IZPIT IZ MATEMATIKE III

25. januar 2010

1. Poiščite vse točke na ploskvi

$$\vec{r}(u,v) = \left(v\cos u, v\sin u, \frac{\cos u}{v}\right),$$

v katerih je tangentna ravnina vzporedna ravnini $x + \sqrt{3}y + 2z = 0$.

Rešitev. Vemo, da za normalo tangentne ravnine velja:

$$\begin{aligned}
\vec{r}(u,v) &= \left(v\cos u, v\sin u, \frac{\cos u}{v}\right)\\
\vec{r_u}(u,v) &= \left(-v\sin u, v\cos u, -\frac{\sin u}{v}\right)\\
\vec{r_v}(u,v) &= \left(\cos u, \sin u, -\frac{\cos u}{v^2}\right)\\
\vec{n}(u,v) &= \vec{r_u}(u,v) \times \vec{r_v}(u,v) =\\
&= \left(\frac{\sin^2 u - \cos^2 u}{v}, -\frac{2\sin u\cos u}{v}, -v(\sin^2 u + \cos^2 u)\right) =\\
&= \left(-\frac{\cos(2u)}{v}, -\frac{\sin(2u)}{v}, -v\right)
\end{aligned}$$

Zaradi vzporednosti omenjenih ravnin tako velja

$$\left(-\frac{\cos(2u)}{v}, -\frac{\sin(2u)}{v}, -v\right) = k(1, \sqrt{3}, 2),$$

kar nam da sledeč sistem enačb:

$$-\frac{\cos(2u)}{v} = k, \quad -\frac{\sin(2u)}{v} = \sqrt{3}k, \quad -v = 2k.$$

Če v iz tretje enačbe vstavimo v prvi dve, dobimo

$$\cos(2u) = 2k^2, \quad \sin(2u) = 2\sqrt{3}k^2.$$

Ob upoštevanju dejstva $\sin^2(2u) + \cos^2(2u) = 1$ dobimo

$$\begin{aligned}
1 &= 12k^4 + 4k^4\\
k^4 &= \frac{1}{16}\\
0 &= \left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\left(k^2 + \frac{1}{4}\right)\\
k_{1,2} &= \pm\frac{1}{2} \Rightarrow v_{1,2} = \mp 1
\end{aligned}$$

Za u nam tako ostane rešiti še sistem

$$\cos(2u) = \frac{1}{2}, \quad \sin(2u) = \frac{\sqrt{3}}{2},$$

ki ima rešitev

$$2u_\ell = \frac{\pi}{3} + 2\ell\pi \ (\ell \in \mathbb{Z}) \quad \Longrightarrow \quad u_\ell = \frac{\pi}{6} + \ell\pi \ (\ell \in \mathbb{Z})$$

Ker v $\vec{r}(u, v)$ nastopata $\sin u$ in $\cos u$, ki imata periodo enako 2π, je dovolj vzeti le dva predstavnika te rešitve $u_0 = \frac{\pi}{6}$, $u_1 = \frac{7\pi}{6}$.

Načeloma smo tako dobili štiri rešitve

$$\vec{r}\left(\frac{\pi}{6}, 1\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$
$$\vec{r}\left(\frac{7\pi}{6}, 1\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$
$$\vec{r}\left(\frac{\pi}{6}, -1\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$
$$\vec{r}\left(\frac{7\pi}{6}, -1\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

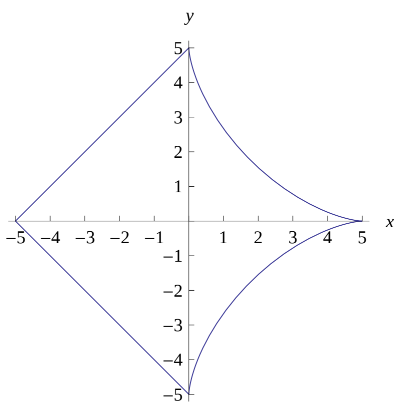
a opazimo, da sta med njimi v resnici le dve različni točki

$$T_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad T_2\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

2. Vzemimo dve krivulji v ravnini: astroido $\vec{r}(t) = (5\cos^3 t, 5\sin^3 t)$ in kvadrat $|x|+|y| = 5$. Iz njiju tvorimo krivuljo C, ki je za $x \geq 0$ enaka dani astroidi in za $x \leq 0$ enaka danemu kvadratu. Izračunajte krivuljni integral 1. vrste

$$\int_C x\, ds.$$

Rešitev. Parametrizirali bomo posebej kvadrat in posebej astroido. Skica krivulje:



Kvadrat: ločiti treba $y \geq 0$ in $y \leq 0$.

- $x = t$, $y = t + 5$, $-5 \leq t \leq 0$, $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2}$.
$$I_1 = \int_{-5}^{0} t\sqrt{2}\, dt = \frac{t^2\sqrt{2}}{2}\Big|_{-5}^{0} = -\frac{25\sqrt{2}}{2}$$
- $x = t$, $y = -t - 5$, $-5 \leq t \leq 0$, $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2}$.
$$I_2 = \int_{-5}^{0} t\sqrt{2}\, dt = \frac{t^2\sqrt{2}}{2}\Big|_{-5}^{0} = -\frac{25\sqrt{2}}{2}$$

Astroida:

$$\begin{aligned} x &= 5\cos^3 t,\ y = 5\sin^3 t \\ \dot{x} &= -15\cos^2 t \sin t,\ \dot{y} = 15\sin^2 t \cos t \\ \sqrt{\dot{x}^2 + \dot{y}^2} &= \sqrt{225\cos^4 t \sin^2 t + 225\sin^4 t \cos^2 t} = 15|\sin t \cos t| \end{aligned}$$

Ker je izraz pod dobljeno absolutno vrednostjo za $0 \leq t \leq \frac{\pi}{2}$ pozitiven, za $-\frac{\pi}{2} \leq t \leq 0$ pa negativen, treba ločiti ta dva primera (uvedli bomo substitucijo $\cos t = u$):

$$\begin{aligned} I_3 &= \int_0^{\frac{\pi}{2}} 5\cos^3 t(15\sin t\cos t)\, dt = -\int_1^0 75u^4 du = -15u^5\Big|_1^0 = 15 \\ I_4 &= \int_{-\frac{\pi}{2}}^{0} 5\cos^3 t(-15\sin t\cos t)\, dt = \int_0^1 75u^4 du = 15u^5\Big|_0^1 = 15 \end{aligned}$$

Celoten integral tako pride $I_1 + I_2 + I_3 + I_4 = 30 - 25\sqrt{2}$.

3. Izračunajte ploskovni integral 1. vrste
$$\iint_S x^2\, dS,$$
kjer je ploskev S določena z $x^2 + y^2 + z^2 = 4$, $z \geq 0$, $x \geq 0$.

Rešitev. Parametrizirajmo ploskev:
$$\vec{r}(\varphi, \vartheta) = (2\cos\varphi\cos\vartheta, 2\sin\varphi\cos\vartheta, 2\sin\vartheta)$$
kjer zaradi omejitev $x \geq 0$ in $z \geq 0$ velja $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ in $0 \leq \vartheta \leq \frac{\pi}{2}$.
$$\begin{aligned} \vec{r_\varphi} &= (-2\sin\varphi\cos\vartheta, 2\cos\varphi\cos\vartheta, 0) \\ \vec{r_\vartheta} &= (-2\cos\varphi\sin\vartheta, -2\sin\varphi\sin\vartheta, 2\cos\vartheta) \\ E &= \vec{r_\varphi}\cdot\vec{r_\varphi} = 4\cos^2\vartheta \\ F &= \vec{r_\varphi}\cdot\vec{r_\vartheta} = 0 \\ G &= \vec{r_\vartheta}\cdot\vec{r_\vartheta} = 4 \\ \sqrt{EG - F^2} &= \sqrt{16\cos^2\vartheta} = 4|\cos\vartheta| = 4\cos\vartheta \end{aligned}$$

zadnji enačaj velja, saj je $\cos\vartheta$ vedno pozitiven za naš razpon kota ϑ. Tako dobimo, da je iskani integral enak (tekom računanja uvedemo substitucijo $u = \sin\vartheta$)

$$\begin{aligned}
\iint_S x^2\,dS &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} 4\cos^2\varphi\cos^2\vartheta \cdot 4\cos\vartheta \cdot d\vartheta = \\
&= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos^2\varphi\cos^3\vartheta d\vartheta = \\
&= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos^2\varphi(1-\cos^2\vartheta)\cos\vartheta d\vartheta = \\
&= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{1} \cos^2\varphi(1-u^2)du = \\
&= \ldots = \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi\,d\varphi = \frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big(1+\cos(2\varphi)\big)d\varphi = \\
&= \ldots = \frac{16\pi}{3}
\end{aligned}$$

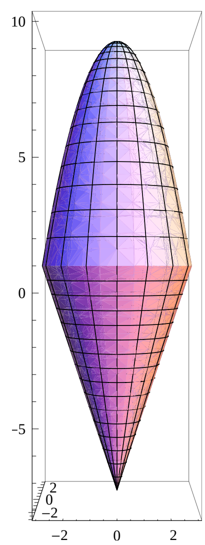
4. S pomočjo Gaussovega izreka izračunajte pretok vektorskega polja

$$\vec{V} = \big(\sin(xy) - e^{\sin x},\ 2x^2y + y\cos(x)e^{\sin x},\ 3x^2z - yz\cos(xy)\big)$$

skozi ploskev, ki je rob telesa določenega z neenačbami:

$$z \geq 3\sqrt{x^2+y^2} - 8, \quad z \leq 10 - (x^2+y^2), \quad x \geq 0.$$

Rešitev.



$$\vec{V} = \vec{V} = \left(\sin(xy) - e^{\sin x},\ 2x^2y + y\cos(x)e^{\sin x},\ 3x^2z - yz\cos(xy)\right)$$

$$\operatorname{div}\vec{V} = \frac{\partial\left(\sin(xy) - e^{\sin x}\right)}{\partial x} + \frac{\partial\left(2x^2y + y\cos(x)e^{\sin x}\right)}{\partial y} + \frac{\partial\left(3x^2z - yz\cos(xy)\right)}{\partial z} =$$

$$= y\cos(xy) - \cos(x)e^{\sin x} + 2x^2 + \cos(x)e^{\sin x} + 3x^2 - y\cos(xy) = 5x^2$$

Uvedemo cilindrične koordinate. Presečisče ploskev:

$$\begin{aligned} 3r - 8 &= 10 - r^2 \\ (r-3)(r+6) &= 0 \\ r_1 &= 3 \\ r_2 &= -6 \end{aligned}$$

kjer zaradi pozitivnosti dobimo le rešitev $r = 3$ Tako se nam iskani pretok vektorskega polja po predlaganem Gaussovem izreku prevede do

$$\begin{aligned} ... &= \iiint_V \operatorname{div}\vec{V} dxdydz = \iiint_V 5x^2 dxdydz = \\ &= 5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^3 dr \int_{3r-8}^{10-r^2} \left(r^2\cos^2\varphi\right) r dz = \\ &= ... = 5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{r^6}{6} - \frac{3r^5}{5} + \frac{18r^4}{4}\right)\cos^2\varphi \Big|_0^3 = \\ &= 486\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi\, d\varphi = 243\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + \cos(2\varphi)\right) d\varphi = \\ &= 243\pi \end{aligned}$$

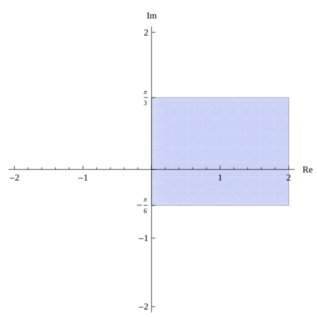
5. Narišite območje

$$D = \left\{ z = x + iy \mid x \geq 0,\ -\frac{\pi}{6} \leq y \leq \frac{\pi}{3} \right\}$$

in sliko tega območja s preslikavo

$$f(z) = \frac{e^{3z} - i}{e^{3z}}.$$

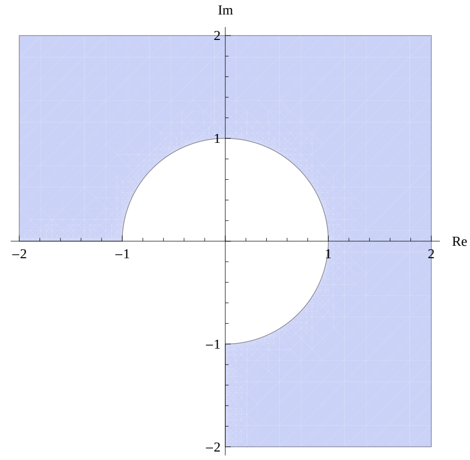
Rešitev. Začetno območje D:



Najprej na območju D uporabimo preslikavo

$$z \mapsto w = e^{3z} = e^{3x+3iy} = e^{3x}\big(\cos(3y) + i\sin(3y)\big)$$

Ker je $x \geq 0$, velja $r = e^{3x} \geq 1$ in ker je $y \in \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$, velja $\varphi = 3y \in \left[-\frac{\pi}{2}, \pi\right]$.



Sedaj pa uporabimo preslikavo $w \mapsto \frac{w-i}{w}$. Ker je to lomljena racionalna funkcija, vemo, da preslika družino premic in krožnic v družino premic in krožnic. Zato je za vsak del dobljene premice oz. krožnice dovolj pogledati sliko treh točk. Tako recimo hitro dobimo da velja

$$\begin{aligned} -1 &\mapsto 1+i \\ \infty &\mapsto 1 \\ 1 &\mapsto 1-i \\ -i &\mapsto 2 \\ i &\mapsto 0 \end{aligned}$$