Fourierov transform

Fourierova transformacija in inverzna transformacija:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \qquad f(t) = F^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

Eksistenca:

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

Lastnosti:

$$F[af(t) + bg(t)] = aF(f) + bF(g)$$

$$F[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$F\left[\overline{f(t)}\right] = \overline{F(-\omega)}$$

$$F[f(t-a)] = e^{ia\omega}F[f(t)] = e^{ia\omega}F(\omega)$$

$$F\left[e^{iat}f(t)\right] = F(\omega + a)$$

Konvolucija:

$$h(t) = \int_{-\infty}^{\infty} f(t-u)g(u)du$$

$$F[h(t)] = F[f(t)] \cdot F[g(t)]$$

Parsevalova enačba:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Odvod:

$$F^{(n)}(\omega) = F\left[(it)^n f(t)\right] \qquad F\left[f^{(n)}(t)\right] = (-i\omega)^n F[f(t)]$$

Sinusna transformacija in inverzna transformacija:

$$F_S[f(t)] = \int_0^{\infty} f(t)\sin\omega t dt \qquad F^{-1}[F_S(\omega)] = \frac{2}{\pi}\int_0^{\infty} F(\omega)\sin\omega t d\omega$$

Kosinusna transformacija in inverzna transformacija:

$$F_C[f(t)] = \int_0^{\infty} f(t)\cos\omega t dt \qquad F^{-1}[F_C(\omega)] = \frac{2}{\pi}\int_0^{\infty} F(\omega)\cos\omega t d\omega$$

Parsevalova enačba za sinusno in kosinusno transformacijo

$$\int_0^{\infty} |f(t)|^2 dt = \frac{2}{\pi}\int_0^{\infty} |F(\omega)|^2 d\omega$$

Lastnosti sinusne in kosinusne transformacije in predpostavke:

$$F_S[f'(t)] = -\omega F_C[f(t)] \qquad \lim_{t\to\infty} f(t) = 0$$

$$F_C[f'(t)] = -f(0) + \omega F_S[f(t)] \qquad \lim_{t\to\infty} f'(t) = 0$$

$$F_S[f''(t)] = \omega f(0) - \omega^2 F_S[f(t)] \qquad \lim_{t\to\infty} f(t)\sin\omega t = 0$$

$$F_C[f''(t)] = -f'(0) - \omega^2 F_C[f(t)] \qquad \lim_{t\to\infty} f(t)\cos\omega t = 0$$

Laplaceov transform

Laplaceova transformacija:

$$F(s)=\int_0^\infty e^{-st}dt$$

Eksistenca:

$$|f(t)|\le Me^{\alpha t}$$

Izrek o premiku:

$$L\left[e^{at}f(t)\right]=F(s-a)$$

Tabela transformacij:

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$t^a,\ a>0$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\cosh at$	$\frac{s}{s^2-a^2}$

Transformacija odvoda:

$$L\left[f^{(n)}\right]=s^nL(f)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$$

Transformacija integrala:

$$L\left[\int_0^t f(\tau)d\tau\right]=\frac{1}{s}L\left[f(t)\right]=\frac{F(s)}{s}$$

Kompleksna inverzna ali *Bromwicheva* formula :

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}F(s)ds$$

Inverzna transformacija in residui:

$$f(t)=\sum res\ e^{st}F(s)$$

-osnovna formula:

$$\underset{z=z_0}{res}\,f(z)=\lim_{z\to z_0}f(z)(z-z_0)$$

-pol prve stopnje:

$$\underset{z=z_0}{res}\frac{P(z)}{Q(z)}=\frac{P(z_0)}{Q'(z_0)}$$

-pol *m*-te stopnje:

$$\underset{z=z_0}{res}\,f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Odvod transformiranke:

$$F'(s)=-L\left[tf(t)\right]$$

Integral transformiranke:

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(\sigma)d\sigma$$

Eksistenca integrala:

$$\lim_{t\to 0+}\frac{f(t)}{t}\ \text{obstaja}$$

Konvolucija:

$$h(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$H(s)=F(s)\cdot G(s)$$

Enotina stopnica:

$$u_a=\begin{cases}0, t<a\\ 1, t>a\end{cases}$$

Izrek o pomiku:

$$e^{-as}F(s)=L\left[f(t-a)u_a(t)\right]$$

Transformacija periodičnih funkcij:

$$f(t+p)=f(t)\qquad L\left[f(t)\right]=\frac{1}{1-e^{-sp}}\int_0^p e^{-st}f(t)dt$$

Specialne funkcije

Gama funkcija:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt\,,\; x > 0$$

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

Lastnosti:

$$\Gamma(n+1) = n!$$

$$\lim_{x\to 0}\Gamma(x) = \lim_{x\to 0}\frac{\Gamma(x+1)}{x} = \infty$$

$$\lim_{x\to\infty}\Gamma(x) = \infty$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Beta funkcija:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt\,,\; x > 0\,,\; y > 0$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Lastnosti:

$$B(x,y) = B(y,x)$$

Legendrova diferencialna enačba: n=0:

$$(1-x)^2 y'' - 2xy' + n(n+1)y = 0 \qquad (1-x)^2 y'' - 2xy' = 0$$

Besselova diferencialna enačba: n=0:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \qquad xy'' + y' + xy = 0$$

Legendrovi polinomi:

$$P_n(x) = \sum_{k=0}^{N}(-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \qquad N = n/2 \text{ za sode } n \qquad N = (n-1)/2 \text{ za lihe } n$$

Rodriguesova formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$$

Besselova funkcija prve vrste reda ν:

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(m+\nu+1)}$$

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\Gamma(m-\nu+1)}$$

če velja $\nu = n \in N$:

$$J_n(x) = x^n \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad n = 1,2,...$$

Lastnosti:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x) \qquad \frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$$

Rekurtančni formuli

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) \qquad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$$

Eulerjeva konstanta:

$$\gamma = \lim_{s\to\infty}\left(1 + \frac{1}{2} + ... + \frac{1}{s} - \ln s\right)$$

Besselova funkcija druge vrste reda 0:

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right]$$

Lastnosti:

$$\lim_{n \to 0} Y_0(x) = -\infty$$

Besselova funkcija druge vrste reda ν:

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos \nu\pi - J_{-\nu}(x) \right]$$

Lastnosti:

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$