2. Kolokvij MATEMATIKA IV

7.6.2012

Bolonjski študij

1. (30%) Z vpeljavo neodvisnih spremenljivk $u = x$, $v = xy$ poiščite tisto rešitev $z(x, y)$ diferencialne enačbe

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x^3 y \quad ,$$

ki zadošča pogoju $z(x, x) = 0$!

2. (40%) Poiščite rešitev $u(x, t)$ parcialne diferencialne enačbe

$$\begin{aligned} u_{xx} &= u_t \\ u(0,t) &= 0 \\ u(\pi,t) &= 0 \\ u(x,0) &= x \quad ! \end{aligned}$$

3. (30%) Poiščite ekstremalo funkcionala

$$I(y) = \int_1^2 y^2 y'^2 \, dx$$

$$\begin{aligned} y(1) &= 1 \\ y(2) &= 2 \quad ! \end{aligned}$$

2. Kolokvij MATEMATIKA IV

7.6.2012

Pred-Bolonjski program

1. (30%) Z vpeljavo neodvisnih spremenljivk $u = x$, $v = xy$ poiščite tisto rešitev $z(x,y)$ diferencialne enačbe

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x^3y \quad ,$$

ki zadošča pogoju $z(x,x) = 0$!

2. (40%) Poiščite rešitev $u(x,t)$ parcialne diferencialne enačbe

$$\begin{aligned} u_{xx} &= u_t \\ u(0,t) &= 0 \\ u(\pi,t) &= 0 \\ u(x,0) &= x \quad ! \end{aligned}$$

3. Vržemo tri kocke, slučajna spremenljivka $X =$ število padlih **šestic**.

   (a) (10%) Podajte verjetnostno funkcijo slučajne spremenljivke X!

   (b) (20%) Po prvem metu obdržimo kocke padle na šestico, jih še enkrat vržemo in slučajna spremenljivka $Y =$ število šestic v tem metu. Podajte verjetnostno funkcijo slučajne spremenljivke Y!

Rešitve

1. naloga

$z_x = z_u \cdot 1 + z_v \cdot y$

$z_y = z_u \cdot 0 + z_v \cdot x$

$x(z_u + z_v y) - y z_v x = 2x^3 y$

$z_u = 2x^2 y$

$z_u = 2uv$

$z = \int 2uv dv = u^2 v + C(v)$

$z = x^3 y + C(xy)$

$z(x, x) = 0 \quad \to \quad x^4 + C(x^2) = 0 \quad \to \quad C(t) = -t^2$

$$\boxed{z = x^3 y - x^2 y^2}$$

2. naloga

$$u = F(x)G(t)$$
$$F''(x)G(t) = F(x)G'(t)$$
$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = -\lambda^2$$

$$F''(x) + \lambda^2 F(x) = 0$$
$$k^2 + \lambda^2 = 0$$
$$k_{1,2} = \pm\lambda i$$
$$F(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$x = 0 \quad \rightarrow \quad A = 0$$
$$x = \pi \quad \rightarrow \quad \sin(\lambda\pi) = 0 \quad \rightarrow \quad \lambda_n = n$$
$$F_n(x) = B_n \sin(nx)$$

$$\frac{G'(t)}{G(t)} = -n^2$$
$$\int \frac{dG}{G} = \int (-n^2)dt$$
$$\ln G = -n^2 t + \ln C$$
$$G_n(t) = C_n e^{-n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}$$

$$t = 0 \quad \rightarrow \quad \sum_{n=1}^{\infty} b_n \sin(nx) = x$$

Funkcijo x razvijemo v *Fourierovo sinusno* vrsto na intervalu $(0,\pi)$

Iz priročnika prepišemo - glej *Bronstein* stran 853, primer 3

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

S primerjavo sklepamo na koeficiente b_n :

$$b_n = (-1)^{n+1}\frac{2}{n}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx)e^{-n^2 t}}$$

3. naloga

V funkcionalu pod integralom manjka neodvisna spremenljivka x.

Eulerjevo diferencialno enačbo rešujemo v obliki $f - y' f_{y'} = C$.

$$y^2 y'^2 - y' y^2 2y' = C = -\frac{A^2}{4}$$

$$y'^2 y^2 = \frac{A^2}{4}$$

$$2yy' = A$$

$$y^2 = Ax + B$$

$$1 = A + B \ , \ 4 = 2A + B$$

$$A = 3 \ , \ B = -2$$

$$\boxed{y = \sqrt{3x - 2}}$$

3. naloga

a)

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 \\ (\frac{5}{6})^3 & 3(\frac{5}{6})^2\frac{1}{6} & 3\frac{5}{6}(\frac{1}{6})^2 & (\frac{1}{6})^3 \end{pmatrix}$$

b)

Označimo $X_i = (X = i)$, $Y_k = (Y = k)$ in uporabimo formulo

$$P(Y_k) = \sum_{i=0}^{3} P(X_i)P(Y_k/X_i)$$

$$P(Y_3) = P(X_3)P(Y_3/X_3) = (\tfrac{1}{6})^3 \cdot (\tfrac{1}{6})^3 = \tfrac{1}{6^6}$$

$$P(Y_2) = P(X_3)P(Y_2/X_3) + P(X_2)P(Y_2/X_2) = \\ (\tfrac{1}{6})^3 \cdot 3\tfrac{5}{6}(\tfrac{1}{6})^2 + 3\tfrac{5}{6}(\tfrac{1}{6})^2 \cdot (\tfrac{1}{6})^2 = \tfrac{15}{6^6} + \tfrac{15}{6^5}$$

$$P(Y_1) = P(X_3)P(Y_1/X_3) + P(X_2)P(Y_1/X_2) + P(X_1)P(Y_1/X_1) = \\ (\tfrac{1}{6})^3 \cdot 3(\tfrac{5}{6})^2\tfrac{1}{6} + 3\tfrac{5}{6}(\tfrac{1}{6})^2 \cdot 2\tfrac{1}{6}\tfrac{5}{6} + 3(\tfrac{5}{6})^2\tfrac{1}{6} \cdot \tfrac{1}{6} = \tfrac{75}{6^6} + \tfrac{150}{6^5} + \tfrac{75}{6^4}$$

$$P(Y_0) = P(X_3)P(Y_0/X_3)+P(X_2)P(Y_0/X_2)+P(X_1)P(Y_0/X_1)+P(X_0) = \\ (\tfrac{1}{6})^3\cdot(\tfrac{5}{6})^3+3\tfrac{5}{6}(\tfrac{1}{6})^2\cdot(\tfrac{5}{6})^2+3(\tfrac{5}{6})^2\tfrac{1}{6}\cdot\tfrac{5}{6}+(\tfrac{5}{6})^3 = \tfrac{125}{6^6}+\tfrac{375}{6^5}+\tfrac{375}{6^4}+\tfrac{125}{6^3}$$

Samo za preizkus. **Ni potrebno za oceno**:

$$p_3 = P(Y_3) = \tfrac{1}{6^6}$$

$$p_2 = P(Y_2) = \tfrac{15}{6^6}(1+6) = 3\tfrac{35}{6^6}$$

$$p_1 = P(Y_1) = \tfrac{75}{6^6}(1+2\cdot 6+6^2) = \tfrac{75}{6^6}(1+6)^2 = 3\tfrac{35^2}{6^6}$$

$$p_0 = P(Y_0) = \tfrac{125}{6^6}(1+3\cdot 6+3\cdot 6^2+6^3) = \tfrac{125}{6^6}(1+6)^3 = \tfrac{35^3}{6^6}$$

$$Y : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{35^3}{6^6} & 3\frac{35^2}{6^6} & 3\frac{35}{6^6} & \frac{1}{6^6} \end{pmatrix}$$

$$p_0+p_1+p_2+p_3 = \tfrac{35^3}{6^6} + 3\tfrac{35^2}{6^6} + 3\tfrac{35}{6^6} + \tfrac{1}{6^6} = \tfrac{(35+1)^3}{6^6} = 1$$