Izpit iz Matematike III

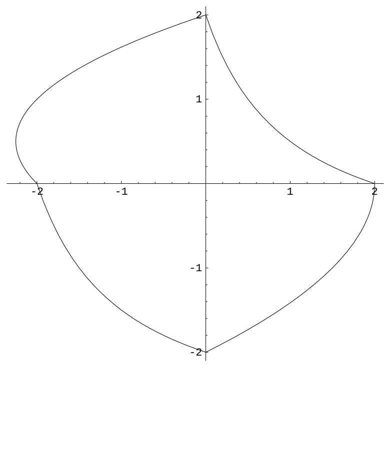
6. junij 2006

1. Vzemimo ploskev
$$\vec{r}(u,v) = \big(2u\cos v - \sqrt{2},\ 2u\sin v + 1,\ 4u - 4v - 4\big)$$
in koordinatni krivulji
$$\vec{r}_1(t) = \big(2\cos t - \sqrt{2},\ 2\sin t + 1,\ -4t\big)$$
$$\vec{r}_2(t) = \big(t\sqrt{2} - \sqrt{2},\ t\sqrt{2} + 1,\ 4t - \pi - 4\big).$$
Poiščite presečišče koordinatnih krivulj $\vec{r}_1, \vec{r}_2$ in enačbo tangentne ravnine na ploskev $\vec{r}(u,v)$ v tem presečišču.

Rešitev. Presečišče je pri $u = 1, v = \frac{\pi}{4}$, torej $T\big(0, \sqrt{2}+1, -\pi\big)$.
$$\vec{r}_u(u,v) = (2\cos v, 2\sin v, 4)$$
$$\vec{r}_v(u,v) = (-2u\sin v, 2u\cos v, -4)$$
$$\vec{r}_u\Big(1, \frac{\pi}{4}\Big) = \big(\sqrt{2}, \sqrt{2}, 4\big)$$
$$\vec{r}_v\Big(1, \frac{\pi}{4}\Big) = \big(-\sqrt{2}, \sqrt{2}, -4\big)$$
$$\vec{n}\Big(1, \frac{\pi}{4}\Big) = \vec{r}_u\Big(1, \frac{\pi}{4}\Big) \times \vec{r}_v\Big(1, \frac{\pi}{4}\Big) = \big(-8\sqrt{2}, 0, 4\big)$$
Ravnina se torej glasi $-8\sqrt{2}x + 4z = -4\pi$ oziroma $2\sqrt{2}x - z = \pi$.

2. Zamenjajte vrstni red integriranja
$$\int_{-\frac{9}{4}}^{-2} dx \int_{\frac{1-\sqrt{9+4x}}{2}}^{\frac{1+\sqrt{9+4x}}{2}} dy \ + \int_{-2}^{0} dx \int_{\frac{3}{x+3}-3}^{\frac{1+\sqrt{9+4x}}{2}} dy \ + \int_{0}^{2} dx \int_{-\sqrt{4-2x}}^{\frac{3}{x+1}-1} dy.$$
Nato integral tudi izračunajte.

Rešitev. Skicirano območje pride



Torej se po zamenjavi vrstnega reda integracije integral glasi:

$$\begin{aligned}
\ldots &= \int_{-2}^{0} dy \int_{\frac{3}{y+3}-3}^{2-\frac{y^2}{2}} dx + \int_{0}^{2} dy \int_{y^2-y-2}^{\frac{3}{y+1}-1} dx = \\
&= \left(5y - \frac{y^3}{6} - 3\log(y+3)\right)\Big|_{-2}^{0} + \left(3\log(y+1) + y - \frac{y^3}{3} + \frac{y^2}{2}\right)\Big|_{0}^{2} = \\
&= 10
\end{aligned}$$

3. S pomočjo Gaussove formule izračunajte pretok vektorskega polja

$$\vec{V} = \left(x - e^{yz} + 2xz,\ \sin(xz) - \frac{y}{3},\ \arctan(x^2y^2) - z^2 - \frac{2z}{5}\right)$$

skozi zaključeno ploskev, ki je rob telesa, dobljenega s preseki

$$x^2 + y^2 \le 1, \quad z \le 2 - x^2 - 2y^2 \ \text{ in } \ z \ge x^2 + y^2 - 3.$$

Rešitev. Divergenca vektorskega polja $\vec{V}$ pride

$$\operatorname{div} \vec{V} = 1 + 2z - \frac{1}{3} - 2z - \frac{2}{5} = \frac{4}{15}$$

in zato se integral po uvedbi cilindričnih koordinat prevede do

$$\frac{4}{15} \int_0^{2\pi} d\varphi \int_0^1 dr \int_{r^2-3}^{2-r^2-r^2\sin^2\varphi} r\, dz = \ldots = \pi$$

4. S pomočjo kompleksne integracije izračunajte

$$\int_0^\infty \frac{18}{(x^2+4)(x^2+1)^2}\, dx.$$

Rešitev.

$$\begin{aligned}
&\int_0^\infty \frac{18}{(x^2+4)(x^2+1)^2}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{18}{(x^2+4)(x^2+1)^2}\, dx = \\
&= 9 \cdot 2\pi i \left(\operatorname{Res}_{x=2i} \frac{1}{(x^2+4)(x^2+1)^2} + \operatorname{Res}_{x=i} \frac{1}{(x^2+4)(x^2+1)^2} \right) = \\
&= 18\pi i \left(-\frac{i}{36} - \frac{i}{36} \right) = \pi
\end{aligned}$$

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