, za katero velja $\Delta u=0$ je harmonična fja

Če je $f(z)=u(x,y)+iv(x,y)$ harmonična, sta harmonični tudi u in v

Konstrukcija $f(z)=e^z$

3 pogoji: 1) e^z razširitev e^x, če $\operatorname{Im} z=0$, naj $e^z=e^x$

2) $\frac{de^z}{dz}=e^z$

3) e^z analitična

$e^z=u(x,y)+iv(x,y)\ /\frac{\partial}{\partial z}$

$u_x(x,y)+iv_x(x,y)\underset{\uparrow}{=}u(x,y)+iv(x,y)$

1\. pogoj

$\begin{matrix}u_x=u\\ v_x=v\end{matrix}\implies u(x,y)=e^xG(y)$

zaradi 3 pogoja: $u_x=v_y$

$u_y=-v_x\implies v_x=-u_y=-e^xG'(y)$ ker $u_y=-v_x\ /dy$

$v_{xy}=-e^xG''(y)=-u_{yy}$ $v_{xy}=-u_{yy}$

$v_y=u_x=u=e^xG(y)$

$$e^xG(y)=-e^xG''(y)$$
$$e^x(G(y)+G''(y))=0\implies G(y)+G''(y)=0\implies G(y)=A\cos y+B\sin y$$

$e^z=u+iv=e^x(A\cos y+B\sin y)+ie^x(-A\sin y+B\cos y)$ ker $v=\int -u_y$

$e^z = e^x(A\cos y + B\sin y + i(A\sin y - B\cos y))$

Če $y=0$ mora (R. pogoj) $e^z = e^x$

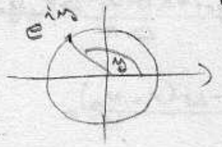
$A\cos 0 + B\sin 0 + i(A\sin 0 - B\cos 0) = 1$

$A = 1$

$B = 0$

$\boxed{e^z = e^x(\cos y + i\sin y)} = e^{x+iy}$

$e^{iy} = \cos y + i\sin y \Rightarrow |e^{iy}| = \cos^2 y + \sin^2 y = 1$



$e^{iy} = e^{i(y+2\pi k)} \Rightarrow e^z$ ni injektivna

$\cos z, \sin z, \cosh z, \sinh z, \log z$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ enake zveze kot realne fje

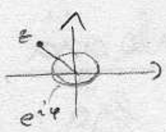
$\sinh = \dfrac{e^z - e^{-z}}{2}$

$\cosh = \dfrac{e^z + e^{-z}}{2}$

Logaritem:

$w = \log z$

$e^w = z$ $\qquad w = u + iv, \ z = re^{i\varphi}$



$\varphi = \arg z$
$r = |z|$

$e^{u+iv} = re^{i\varphi}$

$e^u e^{iv} = re^{i\varphi} \Rightarrow e^u = r \Rightarrow u = \log r = \log|z|$

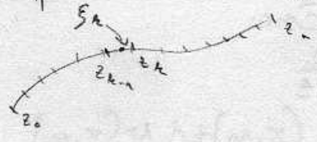
$e^{iv} = e^{i\varphi} \Rightarrow v = \varphi + 2k\pi$ - ker e^z ni injektivna!

$\log z = \log|z| + i(\varphi + 2k\pi) \quad k \in \mathbb{Z}$

Integracija v kompl. ravnini

C krivulja v kompl. ravnini $\quad f: \mathbb{C} \to \mathbb{C}$ fja kompl. sprem.

$\int_C f(z)dz = \lim_{\substack{n\to\infty \\ \Delta z_k \to 0}} \sum_{k=1}^{n} f(\xi_k)\Delta z_k$



$f(z) = u(x,y) + v(x,y)$

$z = x + iy \Rightarrow dz = dx + idy$

$\int f(z)dz = \int_C (u+iv)(dx+idy) = \int_C u dx - v dy + i\int v dx + u dy = \int_C (u,-v)d\vec{r} + i\int (u,v)d\vec{r}$

$d\vec{r} = (dx, dy)$
parametrizirana C

Velja:

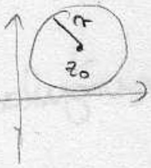
$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad C_1 \cup C_2 = C \quad C_1 \cap C_2 = 0$

$\int_C k f(z)dz = k\int_C f(z)dz$

$\int_C (f(z) + g(z))dz = \int_C f(z)dz + \int_C g(z)dz$

Integral fje $\frac{1}{(z-z_0)^m}$

$\oint_C \frac{dz}{(z-z_0)^m}$ C naj bo krožnica s polmerom r in središčem z_0



$z-z_0 = re^{i\varphi}$

$dz = re^{i\varphi} i\, d\varphi$

$$\int_0^{2\pi} \frac{re^{i\varphi} i\, d\varphi}{(re^{i\varphi})^m} = \frac{1}{r^{m-1}} \int_0^{2\pi} i e^{i\varphi(-(m-1))} d\varphi =$$

$$= \frac{i}{r^{m-1}} \int_0^{2\pi} \left(\cos(\varphi(m-1)) - i \sin(\varphi(m-1))\right) d\varphi$$

Če $m=1 \Rightarrow I = \frac{i}{r^0} \int_0^{2\pi} 1\, d\varphi = 2\pi i$

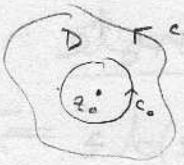
Če $m \neq 1 \Rightarrow I = 0$, ker sin in cos periodični fji

npr $m=2$ $\quad I = \frac{i}{r} \int_0^{2\pi} (\cos\varphi - i \sin\varphi) d\varphi = \frac{i}{r}(\sin\varphi + i\cos\varphi)\Big|_0^{2\pi} =$

$= \frac{i}{r}(i-i) = 0$

Cauchyeva integralska formula

Naj bo f analitična na območju D in C krivulja C v D $z_0 \in D - C$. Velja:



$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

Dokaz:

$$\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_0} \frac{f(z)}{z-z_0} dz = \oint_{C_0} \frac{f(z_0) + f(z) - f(z_0)}{z-z_0} dz = \oint_{C_0} \frac{f(z_0)}{z-z_0} dz + \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz =$$

$$= \underbrace{f(z_0) 2\pi i}_{\text{glej int fje } \frac{1}{(z-z_0)^m}} + \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz$$

ker f analitična je za majhen ρ $|f(z)-f(z_0)| \leq \varepsilon$

$$\oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz \leq \oint_{C_0} \frac{|f(z)-f(z_0)|}{\underbrace{|z-z_0|}_{\rho}} dz \leq \oint_{C_0} \frac{\varepsilon}{\rho} dz = \frac{\varepsilon}{\rho} \oint_{C_0} dz = \frac{\varepsilon}{\rho} 2\pi\rho \xrightarrow{\text{če } \rho \to 0} 0$$

ker krivulja poljubna

Integracijska formula za $f^{(m)}(z_0)$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(\frac{1}{2\pi i} \oint \frac{f(z)}{z-(z_0+\Delta z)} dz - \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0} dz \right) =$$

$$= \lim_{\Delta z \to 0} \frac{1}{\cancel{\Delta z}} \left(\frac{1}{2\pi i} \oint \frac{f(z)(z-z_0) - f(z)(z-z_0-\Delta z)}{(z-z_0)(z-z_0-\Delta z)} dz \right) = \lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)(z-z_0-\Delta z)} dz =$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^2} dz$$

Podobno bi izpeljali za višje odvode, dobimo

$$f^{(m)} z_0 = \frac{m!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{m+1}} dz$$

Če f na D analitična, avtomatično tudi neskončnokrat odvedljiva

Laurentova vrsta

Taylorjeva vrsta za $f(z)$ okoli z_0:

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$ (Poglej si izpeljavo) f mora biti analitična

Če hočemo razviti f okoli točke, v kateri je singularnost, uporabimo Laurentovo vrsto!

f analitična na koncentričnih krožnicah C_1 in C_2 in območju med njima

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{c_n}{(z-z_0)^n},\ \text{kjer}\quad a_n=\frac{1}{2\pi i}\oint_C\frac{f(w)}{(w-z_0)^{n+1}}dw$$

$$c_n=\frac{1}{2\pi i}\oint_C f(w)(w-z_0)^{n-1}dw$$

ali

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,\ \text{kjer}\quad a_n=\frac{1}{2\pi i}\oint_C\frac{f(w)}{(w-z_0)^{n+1}}dw$$

Dokaz:

$$f(z)=\underbrace{\oint_{C_1}\frac{f(w)}{w-z}dw}_{\text{reg. del}}-\oint_{C_2}\frac{f(w)}{w-z}dw$$

$$\frac{1}{w-z}=\frac{1}{w-z_0-(z-z_0)}=-\frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}}=$$

$$=-\frac{1}{z-z_0}\left(1+\frac{w-z_0}{z-z_0}+\left(\frac{w-z_0}{z-z_0}\right)^2+\dots\right)$$

Residuum, singularnosti vrste

$$\underset{z=z_0}{\operatorname{res}} f(z)=c_1=\frac{1}{2\pi i}\oint_C f(z)dz \Rightarrow \oint_C f(z)dz=2\pi i c_1$$

Residuum koeficient pri $\frac{1}{z-z_0}$ pri razvoju v Laurentovo vrsto.

3 vrste singularnosti:

1) odpravljiva singularnost - pri ~~razvoju~~ razvoju v L.v. dobimo samo regularni del

2) pol n-tega reda - pri razvoju v L.v. v glavnem delu končno mnogo neničelnih členov, največja potenca $\frac{1}{(z-z_0)^n}$

3) bistvena singularnost - pri razvoju v L.v. dobimo v glavnem delu neskončno členov

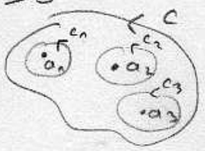
glej se zvezek o residuih

Izrek o residuih

Naj bo f analitična na D, razen v končno mnogo singularnostih znotraj D. C naj bo rob D:

$$\oint_C f(z)dz = 2\pi i \sum_{i=1}^{m} \operatorname*{Res}_{z=a_i} f(z)$$

Dokaz:



$$\oint_{C\cup -C_1\cup -C_2\cup -C_3} f(z)dz = 0$$

$$\|$$

$$\oint_C f(z)dz + \oint_{-C_1} f(z)dz + \oint_{-C_2} f(z)dz + \oint_{-C_3} f(z)dz =$$

$$= \oint_C f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \oint_{C_3} f(z)dz = 0$$

$$\oint_C f(z)dz = \underbrace{\oint_{C_1} f(z)dz}_{\operatorname{Res}_{a_1}} + \underbrace{\oint_{C_2} f(z)dz}_{\operatorname{Res}_{a_2}} + \underbrace{\oint_{C_3} f(z)dz}_{\operatorname{Res}_{a_3}}$$

Računanje residuov:

1) v točki a pol 1. reda

$$f(z) = \frac{c_1}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots \qquad (z-a)$$

$$(z-a)f(z) = c_1 + a_0(z-a) + a_1(z-a)^2 + a_2(z-a) + \dots \qquad / \lim_{z\to a}$$

$$\lim_{z\to a}(z-a)f(z) = c_1$$

2) a pol m-tega reda

$$f(z) = \frac{c_m}{(z-a)^m} + \frac{c_{m-1}}{(z-a)^{m-1}} + \dots + \frac{c_1}{z-a} + a_0 + a_1(z-a) + \dots \qquad / (z-a)^m$$

$$(z-a)^m f(z) = c_m + c_{m-1}(z-a) + \dots + c_1(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + \dots \qquad / \frac{d}{dz}$$

$$\frac{d}{dz}\left((z-a)^m f(z)\right) = c_{m-1} + \dots + (m-1)c_1(z-a)^{m-2} + m a_0(z-a)^{m-1} + (m+1)a_1(z-a)^m + \dots$$

$$\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) = (m-1)!\,c_1 + m!\,a_0(z-a) + \dots \qquad / \lim_{z\to a}$$

$$c_1 = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)$$

3) a bistvena singularnost

Res izračunamo z razvojem v L.v., pogledamo člen pri $(z-a)^{-1}$

Računanje realnih int v kompleksnem

1) $\int_0^{2\pi} R(\cos\varphi, \sin\varphi)\,d\varphi$, R racionalna fja

uvedemo novo sprem $z = e^{i\varphi}$

krivulja je krožnica s polmerom 1 (ker gre φ od 0 do 2π)

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} = \frac{z+\frac{1}{z}}{2}$$

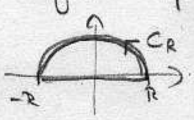
$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

$$dz = ie^{i\varphi}d\varphi = z\cdot i\,d\varphi \Rightarrow d\varphi = \frac{dz}{z\cdot i}$$

uporabimo izrek o residuih

2) $\int_{-\infty}^{\infty} f(x)\,dx$, $f(x)$ rac. fja, stopnja imenovalca vsaj za dva višja od števca

nova sprem. $z = x$, za krivuljo izberemo pol krožnice in premer, ki gre proti ∞



$$\int_{C_R} f(z)\,dz = 2\pi i \sum_{\text{znotraj } C_R} \operatorname{Res} f(z)$$

$$\| $$

$$\int_{-R}^{R} f(z)\,dz + \underbrace{\int_{\tilde C_R} f(z)\,dz}_{\text{pol krožnice}} \xrightarrow{R\to\infty} 0$$

$$\| $$

$$\int_{-R}^{R} f(x)\,dx$$

ker je imenovalec za 2 večji od števca je to res za velike x-e

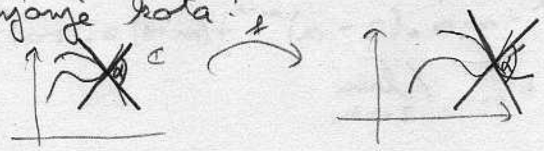
$$\left|\int_{\tilde C_R} f(z)\,dz\right| \le \int_{\tilde C_R} |f(z)|\,|dz| \le k\frac{1}{R^2}\underbrace{\int_{\tilde C_R} dz}_{\text{dolžina polovice kroga}} = k\frac{1}{R^2}\pi R \xrightarrow{R\to\infty} 0$$

Torej $\int_{-\infty}^{\infty} f(x)\,dx = \int_{C_R} f(z)\,dz = 2\pi i \sum \operatorname{Res} f(z)$

Konformne preslikave

Če f preslikava, ki ohranja kote in orientacijo, pravimo, da je konformna preslikava

ohranjanje kota:



Če f analitična, potem konformna v vseh točkah, razen kjer $f'(z)=0$

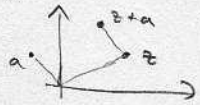
Lomljena lin. preslikava

Lomljena lin. presl. je kompozitum translacije, rotacije, skrčitve (raztega) in inverzije oblike

$f(z)=\frac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{C}$

Translacija:

$f(z)=z+a \quad a\in\mathbb{C}$



kompl. števila prestavi za a

rotacija:

$f(z)=e^{i\varphi_0}\cdot z = e^{i\varphi_0}|z|e^{i\varphi_z} = |z|e^{i(\varphi_0+\varphi_z)}$

kompl. števila zavrti za a

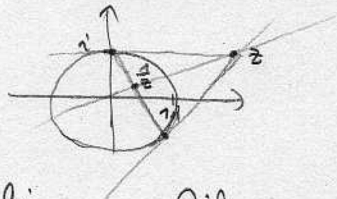
skrčitev (raztega):

$f(z)=r\cdot z = r|z|e^{i\varphi} \quad r\in\mathbb{R}$

z raztegne ali skrči za faktor r (če r negativen, spremeni smer)

inverzija:

$f(z)=\frac{1}{z}$



Lomljena lin. preslikava preslika množico premic in krožnic v množico premic in krožnic (premica → krožnica ali premica, krožnica → krožnica ali premica