

# Matematika 4

## 5. vaja

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## Enačba za nihanje vpete strune

$$\mu \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial u(x,t)}{\partial x} \right)$$

Struna je vpetá v  $x = 0$  in  $x = l$ , dolžina je  $l$ .

Robni pogoji:  $u(0, t) = u(l, t) = 0$ .

Napetost strune:  $T = \textit{konst.}$

$c^2 = \frac{T}{\mu}$ , kjer je  $\mu$  dolžinska gostota strune.

- ▶  $\frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2}$
- ▶ Ločitev spremenljivk  $u(x, t) = X(x)T(t)$ .
- ▶ Lastne funkcije:  $-X'' = \lambda^2 X$ ,  $X(x) = \sin(\lambda x)$ .
- ▶ Lastne vrednosti:  
zaradi robnih pogojev je  $\lambda l = n\pi$ ,  $n \in \mathbb{N}$ ,  $\lambda_n = \frac{n\pi}{l}$ .
- ▶ Časovni del  $T'' + \omega_n^2 T = 0$ .  $\omega_n = c\lambda_n$ .
- ▶  $T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$
- ▶  $u_n(x, t) = \sin(\lambda_n x) (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))$ .

## Enačba za nihanje krožeče strune

$$\mu \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial u(x,t)}{\partial x} \right)$$

Struna dolžine  $l$  je vpeta v  $x = 0$ .

Robni pogoji:  $u(0, t) = 0$ ,  $|u(l, t)| < \infty$ .

Napetost strune:  $dT = x\omega^2\mu dx$ ,  $T(x) = \mu\omega^2 \int_x^l \xi d\xi$ ,

$T(x) = \frac{1}{2}\mu\omega^2(l^2 - x^2)$ ,  $c^2 = \frac{1}{2}\omega^2$ .

- ▶ Uvedemo novo spremenljivko  $\xi = \frac{x}{l}$ .
- ▶  $c^{-2}u_{tt} = ((1 - \xi^2)u_\xi)_\xi$ .
- ▶ Robni pogoji  $u(0, t) = 0$ ,  $|u(1, t)| < \infty$ .
- ▶ Ločitev spremenljivk  $u(\xi, t) = \Xi(\xi)T(t)$ .
- ▶  $-((1 - \xi^2)\Xi')' = \lambda\Xi$ ,  $(1 - \xi^2)\Xi'' - 2\xi\Xi' + \lambda\Xi = 0$ .

- ▶ Rešitev je lihi Legendreev polinom, ker je  $u(0, t) = 0$ .
- ▶ Lastne vrednosti  $\lambda_n = 2n(2n - 1)$ .
- ▶ Lastne funkcije  $\Xi_n(\xi) = P_{2n-1}(\xi)$ .
- ▶ Časovni del:  $T'' + c^2\lambda T = 0$ ,  $\omega_n = c\sqrt{\lambda_n} = \omega\sqrt{n(2n - 1)}$ .  
 $T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$ .
- ▶  $u_n(x, t) = P_{2n-1}(x/l) (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))$ .

## Enačba za nihanje obešene strune

$$\mu \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial u(x,t)}{\partial x} \right)$$

Obešena struna, vpeta v  $x = l$  in dolžine  $l$ . Os  $x$  je navpična.

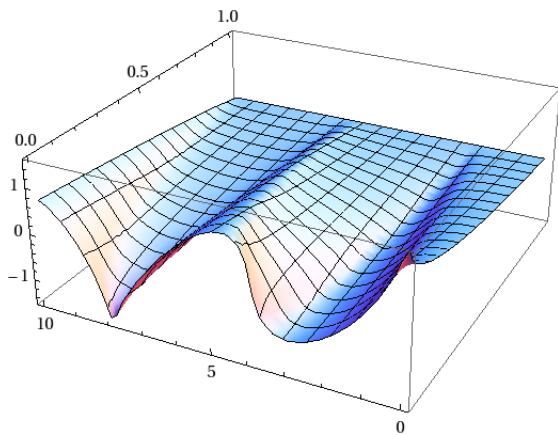
Robni pogoji:  $|u(0, t)| < \infty$ ,  $|u(l, t)| = 0$ .

Napetost strune:  $T(x) = g\mu x$ ,  $c^2 = g$ .

- ▶  $\frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial}{\partial x} \left( x \frac{\partial u(x, t)}{\partial x} \right)$ .
- ▶ Uvedemo novo spremenljivko  $x = \xi^2$ .
- ▶  $\left(\frac{2}{c}\right)^2 u_{tt} = \frac{1}{\xi} (\xi u_{\xi})_{\xi}$ .
- ▶ Ločimo spremenljivki  $u(\xi, t) = \Xi(\xi) T(t)$ ,  $-\frac{1}{\xi} (\xi \Xi')' = \lambda^2 \Xi$ .
- ▶  $\Xi'' + \frac{1}{\xi} \Xi' + \lambda^2 \Xi = 0$ .

- ▶ Robni pogoji  $\Xi(\sqrt{l}) = 0$ ,  $|\Xi(0)| < \infty$ .
- ▶ Diferencialno enačbo reši Besslova funkcija  $\Xi(\xi) = J_0(\lambda\xi)$ .
- ▶ Lastne vrednosti:  $\lambda_n^2 = \xi_{0,n}^2/l$ , kjer je  $\xi_{0,n}$   $n$ -ta ničla Besslove funkcije  $J_0(\xi)$ .
- ▶ Lastne funkcije:  $\Xi_n(\xi) = J_0(\lambda_n\xi)$ .
- ▶ Časovni del:  $T'' + \left(\frac{c}{2}\right)^2 \lambda^2 T = 0$ ,  $\omega_n = \frac{1}{2}c\lambda_n$ .  
 $T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$ .
- ▶  $u_n(x, t) = J_0(\lambda_n \sqrt{x}) (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))$ .

# Nihanje obešene strune



# Nihanje vpete okrogle membrane $\Delta u(r, t) = \frac{1}{c^2} \frac{\partial^2 u(r, t)}{\partial t^2}$

Polmer membrane je  $a$ . Gledamo lastna nihanja, ki niso odvisna od kota.  $\Delta u = u_{rr} + \frac{1}{r} u_r$ . Robna pogoja  $u(a, t) = 0$  in  $|u(0, t)| < \infty$ .

- ▶ Enačba  $u_{rr} + \frac{1}{r} u_r = \frac{1}{c^2} u_{tt}$ .
- ▶ Ločimo spremenljivke  $u(r, t) = R(r)T(t)$ .
- ▶  $-(R'' + \frac{1}{r} R') = \lambda^2 R$ ,  $T'' + c^2 \lambda^2 T = 0$ .
- ▶ Vpeljemo novo neznanko  $\xi = r\lambda$ ,  
$$\frac{d^2 R(\xi)}{d\xi^2} + \frac{1}{\xi} \frac{dR(\xi)}{d\xi} + R(\xi) = 0.$$
- ▶ Ker je  $|R(\xi)| < \infty$ , je  $R(\xi) = J_0(\xi)$ ,  $R(r) = J_0(\lambda r)$ .
- ▶ Upoštevamo še drugi robni pogoj  $R(\lambda a) = 0$ ,  $\lambda a = \xi_{0,n}$ ,  
 $n$ -ta ničla Besslove funkcije  $J_0(\xi)$ ,  $\lambda_n = \frac{\xi_{0,n}}{a}$ .
- ▶  $R_n(r) = J_0(\lambda_n r)$ ,  $T'' + c^2 \lambda_n^2 T = 0$ ,  $\omega_n = c \lambda_n$ .
- ▶  $u_n(r, t) = R_n(r)T_n(t) = J_0(\lambda_n r) (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))$ .



# Nihanja pravokotne membrane $\Delta u(x, y, t) = \frac{1}{c^2} \frac{\partial^2 u(x, y, t)}{\partial t^2}$

Stranici sta  $a$  in  $b$ .  $\Delta u = u_{xx} + u_{yy}$ . Robni pogoji  
 $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ .

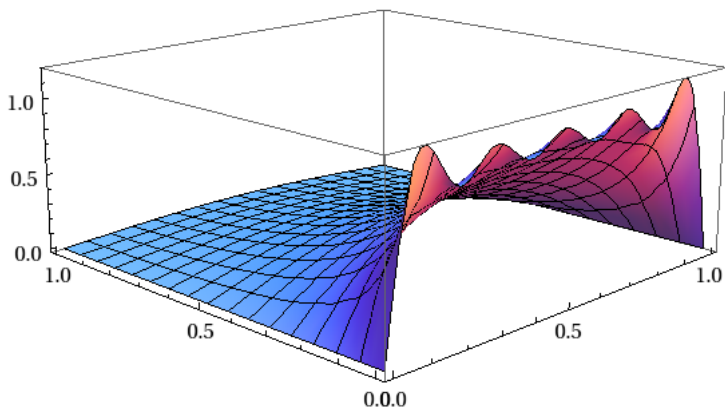
- ▶ Enačba  $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$ .
- ▶ Ločimo spremenljivke  $u(x, y, t) = X(x)Y(y)T(t)$ .
- ▶ Enačba  $\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \frac{T''}{T}$ , robni pogoji  
 $X(0) = X(a) = Y(0) = Y(b) = 0$ .
- ▶ Prostorski del:  $X'' + \lambda_m^2 X = 0$ ,  $Y'' + \mu_n^2 Y = 0$ . Lastni vrednosti  $\lambda_m = \frac{m\pi}{a}$  in  $\mu_n = \frac{n\pi}{b}$ ,  $m, n \in \mathbb{N}$ .
- ▶ Časovni del  $T'' + \omega_{mn}^2 T = 0$ ,  $\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ .
- ▶ Lastne funkcije  $u_{mn}(x, y, t) = \sin(\lambda_m x) \sin(\mu_n y) (A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t))$ .

Dirichletov robni problem:  $\Delta u = 0$  za polpas,  $0 < x < 1$ ,  
 $0 < y < \infty$ .

Robni pogoji  $u(0, y) = 0$ ,  $u(1, y) = 0$ ,  $u(x, 0) = 1$   
 $\lim_{y \rightarrow \infty} u(x, y) = 0$ .

- ▶ Enačba  $u_{xx} + u_{yy} = 0$ .
- ▶ Ločimo spremenljivke  $u(x, y) = X(x)Y(y)$ .
- ▶ Enačba  $\frac{X''}{X} + \frac{Y''}{Y} = 0$ , robni pogoji  $X(0) = X(1) = 0$ ,  $|Y(y)|$  je omejena funkcija  $0 < y < \infty$ .
- ▶  $X'' + \lambda_n^2 X = 0$ ,  $\lambda_n = n\pi$   $n \in \mathbb{N}$ .
- ▶  $X_n(x) = \sin(n\pi x)$  in  $Y_n(y) = A_n e^{-n\pi y} + B_n e^{n\pi y}$ .
- ▶ Koeficienti  $B_n = 0$ , ker je funkcija  $Y(y)$  omejena.
- ▶  $u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n\pi y}$ .
- ▶  $u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = 1$ ,  $A_n = 2 \frac{1 - (-1)^2}{n\pi}$ .

Graf k prejšnji nalogi  $z = \sum_{n=1}^{10} u_n(x, y)$



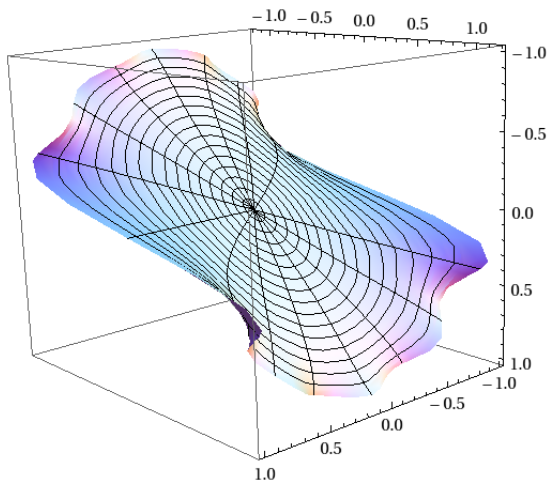
# Dirichletov robni problem, $\Delta u = 0$ za krog, $0 \leq r \leq 1$ .

Robni pogoji:  $|u(r, \phi)| < \infty$ ,  $u(1, \phi) = f(\phi)$ .

- ▶ Enačba:  $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = 0$ .
- ▶ Ločimo spremenljivke:  $u(r, \phi) = R(r)\Phi(\phi)$ .
- ▶ Enačba:  $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = 0$ .
- ▶ Robni pogoji:  $|R(r)|$  je omejena funkcija,  $\Phi(\phi)$  je periodična funkcija s periodo  $2\pi$ .
- ▶ Enačbi:
  - ▶  $\Phi'' + \lambda_n\Phi = 0$ ,  $\lambda_n = n^2$ ,  $n \in \mathbb{N}$  in
  - ▶  $r^2R''(r) + rR'(r) - n^2R(r) = 0$ .
- ▶  $\Phi_n(\phi) = A_n \sin(n\phi) + B_n \cos(n\phi)$  in  $R_n(r) = r^n$ .
- ▶  $u(r, \phi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\phi) + B_n \sin(n\phi)) r^n$ .
- ▶  $u(1, \phi) = f(\phi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\phi) + B_n \sin(n\phi))$ .

## Graf k prejšnji nalogi

$$z = \sum_{n=1}^{10} u_n(x, y), \quad f(\phi) = \text{sign}(\phi), \quad -\pi \leq \phi \leq \pi.$$



## Splošna nihanja vpete okrogle membrane:

$$\Delta u = \frac{1}{c^2} u_{tt} \quad 0 \leq r \leq a.$$

Robni pogoji:  $|u(0, \phi, t)| < \infty$ ,  $u(a, \phi, t) = 0$ ,  
 $u(r, \phi + 2\pi, t) = u(r, \phi, t)$ , periodičnost.

Začetni pogoji:  $u(r, \phi, 0) = f(r, \phi)$  in  $u_t(r, \phi, 0) = g(r, \phi)$

- ▶ Enačba:  $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = \frac{1}{c^2} u_{tt}$ .
- ▶ Ločimo spremenljivke:  $u(r, \phi, t) = R(r)\Phi(\phi)T(t)$ .
- ▶ Enačba:  $\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Phi''}{\Phi} = \frac{1}{c^2} \frac{T''}{T}$ .
- ▶ Robni pogoji:  $|R(0)| < \infty$ ,  $R(a) = 0$  in  $\Phi(\phi)$  je periodična funkcija s periodo  $2\pi$ .

► Enačbe:

- $\Phi'' + m^2\Phi = 0, \quad m \in \mathbb{N},$
- $T'' + c^2\lambda^2 T = 0,$
- $r^2 R''(r) + rR'(r) + (\lambda^2 r^2 - m^2)R(r) = 0,$
- Vpeljemo novo spremenljivko  $\xi = \lambda r$  dobimo enačbo
- $\xi^2 R''(\xi) + \xi R'(\xi) + (\xi^2 - m^2)R(\xi) = 0.$   
Rešitev je  $R(\xi) = AJ_m(\xi) + BN_m(\xi).$
- $\lambda_{mn} = \xi_{mn}/a, \omega_{mn} = c\lambda_{mn}$   
 $\xi_{mn}$  je  $n$ -ta ničla Besslove funkcije  $J_m(\xi).$

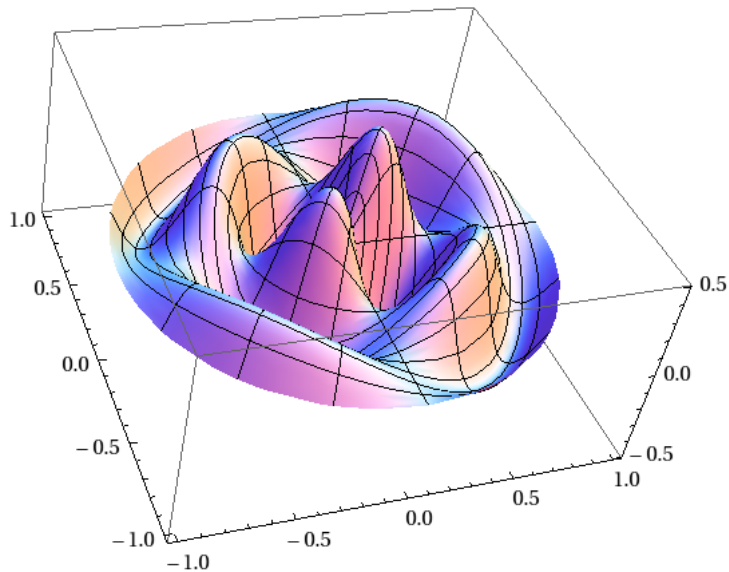
► Rešitve:

- $\Phi_m(\phi) = \sin(m\phi),$
- $R_{mn}(r) = J_m(\lambda_{mn}r)$
- $T_{mn} = A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t).$

►  $u(r, \phi, t) =$

$$\sum_{m,n} \sin(m\phi) J_m(\lambda_{mn}r) (A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t)).$$

Graf lastne funkcije  $u_{2,3}(r, \phi, 0)$ .





## Prevajanje toplote po palici z izoliranima krajiščema

$$u_{xx} = \frac{1}{\sigma^2} u_t, \quad u_x(0, t) = u_x(a, t) = 0, \quad u(x, 0) = f(x).$$

▶  $u_{xx}(x, t) = \frac{1}{\sigma^2} u_t(x, t). \quad u(x, t) = X(x)T(t).$

▶  $\frac{X''}{X} = \frac{1}{\sigma^2} \frac{T'}{T}.$

▶ Krajevni del:  $\frac{X''}{X} = -\lambda^2$

▶ Enačba:  $X'' + \lambda^2 X = 0.$

▶  $X(x) = A \cos(\lambda x) + B \sin(\lambda x).$

▶ Robni pogoji:  $X'(x) = -A\lambda \sin(\lambda x) + B\lambda \cos(\lambda x).$

▶  $X(0) = 0 \rightarrow B = 0, \quad X(a) = 0 \rightarrow \lambda a = n\pi, \quad n \in \mathbb{N}.$

▶ Lastne funkcije  $X_n(x) = \cos\left(\frac{n\pi}{a} x\right).$

- ▶ Časovni del:  $\frac{1}{\sigma^2} \frac{T'}{T} = - \left( \frac{n\pi}{a} \right)^2$ .  $\mu_n = \sigma \frac{n\pi}{a}$ .
  - ▶ Enačba  $T' + \mu_n^2 T = 0$ .
  - ▶  $T_n(t) = A_n e^{-\mu_n^2 t}$ .
- ▶ Lastne funkcije:  $u_n(x, t) = A_n \cos\left(\frac{n\pi}{a}x\right) e^{-\mu_n^2 t}$ .
- ▶ Začetni pogoj:  $f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)$ .
- ▶  $A_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi}{a}x\right) dx$ ,  $n > 0$ .
- ▶ Rešitev:  $u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right) e^{-\mu_n^2 t}$ .
- ▶ Stacionarna rešitev  $\lim_{t \rightarrow \infty} u(x, t) = A_0 = \frac{1}{a} \int_0^a f(x) dx$ .

## Temperatura na krajiščih je konstantna.

$$u_{xx} = \frac{1}{\sigma^2} u_t, \quad u(0, t) = \tau_0, \quad u(a, t) = \tau_a \quad \text{in} \quad u(x, 0) = f(x).$$

- ▶ Piščimo stacionarno rešitev,  $\bar{u}_t = 0$ , to je rešitev neodvisna od časa,  $\bar{u}_{xx} = 0$ .
- ▶ Rešitev je linearna funkcija:  $\bar{u}(x) = \tau_0 + \frac{\tau_a - \tau_0}{a} x$ .
- ▶  $u(x, t) = \bar{u}(x) + v(x, t)$ ,  $v(0, t) = v(a, t) = 0$ .
- ▶  $v(x, t) = X(x)T(t)$ .  $\frac{X''}{X} = \frac{1}{\sigma^2} \frac{T'}{T} = -\lambda^2$ .
- ▶  $X(x) = A \cos(\lambda x) + B \sin(\lambda x)$ ,  $X(0) = X(a) = 0$ .
- ▶  $v_n(x, t) = A_n \sin\left(\frac{n\pi}{a} x\right) e^{-\left(\sigma \frac{n\pi}{a}\right)^2 t}$ .
- ▶ Rešitev:  $u(x, t) = \tau_0 + \frac{\tau_a - \tau_0}{a} x + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) e^{-\left(\sigma \frac{n\pi}{a}\right)^2 t}$ .
- ▶  $A_n = \frac{2}{a} \int_0^a \left( f(x) - \left( \tau_0 + \frac{\tau_a - \tau_0}{a} x \right) \right) \sin\left(\frac{n\pi}{a} x\right) dx$ .

Reši Poissonov robni problem  $\Delta u = f$  v prostoru. Iščemo rešitev, ki je odvisna le od  $r = \sqrt{x^2 + y^2 + z^2}$

$$\Delta u = f, \quad u(1) = 1, \quad u(0) < \infty, \quad f(r) = \begin{cases} 1 & 0 \leq r \leq 1 \\ 0 & \text{drugod} \end{cases}$$

- ▶ Laplaceov operator  $\Delta u(r) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r}$ .
- ▶  $r^2 u''(r) + 2ru'(r) = \begin{cases} r^2 & 0 \leq r < 1 \\ 0 & r \leq 1 < \infty \end{cases}$ ,  $u(1) = 1$ ,  $u(0) < \infty$ .
- ▶ Eulerjeva diferencialna enačba,  $r = e^x$ .
- ▶  $u''(x) + u'(x) = \begin{cases} e^{2x} & -\infty < x \leq 0 \\ 0 & 0 < x < \infty \end{cases}$ ,  
 $u(0) = 1$ ,  $\lim_{x \rightarrow -\infty} u(x) < \infty$ .

$$\blacktriangleright u(r) = \frac{-20c_1 + r^5\theta(r) - (r^5 - 5r + 4)\theta(r-1)}{20r} + c_2$$

$$\blacktriangleright u(r) = \frac{1}{20}r^4\theta(r) - \frac{1}{20r}(r^5 - 5r + 4)\theta(r-1)$$

Reši Poissonov robni problem  $\Delta u = f$  v prostoru. Iščemo rešitev, ki je odvisna le od  $r = \sqrt{x^2 + y^2 + z^2}$

$$\Delta u = 1, \quad 1 < r < 2, \quad u(1) = 0, \quad u(2) = 0$$

- ▶ Laplaceov operator  $\Delta u(r) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r}$ .
- ▶  $r^2 u''(r) + 2ru'(r) = r^2$ ,  $u(1) = 0$ ,  $u(2) = 0$ .
- ▶ Splošna rešitev  $u(r) = c_1 + c_2 r^{-1} + \frac{1}{6} r^2$ .
- ▶  $u(r) = \frac{r^3 - 7r + 6}{6r}$ .

# Reševanje parcialnih diferencialnih enačb s pomočjo Laplaceove transformacije.

Reši diferencialno enačbo

$$\frac{\partial u(x, t)}{\partial x} + x \frac{\partial u(x, t)}{\partial t} = 0, \quad t > 0, \quad u(x, 0) = 0, \quad u(0, t) = t^2.$$

► Laplaceova transformacija enačbe:

►  $\frac{dU(x, s)}{dx} + x (sU(x, s) - u(x, 0)) = 0.$

►  $\int \frac{dU}{U} = - \int xs \, dx \rightarrow U(x, s) = F(s)e^{-s\frac{x^2}{2}}.$

►  $u(x, t) = f\left(t - \frac{x^2}{2}\right), \quad u(0, t) = t^2, \quad \rightarrow$

►  $u(x, t) = \left(t - \frac{x^2}{2}\right)^2 \theta\left(t - \frac{x^2}{2}\right).$

# Reševanje parcialnih diferencialnih enačb s pomočjo Laplaceove transformacije.

Reši diferencialno enačbo

$$\frac{\partial u(x, t)}{\partial x} + 2x \frac{\partial u(x, t)}{\partial t} = 2x, \quad t > 0, \quad u(x, 0) = 1, \quad u(0, t) = 1.$$

► Laplaceova transformacija enačbe:

$$\frac{dU(x, s)}{dx} + 2x(sU(x, s) - u(x, 0)) = 2 \frac{x}{s}.$$

$$\frac{dU(x, s)}{dx} + 2xsU(x, s) = \frac{2x}{s} + 2x \rightarrow$$

$$U(x, s) = F(s)e^{-sx^2} + \frac{1+s}{s^2}.$$

$$u(x, t) = f(t - x^2) + 1 + t$$
$$u(x, 0) = f(t) + 1 + t = 1, \rightarrow f(t) = -t.$$

$$u(x, t) = -(t - x^2)\theta(t - x^2) + (1 + t)\theta(t).$$