Matematika 4

4. vaja

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Reši parcialno diferencialno enačbo $\frac{\partial^2}{\partial x^2} u(x,y) = 0$

- $$\frac{\partial^2}{\partial x^2} u(x,y) = 0.$$
- $\frac{\partial}{\partial x} u(x,y) = f(y)$.
- $u(x,y) = f(y)x + g(y)$.

Reši parcialno diferencialno enačbo $\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0$

- $$\frac{\partial^2}{\partial x \partial y} u(x, y) = 0.$$
- $\frac{\partial}{\partial x} u(x, y) = f'(x)$.
- $u(x, y) = f(x) + g(y)$.

Reši parcialno diferencialno enačbo $\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial u(x,y)}{\partial x} = 0$

- $\frac{\partial^2}{\partial x \partial y} u(x,y) + \frac{\partial}{\partial x} u(x,y) = 0.$
- Uvedemo novo spremenljivko $u(x,y)_x = v(x,y)$.
- $v(x,y)_y + v(x,y) = 0 \rightarrow v(x,y) = f'(x)e^{-y}.$
- $u_x(x,y) = f'(x)e^{y} \rightarrow u(x,y) = f(x)e^{-y} + g(y).$
- $u(x,y) = f(x)e^{-y} + g(y).$

Reši parcialno diferencialno enačbo $\frac{\partial^2 u(x,y)}{\partial x^2} + u(x,y) = 0$

- $\dfrac{\partial^2}{\partial x^2} u(x,y) + u(x,y) = 0.$
- $u(x,y) = f(y)\cos x + g(y)\sin x.$

Reši parcialno diferencialno enačbo $\frac{\partial u(x,y)}{\partial y} = x$

- $\frac{\partial}{\partial y} u(x, y) = x$
- $u(x, y) = x\,y + f(x)$.

Reši parcialno diferencialno enačbo $\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial u(x,y)}{\partial x} + x + y = 0$

- $\frac{\partial^2 u}{\partial x \partial y} u(x,y) + \frac{\partial u}{\partial x} u(x,y) + x + y = 0.$
- Uvedemo novo spremenljivko $u(x,y)_x = v(x,y) \rightarrow v(x,y)_y + v = -x - y.$
- Rešitev homogene enačbe $v_{hy} + v_h = 0$ je $v_h = f'(x) e^{-y}.$
- Partikularna rešitev $\overline{v}(x,y) = C(y) e^y.$
- $\overline{v}(x,y)_y + \overline{v}(x,y) = -x - y \rightarrow C'(y) e^{-y} - C(y) e^{-y} + C(y) e^{-y} = -x - y \rightarrow C'(y) = (-x - y) e^y \rightarrow$
- $C(y) = -x e^y - y e^y + e^y \rightarrow \overline{v}(x,y) = -x - y + 1.$
- $u(x,y)_x = v(x,y) = f'(x) e^{-y} - x - y + 1 \rightarrow$
- $u(x,y) = f(x) e^{-y} - \frac{x^2}{2} - x\,y - x + g(y).$

Reši parcialno diferencialno enačbo $\frac{\partial u(x,y)}{\partial x} = 2x\,y\,u(x,y)$

- Vzamemo, da je y konstanta.
- Navadna diferencialna enačba ima ločljive spremenljivke.
- Pišemo $v(x) = u(x,y) \to$
- $v' = 2x\,y\,v \to \frac{dv}{v} = 2x\,y\,dx$.
- $\ln|v| = x^2 y + \ln|C| \to v(x) = C\,e^{x^2 y}$.
- $u(x,y) = C(y)e^{x^2 y}$.

Reši parcialno diferencialno enačbo $\frac{\partial u(x,y)}{\partial x} + x\frac{\partial^2 u(x,y)}{\partial x^2} = y$

- Uvedemo novo spremenljivko in vzamemo, da je y konstanta.
- $v(x) = u(x,y)_x \rightarrow v + xv' = y \rightarrow x\frac{dv}{dx} = y - v$.
- Ločimo spremenljivke $\frac{dv}{y-v} = \frac{dx}{x}$.
- $\ln|y-v| = \ln|x| + \ln|C| \rightarrow y - v = C\,x \rightarrow v = y - Cx$.
- $u(x,y)_x = y + f(y)x \rightarrow u(x,y) = x\,y + f(y)\frac{x^2}{2} + g(y)$.
- $u(x,y) = x\,y + f(y)\dfrac{x^2}{2} + g(y)$.

Reši parcialno diferencialno enačbo $\frac{\partial^2 u(x,y)}{\partial y^2} + \frac{\partial u(x,y)}{\partial y} = xy$

- Uvedemo novo spremenljivko in vzamemo, da je x konstanta.
- $v(y) = u(x,y)_y \to v' + v = x\,y.$
- Rešitev homogene enačbe je $v(y) = Ce^{-y}$.
- Nastavek za partikularno rešitev $\overline{v}(y) = C(y)e^{-y}$.
- $C'(y)e{-}y = x\,y \to C'(y) = x\,y\,e^y \to C(y) = x(ye^y - e^y)$
- $\overline{v}(y) = x\,y - x \to v(y) = x\,y - x + Ce^{-y}.$
- $u(x,y)_x = x\,y - x + f(x)e^{-y} \to u(x,y) = \frac{1}{2}x\,y^2 - x\,y + f(x)e^{-y} + g(x).$
- $u(x,y) = \dfrac{1}{2}x\,y^2 - x\,y + f(x)e^{-y} + g(x).$

Vpelji nove spremenljivke in reši enačbo $\frac{\partial^2 u(x,y)}{\partial x^2} - 2\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$, $t = x$, $z = x + y$.

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial x}$.
- $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}$.
- $u_x = u_t + u_z \quad u_y = u_z \to u_{xx} = (u_t + u_z)_t + (u_t + u_z)_z = u_{tt} + 2u_{tz} + u_{zz}, \quad u_{yy} = u_{zz} \to u_{xy} = (u_t + u_z)_z = u_{tz} + u_z z$.
- $u_{xx} - 2u_{xy} + u_{yy} = u_{tt} + 2u_{tz} + u_{zz} - 2u_{tz} - 2u_{zz} + u_{zz} = u_{tt} = 0$.
- $u = f(z)t + g(z) \to u(x,y) = f(x+y)x + g(x+y)$.
- $u(x,y) = f(x+y)x + g(x+y)$.

Vpelji nove spremenljivke in reši enačbo $x\frac{\partial u(x,y)}{\partial x} - y\frac{\partial u(x,y)}{\partial y} = 2u(x,y),\ t = x^2,\ z = x\,y$

- $\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial x} + y\frac{\partial u}{z} \to \frac{\partial u}{\partial y} = x\frac{\partial u}{z}$
- $\frac{\partial u}{\partial x} = 2\sqrt{t}\frac{\partial u}{\partial x} + \frac{z}{\sqrt{t}}\frac{\partial u}{z} \to \frac{\partial u}{\partial y} = \sqrt{t}\frac{\partial u}{\partial z}$.
- $\sqrt{t}2\sqrt{t}u_t + \sqrt{t}\frac{z}{\sqrt{t}}u_z - \frac{z}{\sqrt{t}}\sqrt{t}u_z = 2u$
- $2tu_t = 2u \to \frac{du}{u} = \frac{dt}{t} \to u = tf(z)$
- $u(x,y) = x^2 f(xy)$.

Vpelji nove spremenljivke in reši enačbo $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial x \partial y} - 2\frac{\partial^2 u(x,y)}{\partial y^2} = 0$, $t = x + y$, $z = 2x - y$

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- $u_x = u_t + 2u_z \quad u_y = u_t - u_z \to$
- $u_{xx} = (u_t + 2u_z)_t + 2(u_t + 2u_z)_z = u_{tt} + 4u_{tz} + 4u_{zz}$,
  $u_{yy} = (u_t - u_z)_t - (u_t - u_z)_z = u_{tt} - 2u_{tz} + u_{zz}$,
  $u_{xy} = (u_t + 2u_z)_t - (u_t + 2u_z)_z$.
- $u_{xx} - 2u_{xy} + u_{yy} = u_{tt} + 2u_{tz} + u_{zz} - 2u_{tz} - 2u_z z + u_{zz} \to$
  $u_{tt} = 0$.
- $u = f(z)t + g(z) \to u(x,y) = f(2x - y)(x + y) + g(2x - y)$.
- $u(x,y) = f(2x - y)(x + y) + g(2x - y)$.

Poišči rešitev diferencialne enačbe $\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = 0$ v obliki $u(x,y) = X(x)Y(y)$

- Vstavimo $u(x,y) = X(x)Y(y)$,
  $X'(x)Y(y) + X(x)Y'(y) = 0 \rightarrow$
- $\dfrac{X'(x)}{X(x)} + \dfrac{Y'(y)}{Y(y)} = 0.$
- Velja $\dfrac{X'(x)}{X(x)} = -\dfrac{Y'(y)}{Y(y)} = k,$
- kjer je k parameter nodvisen od x in y.
- $\frac{dX}{X} = k\,dx$ in $\frac{dY}{Y} = -k\,dy \rightarrow$
- $X(x) = Ae^{kx}$ in $Y(y) = Be^{-ky} \rightarrow$
- $u(x,y) = Ce^{k(x-y)}$

Poišči rešitev diferencialne enačbe $x^2u_{xy} + 3y^2u = 0$ v obliki $u = XY$

- $x^2X'Y' + 3y^2XY = 0 \rightarrow$
- $x^2\dfrac{X'}{X}\dfrac{Y'}{Y} + 3y^2 = 0 \rightarrow$
- $x^2\frac{X'}{X} = -3y^2\frac{Y}{Y'} = k,$
- kjer je k parameter nodvisen od x in y.
- $\frac{dX}{X} = k\frac{dx}{x^2}$ in $\frac{dY}{Y} = -3y^2\frac{dy}{k}\,dy \rightarrow$
- $X(x) = Ae^{-\frac{k}{x}}$ in $Y(y) = Be^{-\frac{y^3}{k}} \rightarrow$
- $u(x, y) = Ce^{-\frac{k}{x}-\frac{y^3}{k}}$

Poišči rešitev diferencialne enačbe $u_x + yu_y = 0$ v obliki $u = XY$, ki ustreza pogojema $u(1,0) = 1$ in $u(0,1) = 2$.

- $X'Y + yXY' = 0 \to \dfrac{X'}{X} + y\dfrac{Y'}{Y} = 0 \to$
- $\frac{X'}{X} = -y\frac{Y'}{Y} = k$, kjer je k parameter nodvisen od x in y.
- $\frac{dX}{X} = k\,dx$ in $\frac{dY}{Y} = -y\,dy \to$
- $X(x) = Ae^{kx}$ in $Y(y) = Be^{-k\frac{y^2}{2}} \to u(x,y) = Ce^{kx-k\frac{y^2}{2}}$.
- $u(1,0) = Ce^k = 1$ in $u(0,1) = Ce^{-\frac{k}{2}} = 2$
- $k = -\frac{1}{3}\log 4,\ C = 4^{1/3}$.