Izpit Matematika IV

22. avgust 2011

Rešitve

1. naloga

$$sY(s) - 2aY(s) + (a^2+1)\frac{Y(s)}{s} = \frac{1}{s}$$

$$\left[s^2 - 2a + (a^2+1)\right]Y(s) = 1$$

$$Y(s) = \frac{1}{(s-a)^2+1} = F(s-a) \quad , \; F(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \quad \to \quad \boxed{y(t) = e^{at}\sin t}$$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$
$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$
$$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1) x^n + \sum_{n=1}^{\infty} C_{n-1} x^n = 0$$
$$2C_2 + \sum_{n=1}^{\infty} \Big[C_{n+2}(n+2)(n+1) + C_{n-1}\Big] x^n = 0$$

$$C_2 = 0 \quad , \quad C_{n+2} = \frac{-1}{(n+2)(n+1)} C_{n-1}$$

$$y(0) = 1 \quad \to \quad C_0 = 1 \quad , \quad C_3 = -\tfrac{1}{6} \quad , \quad C_6 = \tfrac{1}{180} \quad , \quad \dots$$
$$y'(0) = 0 \quad \to \quad C_1 = 0 \quad \to \quad C_{3k+1} = 0$$
$$C_2 = 0 \quad \to \quad C_{3k+2} = 0$$

$$\boxed{y = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \cdots}$$

3. naloga

$$u(x,t) = F(x)G(t)$$
$$F(x)G'(t) = c^2 F''(x)G(t)$$
$$\frac{1}{c^2}\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda^2$$

$$F''(x) + \lambda^2 F(x) = 0$$
$$F(x) = A\cos\lambda x + B\sin\lambda x$$
$$x = 0 \quad \to \quad A = 0$$
$$x = 10 \quad \to \quad B\sin(\lambda 10) = 0 \quad \to \quad \lambda_n = \tfrac{n\pi}{10}$$
$$F_n(x) = B_n \sin\left(\frac{n\pi}{10}x\right) \quad , \quad n = 1, 2, \ldots$$

$$\frac{1}{c^2}\frac{G'(t)}{G(t)} = -\lambda_n^2$$
$$G_n(t) = C_n e^{-\lambda_n^2 c^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{10}x\right) e^{-(\frac{n\pi c}{10})^2 t}$$
$$t = 0 \quad \to \quad \sum_{n=1}^{\infty} A_n \left(\tfrac{n\pi}{10}x\right) = \sin\tfrac{\pi x}{10}$$
$$A_n = 0 \text{ za vsak } n \text{ razen } A_1 = 1$$

$$\boxed{u(x,t) = \sin\left(\tfrac{\pi x}{10}\right) e^{-(\frac{\pi c}{10})^2 t}}$$

4. naloga

Zapišemo *Eulerjevo* diferencialno enačbo za funkcional

$$F(y) = \int_0^1 \left(y'^2 + x^2 + \lambda y^2\right) dx$$

$2\lambda y - (2y')' = 0$
$y'' - \lambda y = 0$

Karakterisična enačba $r^2 - \lambda = 0$ ima korena $r_{1,2} = \pm\sqrt{\lambda}$
Ločimo tri primere:

$\lambda = k^2 > 0$
$y = Ae^{kx} + Be^{-kx}$
$x = 0 \quad \rightarrow \quad A + B = 0 \quad \rightarrow \quad B = -A$
$x = 1 \quad \rightarrow \quad A\left(e^k - e^{-k}\right) = 0 \quad \rightarrow \quad A = B = 0 \quad \rightarrow \quad y = 0$
Ni ekstremala, ker ne zadošča pogoju $\int_0^1 y^2 = 2$

$\lambda = 0$
$y = Ax + B$
$x = 0 \quad \rightarrow \quad B = 0$
$x = 1 \quad \rightarrow \quad A = 0 \quad \rightarrow \quad y = 0$
Ni ekstremala, ker ne zadošča pogoju $\int_0^1 y^2 = 2$

$\lambda = -k^2 < 0$
$y = A\cos kx + B\sin kx$

$x = 0 \quad \rightarrow \quad A = 0$
$x = 1 \quad \rightarrow \quad k = n\pi \quad \rightarrow \quad y = B\sin(n\pi x)\ ,\ n \in N$
y vstavimo v pogoj $\int_0^1 y^2 = 2$ in določimo konstanto B :

$$\int_0^1 B^2 \sin^2(n\pi x)\, dx = B^2 \frac{1}{2} = 2 \quad \rightarrow \quad B = \pm 2$$

$$\boxed{y = \pm 2\sin(n\pi x)\ ,\ n \in N}$$

5. naloga

$$\int_{-\infty}^{\infty} p(x)\,dx \;=1 \quad \rightarrow \quad A\int_{0}^{1} x(1-x)\,dx \;= A\frac{1}{6}=1 \quad \rightarrow \qquad \boxed{A=6}$$

$$P\big(X\leq \tfrac{1}{2}|\tfrac{1}{3}\leq X\leq \tfrac{2}{3}\big)=P\big(X\leq \tfrac{2}{3}\big)=\int_{0}^{\frac{2}{3}} 6x(1-x)\,dx \;=\boxed{\frac{20}{27}}$$