IZPIT IZ MATEMATIKE III

8. september 2009

1. Podana je krivulja

$$\vec{r}(t) = (\sqrt{2}\operatorname{ch} t, \sin t + \cos t, \sin t - \cos t)$$

in točki $T_1(\sqrt{2}, 1, -1)$, $T_2\big(\frac{5\sqrt{2}}{4}, \sin(\log 2) + \cos(\log 2), \sin(\log 2) - \cos(\log 2)\big)$.

(a) Določite enačbo tangentne premice na krivuljo $\vec{r}(t)$ v točki T_1.

(b) Izračunajte dolžino loka krivulje $\vec{r}(t)$ med točkama T_1 in T_2.

Rešitev.

(a) Točka T_1 je jasno dosežena pri $t = 0$. Tako dobimo

$$\dot{\vec{r}}(t) = (\sqrt{2}\operatorname{sh} t, \cos t - \sin t, \cos t + \sin t)$$
$$\dot{\vec{r}}(0) = (0, 1, 1)$$

Enačba iskane premice se tako glasi $x = \sqrt{2}$, $y - 1 = z + 1$.

(b) Točkama T_1 in T_2 pripadata $t_1 = 0$ in $t_2 = \log 2$. Tako dobimo

$$ds = \sqrt{\dot{\vec{r}}(t) \cdot \dot{\vec{r}}(t)}\, dt = \sqrt{(\sqrt{2}\operatorname{sh} t)^2 + (\cos t - \sin t)^2 + (\cos t + \sin t))^2}\, dt =$$
$$= ... = \sqrt{2\operatorname{sh}^2 t + 2}\, dt = \sqrt{2(\operatorname{sh}^2 t + 1)}\, dt = \sqrt{2\operatorname{ch}^2 t}\, dt = \sqrt{2}\operatorname{ch} t\, dt$$
$$s = \int_0^{\log 2} ds = \int_0^{\log 2} \sqrt{2}\operatorname{ch} t\, dt = \sqrt{2}\operatorname{sh} t\Big|_0^{\log 2} = \sqrt{2}\,\frac{e^{\log 2} - e^{-\log 2}}{2} =$$
$$= \sqrt{2}\,\frac{2 - \frac{1}{2}}{2} = \frac{3\sqrt{2}}{4}$$

2. Izračunajte ploščino območja, določenega z

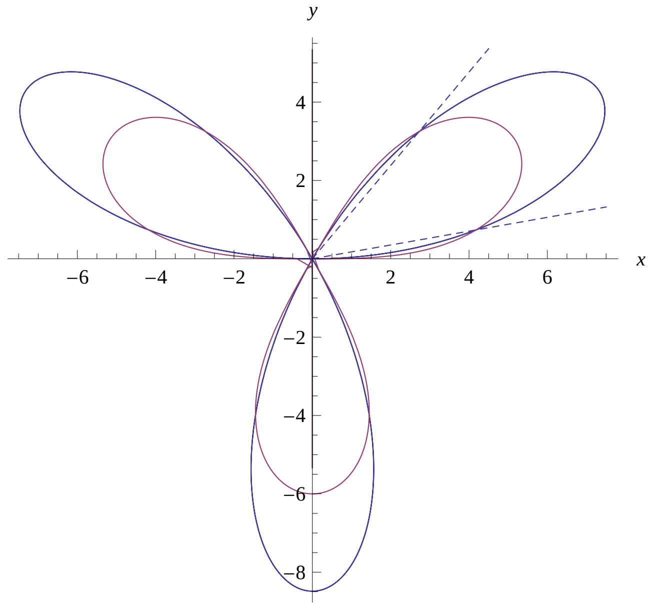
$$6\sqrt{\sin 3\varphi} \le r \le 6\sqrt{2}\sin 3\varphi.$$

Območje najprej skicirajte!

Rešitev. Najprej opazimo, da je zaradi simetričnosti dovolj gledati le prvi kvadrant (in nato rezultat enostavno pomnožimo s 3).

Krivulji se sekata, ko velja

$$6\sqrt{\sin 3\varphi} = 6\sqrt{2}\sin 3\varphi$$
$$\sin 3\varphi = 2\sin^2 3\varphi$$
$$\sin 3\varphi\,(1 - 2\sin 3\varphi) = 0$$



Z upoštevanjem, da gledamo le prvi kvadrant, dobimo

$$\begin{array}{lcl} 3\varphi = 0 & \longrightarrow & \varphi = 0 \\ 3\varphi = \dfrac{\pi}{6} & \longrightarrow & \varphi = \dfrac{\pi}{18} \\ 3\varphi = \dfrac{5\pi}{6} & \longrightarrow & \varphi = \dfrac{5\pi}{18} \end{array}$$

Tako dobimo

$$S = 3\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} d\varphi \int_{6\sqrt{\sin 3\varphi}}^{6\sqrt{2}\sin 3\varphi} r\,dr = 3\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \left(\frac{r^2}{2}\Bigg|_{6\sqrt{\sin 3\varphi}}^{6\sqrt{2}\sin 3\varphi} \right) d\varphi =$$

$$= 54\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \big(2\sin^2 3\varphi - \sin 3\varphi\big) d\varphi = 54\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \big(1 - \cos 6\varphi - \sin 3\varphi\big) d\varphi =$$

$$= 54\left(\varphi - \frac{\sin 6\varphi}{6} + \frac{\cos 3\varphi}{3}\right)\Bigg|_{\frac{\pi}{18}}^{\frac{5\pi}{18}} = 15\pi - 3\pi + \frac{9\sqrt{3}}{2} + \frac{9\sqrt{3}}{2} - 9\sqrt{3} - 9\sqrt{3} =$$

$$= 12\pi - 9\sqrt{3}$$

3. Določite parameter a tako, da bo krivuljni integral

$$\int_C \left(2x - \frac{a\cos z}{x}, \frac{z}{1+y^2z^2} + \frac{2\cos z}{y}, \frac{y}{1+y^2z^2} - (a^2-2)\log(xy)\sin z\right) \cdot d\vec{r}$$

neodvisen od poti in ga za primer, ko je C poljubna krivulja od točke $T_1(1,1,1)$ do točke $T_2(2,\frac{1}{2},0)$, izračunajte.

Rešitev. Vemo, da je krivuljni integral $\int_C \vec{V} \cdot d\vec{r}$ neodvisen od poti, ko velja rot $\vec{V} = \vec{0}$. Torej mora v našem primeru veljati

$$\left(-\frac{(a^2-4)\sin z}{y}, \frac{(a^2+a-2)\sin z}{x}, 0\right) = (0,0,0),$$

kar je res, ko velja $a^2 - 4 = (a-2)(a+2) = 0$ in $a^2 + a - 2 = (a+2)(a-2) = 0$. Tako dobimo $a = -2$.

Za izračun dobljenega integrala najprej izračunamo potencial vektorskega polja $\vec{V}$:

$$\begin{aligned}
\int \left(2x + \frac{2\cos z}{x}\right) dx &= x^2 + 2\cos z \log x + D(y,z) \\
\int \left(\frac{z}{1+y^2z^2} + \frac{2\cos z}{y}\right) dy &= \arctan(yz) + 2\cos z \log y + D(x,z) \\
\int \left(\frac{y}{1+y^2z^2} - 2\log(xy)\sin z\right) dz &= \arctan(yz) + 2\log xy \cos z + D(x,y) \\
u &= x^2 + 2\log xy \cos z + \arctan(yz) + D
\end{aligned}$$

(Upoštevali smo dejstvo $\log x + \log y = \log xy$.) Vemo, da je iskani integral enak

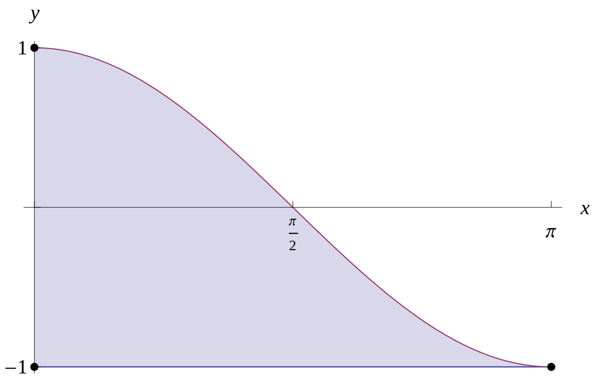
$$u(T_2) - u(T_1) = 4 - \left(1 + \frac{\pi}{4}\right) = 3 - \frac{\pi}{4}$$

4. Vzemimo točke $T_1(0,-1)$, $T_2(\pi,-1)$, $T_3(0,1)$. S pomočjo Greenove formule izračunajte integral

$$\int_C \left(\frac{y^2}{x+1} - 2xy\right) dx + \big(2y\log(x+1) + 6y\cos x\big) dy,$$

kjer je krivulja C pozitivno orientirana in sestavljena iz daljice od točke T_1 do točke T_2, krivulje $y = \cos x$ od točke T_2 do točke T_3 in daljice od točke T_3 do točke T_1.

Rešitev. Označimo $P = \frac{y^2}{x+1} - 2xy$ in $Q = 2y\log(x+1) + 6y\cos x$.



Iskani integral se nam po Greenovi formuli prevede do

$$
\begin{aligned}
\iint_D (Q_x - P_y)\,dxdy &= \int_0^{\pi} dx \int_{-1}^{\cos x} \big(2x - 6y\sin x\big)dy = \\
&= \int_0^{\pi} \big(2xy - 3y^2 \sin x\big)\Big|_{-1}^{\cos x} dx = \\
&= \int_0^{\pi} \big(2x\cos x - 3\cos^2 x \sin x + 2x + 3\sin x\big)\,dx = \\
&= \big(2\cos x + 2x\sin x + \cos^3 x + x^2 - 3\cos x\big)\Big|_0^{\pi} \\
&= \ldots = \pi^2
\end{aligned}
$$

5. Izračunajte kompleksni integral

$$\int_{|z-2-2i|=\frac{5}{2}} \frac{8}{z(z-2)^3(z-2i)}\,dz,$$

kjer je integracija v pozitivni smeri.

Rešitev. Iz teorije vemo, da je dovolj gledati residuume v singularnostih znotraj integracijskega območja, torej le v $z = 2$ in $z = 2i$. Označimo $f(z) := \frac{8}{z(z-2)^3(z-2i)}$.

$$
\begin{aligned}
\mathrm{res}_{z=2i} f(z) &= \lim_{z\to 2i} f(z)(z-2i) = \lim_{z\to 2i} \frac{8}{z(z-2)^3} = \\
&= \frac{8}{2i(2i-2)^3} = \frac{1}{2i(i-1)^3} = \frac{1}{-4+4i} = \frac{-4-4i}{16+16} = -\frac{1}{8} - \frac{i}{8} \\
\mathrm{res}_{z=2} f(z) &= \frac{1}{2}\lim_{z\to 2}\big(f(z)(z-2)^3\big)'' = \frac{1}{2}\lim_{z\to 2}\left(\frac{8}{z(z-2i)}\right)'' = \frac{1}{2}\lim_{z\to 2}\left(\frac{-16z+16i}{(z^2-2iz)^2}\right)' \\
&= \frac{1}{2}\lim_{z\to 2}\frac{-16(z^2-2iz)^2 - (-16z+16i)2(z^2-2iz)(2z-2i)}{(z^2-2iz)^4} = \\
&= \frac{1}{2}\,\frac{-16(4-4i)^2 - (-32+16i)2(4-4i)(4-2i)}{(4-4i)^4} = \\
&= \frac{1}{2}\,\frac{-(1-i)^2 - (-2+i)(1-i)(2-i)}{(1-i)^4} = \frac{-(1-i) - (-2+i)(2-i)}{2(1-i)^3} = \\
&= \ldots = \frac{2-3i}{-4-4i} = \frac{(2-3i)(-4+4i)}{16+16} = \ldots = \frac{1}{8} + \frac{5i}{8}
\end{aligned}
$$

Iskani integral je tako enak

$$
\begin{aligned}
\int_{|z-2-2i|=\frac{5}{2}} f(z)\,dz &= 2\pi i\big(\mathrm{res}_{z=2i} f(z) + \mathrm{res}_{z=2} f(z)\big) = \\
&= 2\pi i\left(-\frac{1}{8} - \frac{i}{8} + \frac{1}{8} + \frac{5i}{8}\right) = 2\pi i\,\frac{i}{2} = \\
&= -\pi
\end{aligned}
$$