MATEMATIKA IV
zapiski z avditornih vaj

Šolsko leto	2008 / 2009
Izvajalec	Kristijan Cafuta
Avtor dokumenta	stromar.si
Sodelavci	stromar.si

UREJANJE DOKUMENTA

VERZIJA	01	**REVIZIJA**	04
DATUM	29. 5. 2009		
ZADNJI POPRAVLJAL		p	
PREGLEDAL			

OPOMBE

NAPAKE SPOROČITE NA MAIL: stromar.si@gmail.com

POPRAVKI

Fourierjeva transformacija

$$\boxed{\mathcal{F}(f) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt}$$

$$\| $$

$$\mathcal{F}(f(t))(\omega)$$

$$\|$$

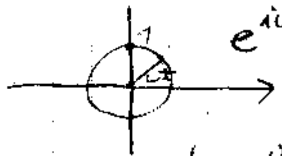
$$F(\omega)$$

Pr. $f(t) = \begin{cases} -1; & \text{ko je } -1 \leq t < 0 \\ 1; & \text{ko je } 0 \leq t \leq 1 \\ 0; & |t| > 1 \end{cases}$ $\qquad \mathcal{F}(f(t)) = ?$

$$\mathcal{F}(f(t)) = \int_{-1}^{0}(-1)e^{i\omega t}\,dt + \int_0^1 1e^{i\omega t}\,dt = -\frac{e^{i\omega t}}{i\omega}\Big|_{-1}^{0} + \frac{e^{i\omega t}}{i\omega}\Big|_0^1 =$$

$$= -\frac{1}{i\omega} + \frac{e^{-i\omega}}{i\omega} + \frac{e^{i\omega}}{i\omega} - \frac{1}{i\omega} =$$

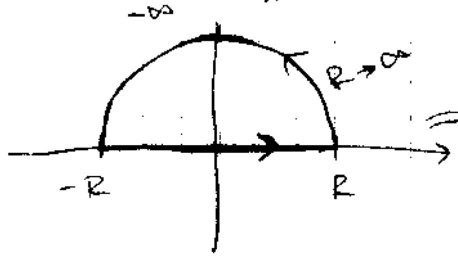
$$= -\frac{2}{i\omega} + \frac{2\cos\omega}{i\omega}$$



$$\left|\frac{e^{i\omega t}}{a^2+t^2}\right| = \frac{1}{a^2+t^2}$$

Pr. $f(t) = \frac{1}{a^2+t^2}$

$$F(\omega) = \int_{-\infty}^{\infty}\frac{1}{a^2+t^2}e^{i\omega t}\,dt$$



$\Rightarrow$ Če je stopnja im. vsaj za 2 večja od števca, lahko rešimo na tak način

$$\frac{e^{i\omega t}}{a^2+t^2} \quad \text{poli: } t = \pm ia$$

$(t+ia)^1(t-ia)^1$ v zg. polravnini je $+ia$ (pol 1. st.)

$$\operatorname*{Res}_{t=ia}\frac{e^{i\omega t}}{a^2+t^2} = \lim_{t\to ia}\frac{e^{i\omega t}}{a^2+t^2}(t-ia) = \lim_{t\to ia}\frac{e^{i\omega t}}{t+ia} =$$

$$= \frac{e^{-\omega a}}{2ia}$$

$$F(\omega) = 2\pi i \operatorname*{Res}_{t=ia}\frac{e^{i\omega t}}{a^2+t^2} = \cancel{2}\pi i\frac{e^{-\omega a}}{\cancel{2}ia} = \frac{\pi e^{-\omega a}}{a}$$

Parsevalova enačba

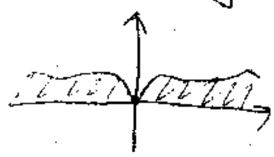
$$\boxed{\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{F}(\omega)|^2\,d\omega}$$

$$f(t)=\begin{cases}1; & -1\le t\le 1\\ 0; & |t|>1\end{cases}\qquad \text{D.N.}\quad \mathcal{F}(f(t))=\frac{2\sin\omega}{\omega}$$

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2dx = \frac{1}{8}\int_{-\infty}^{\infty}\left(\frac{2\sin x}{x}\right)^2dx =$$

sodo fje

$\frac{1}{2}\cdot\frac{1}{4}$



$$= \frac{2\pi}{8}\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{2\pi}{8}\int_{-1}^{1}dt = \frac{2\pi}{8}\,t\Big|_{-1}^{1} = \frac{\pi}{2}$$

Konvolucija

$$\boxed{\begin{aligned} f_1 * f_2 &= \int_{-\infty}^{\infty} f_1(u)\cdot f_2(t-u)\,du\\ \mathcal{F}(f_1 * f_2) &= \mathcal{F}(f_1)\cdot\mathcal{F}(f_2)\\ f_1 * f_2 &= \mathcal{F}^{-1}\big(\mathcal{F}(f_1)\cdot\mathcal{F}(f_2)\big)\end{aligned}}$$

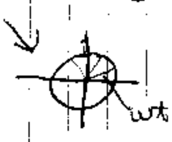
$$f(t)=\begin{cases}e^{-t}; & t>0\\ 0; & t\le 0\end{cases}$$

$$g(t)=\begin{cases}te^{-t}; & t>0\\ 0; & t\le 0\end{cases}$$

a) $\mathcal{F}(f)=\int_0^{\infty}e^{-t}e^{i\omega t}\,dt = \int_0^{\infty}e^{(i\omega-1)t}\,dt = \frac{e^{(i\omega-1)t}}{i\omega-1}\Big|_0^{\infty} =$

$$= -\frac{1}{i\omega-1} = \frac{1}{1-i\omega}$$

$$\lim_{t\to\infty} e^{(i\omega-1)t} = \lim_{t\to\infty} \underbrace{e^{-t}}_{0}e^{i\omega t} = 0$$



b) $f * f = \int_{-\infty}^{\infty} f(u)\cdot f(t-u)\,du = *$

$f(u) = \begin{cases} e^{-u}; & u>0 \\ 0; & u\le 0 \end{cases} \qquad f(t-u) = \begin{cases} e^{-t+u}; & t-u>0 \\ 0; & t-u\le 0 \end{cases}$

$* = \begin{cases} \int_0^t e^{-u}\cdot e^{-t+u}\,du = \int_0^t e^{-t}\,du = e^{-t}\cdot t & ; t>0 \\ 0; \quad u\le 0 \text{ ali } u\ge t \end{cases}$

$e^{-t}\cdot u\Big|_0^t$

c) $\mathcal{F}(g) = ?$

$\mathcal{F}(g) = \int_0^{\infty} t\cdot e^{-t} e^{i\omega t}\,dt = \dots$

opazimo: $g = f * f$, torej $\mathcal{F}(g) = \mathcal{F}(f * f) =$

$= \mathcal{F}(f)\cdot\mathcal{F}(f) = \frac{1}{(1-i\omega)^2}$

$f(t) = e^{-\frac{t^2}{2}}$

$\mathcal{F}(f(t)) = \sqrt{2\pi}\cdot e^{-\frac{\omega^2}{2}}$

$f_a(t) = e^{-at^2}$

$f_b(t) = e^{-bt^2}$

a) $\underline{\mathcal{F}(f_a(t))} = \int_{-\infty}^{\infty} e^{-at^2} e^{i\omega t}\,dt = \int_{-\infty}^{\infty} e^{-a\frac{k^2}{2a}} e^{i\omega\frac{k}{\sqrt{2a}}}\frac{1}{\sqrt{2a}}\,dk =$

$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{i\omega t}\,dt = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}$

$-at^2 = -\frac{k^2}{2} \Rightarrow \boxed{t = \frac{k}{\sqrt{2a}}} \quad dt = \frac{1}{\sqrt{2a}}\,dk$

$= \sqrt{\frac{2\pi}{2a}}\, e^{-\frac{\omega^2}{4a}} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$

b) $\underline{f_a * f_b} = \mathcal{F}^{-1}\left(\mathcal{F}(f_a)\cdot\mathcal{F}(f_b)\right) = \mathcal{F}^{-1}\left(\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}\cdot\sqrt{\frac{\pi}{b}}\, e^{-\frac{\omega^2}{4b}}\right)$

$\mathcal{F}^{-1}\left(\frac{\pi}{\sqrt{ab}}\, e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}\right) = *$

substitucija:

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c} \Rightarrow \frac{1}{c}=\frac{b+a}{ab} \Rightarrow c=\frac{ab}{a+b}$$

$$* = \mathcal{F}^{-1}\left(\sqrt{\frac{\pi}{a+b}}\sqrt{\frac{\pi}{c}}\cdot e^{-\frac{\omega^2}{4c}}\right)=\sqrt{\frac{\pi}{a+b}}\cdot e^{-\frac{ab}{a+b}t^2}$$

$$\mathcal{F}(f)=\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\,dt =$$

$$e^{i\omega t}=\cos(\omega t)+i\sin(\omega t)$$

$$=\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt + i\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

$$\mathcal{F}_s(f)=\int_0^{\infty} f(t)\sin(\omega t)\,dt$$ – sinusna fur. transformacija

$$\mathcal{F}_c(f)=\int_0^{\infty} f(t)\cos(\omega t)\,dt$$ – cosinusna – " –

$$f(t)=\begin{cases} t; & 0<t<1 \\ 0; & t>1\end{cases}$$

~~Reši~~ Transformiraj fjo z sin. in cos. ~~fo~~ Four. transformacijo.

$$\mathcal{F}_s=\int_0^1 t\sin(\omega t)\,dt=\left[-t\frac{\cos(\omega t)}{\omega}\right]_0^1+\frac{1}{\omega}\int_0^1\cos(\omega t)\,dt =$$

$u=t \Rightarrow du=dt$

$dv=\sin(\omega t)dt \rightarrow v=-\frac{\cos(\omega t)}{\omega}$

$$=-\frac{\cos\omega}{\omega}+\left[\frac{1}{\omega}\;\frac{\sin(\omega t)}{\omega}\right]_0^1=-\frac{\cos\omega}{\omega}+\frac{\sin\omega}{\omega^2}$$

$$\mathcal{F}_c=\int_0^1 t\cos(\omega t)\,dt=\frac{t\sin\omega}{\omega}\Big|_0^1-\int_0^1\frac{1}{\omega}\sin(\omega t)\,dt =$$

$u=t \quad dt=du$

$dv=\cos\omega t \quad v=\frac{1}{\omega}\sin\omega t$

$$=\frac{\sin\omega}{\omega}+\frac{\cos\omega t}{\omega^2}\Big|_0^1=\frac{\sin\omega}{\omega}+\frac{\cos\omega}{\omega^2}-\frac{1}{\omega^2}$$

D.N.

$$f_1=\begin{cases} t; & -1<t<1 \\ 0; & |t|>1\end{cases} \qquad \hat{\mathcal{F}}(f_1)$$

$$f_2=\begin{cases} t; & 0<t<1 \\ -t; & -1<t<0 \\ 0; & |t|>1\end{cases} \qquad \hat{\mathcal{F}}(f_2)$$

MA-3-VAJE
4.3.2008

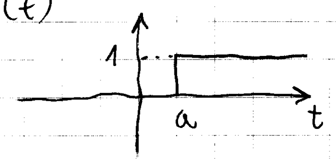
MANJKA PREJSNJA URA (FOURIERJEVA TRANSFORMACIJA)

Laplaceova transformacija

$$\boxed{\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt}$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$; $\mathbb{N}$
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ $\alpha > -1$, $\alpha \in \mathbb{R}$
$e^{\alpha t}$	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+a^2}$

$u_a(t)$



$f(t)$	$F(s)$
$\int_0^t f(u) g(t-u)\, du$	$F(s) \cdot G(s)$

$\mathcal{L}(f_1+f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2)$
$\mathcal{L}(c \cdot f) = c \cdot \mathcal{L}(f)$; c - konst

$f(t)$	$F(s)$
$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
$e^{at} f(t)$	$F(s-a)$
$u_a(t) f(t-a)$	$e^{-as} F(s)$
$f'(t)$	$s F(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} F(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$
$t \cdot f(t)$	$-F'(s)$
$t^n \cdot f(t)$	$(-1)^n F^{(n)}(s)$
$\int_0^t F(u)\, du$	$\frac{F(s)}{s}$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\, du$

$$\mathrm{sh}\, x = \frac{e^x - e^{-x}}{2}$$

- Transformirajmo
- $f(t) = 2t^2 - e^{-t}$

  $\mathcal{L}(f(t)) = 2 \cdot \frac{2!}{s^{2+1}} - \frac{1}{s+1} = \frac{4}{s^3} - \frac{1}{s+1}$

- $f(t) = \sin(t) \cdot \sin(2t) = *$

  pretvorimo po formuli : $\frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

  $* = \frac{1}{2}\left[\cos(-t) - \cos(3t)\right]$

  $\downarrow$

  $\mathcal{L}(f(t)) = \frac{1}{2}\left(\frac{s}{s^2+1^2} - \frac{s}{s^2+9}\right)$

- $f(t) = e^{-t} \cdot t^{+3}$ $\quad \mathcal{L}(t^3) = \frac{2 \cdot 3}{s^4} = \frac{6}{s^4}$

  $\mathcal{L}(f) = \frac{6}{(s+1)^4}$

- $f(t) = \int_0^t \frac{1-e^{-u}}{u}\, du$

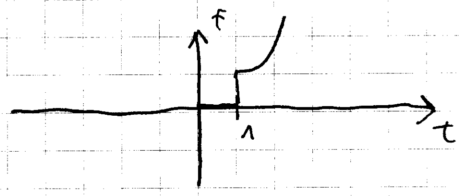
  $\mathcal{L}(f(t)) = \frac{\mathcal{L}\left(\frac{1-e^{-u}}{u}\right)}{s} = *$

  $\mathcal{L}\left(\frac{1-e^{-u}}{u}\right) = \int_s^\infty \mathcal{L}(1-e^{-u})\, dw = \int_s^\infty \left[\frac{1}{w} - \frac{1}{w+1}\right] dw =$

  $= \left[\ln(w) - \ln(w+1)\right]_s^\infty = \ln \frac{w}{w+1}\Big|_s^\infty = -\ln\frac{s}{s+1} = \ln\left(\frac{s+1}{s}\right)$

  $= * = \frac{\ln \frac{s+1}{s}}{s}$

• $f(t) = \begin{cases} 0; & 0 < t < 1 \\ t^2; & t > 1 \end{cases}$



$f(t) = u_1(t) \cdot \underbrace{t^2}$

$\hookrightarrow \; t^2 = \underline{\underline{(t-1)^2}}_{\checkmark} \; \underline{\underline{+2t-1}}_{\times}$ vedno se mora pojavljati $(t-1)$

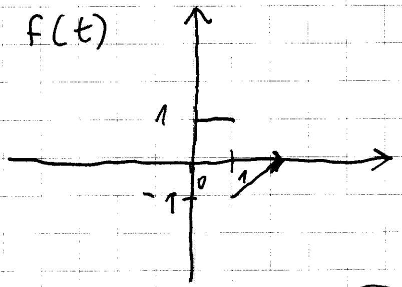
$= (t-1)^2 + \overbrace{2(t-1)+1}$

$$\mathcal{L}(f) = \mathcal{L}\left(u_1(t)\left((t-1)^2 + 2(t-1) + 1\right)\right)$$

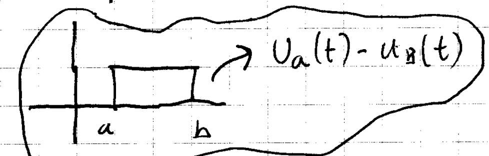
$$= e^{-s}\,\mathcal{L}\left(t^2 + 2t + 1\right) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

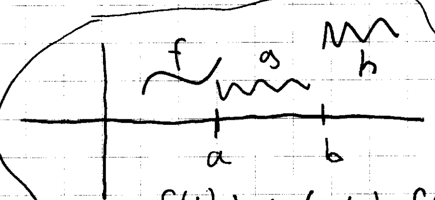
$\left.\begin{array}{l} f(t-1) = (t-1)^2 + 2(t-1) + 1 \\ f(t) = t^2 + 2t - 1 \end{array}\right\}$ pretvorimo

• f(t)



$f(t) = \begin{cases} 1 & 0 < t < 1 \\ t-2 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$





$f(t) + u_a\left(g(t) - f(t)\right) + u_B(t)\left(h(t) - g(t)\right)$

$f(t) = (1 - u_1(t)) \cdot 1 + (u_1(t) - u_2(t))(t-2) = *$

$$* = 1 - \underbrace{u_1}_{} + u_1 t - u_2 t \underbrace{- 2u_1}_{} + 2u_2 =$$

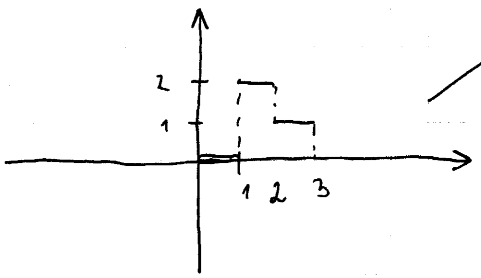
$u_a(t)F(t-a) = e^{-as}F(s)$

$$= 1 - 3u_1 + 2u_2 - u_2 t + u_1 t =$$

$$= 1 - 3u_1 + 2u_2 - u_2\big((t-2)+2\big) + u_1\big((t-1)+1\big) =$$

$$= \frac{1}{s} - 3e^{-s}\cdot \underset{\substack{\uparrow \\ \mathcal{L}(1)}}{\frac{1}{s}} + 2\cdot e^{-2s}\underset{\substack{\uparrow \\ \mathcal{L}(1)}}{\frac{1}{s}} - e^{-2s}\underset{\substack{\uparrow \\ \mathcal{L}(t+2)}}{\left(\frac{1}{s^2}+\frac{2}{s}\right)} +$$

$$+ e^{-s}\underset{\substack{\uparrow \\ \mathcal{L}(t+1)}}{\left(\frac{1}{s^2}+\frac{1}{s}\right)}$$

- 

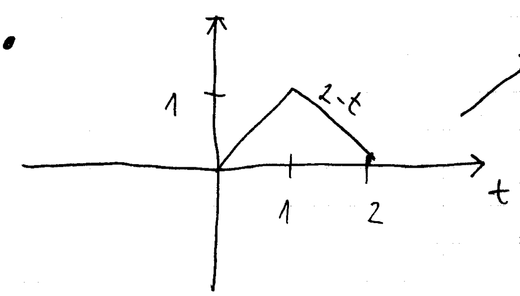
$F(t) = 2u_1 - u_2 - u_3$

$2u_1$: od 1 dalje vrednost 2

$-u_2$: odšteje se 1, f = 1

$-u_3$: odštejemo 1, F = 0

$$F(s) = \underbrace{2\cdot\frac{1}{s}}_{\text{tr. konst}} \cdot e^{-1} - e^{-2s}\frac{1}{s} - e^{-3s}\frac{1}{s}$$

- 

$$F(t) = t(1-u_1) + (2-t)(u_1 - u_2) =$$

$$= t - t\cdot u_1 + (2-t)u_1 - (2-t)u_2 =$$

$$= t + u_1(2-2t) + u_2(t-2)$$

$$= t - 2u_1(t)(t-1) + u_2(t)(t-2)$$

$$F(s) = \frac{1}{s^2} - 2e^{-3}\underset{\nwarrow \mathcal{L}(t)}{\frac{1}{s^2}} + e^{-2s}\underset{\nwarrow \mathcal{L}(t)}{\frac{1}{s^2}}$$

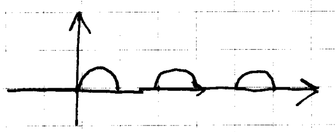
• Če f(t) periodična s periodo T, dobimo

$$\mathcal{L}(f(t)) = \underbrace{\frac{1}{1-e^{-sT}}}_{\text{korekcijski faktor}} \int_0^T e^{-st} f(t)\,dt$$

Oglejmo si:

$$f(t)=\begin{cases}\sin t; & 0<t<\pi\\ 0; & \pi<t<2\pi\end{cases}$$

sledi $T=2\pi$



glej graf fje

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-s2\pi}}\int_0^\pi e^{-st}\sin(t)\,dt =$$

$$= \frac{1}{1-e^{-s2\pi}} \underbrace{\left(\frac{e^{-st}}{s^2+1}\cdot\left(-s\cdot\sin t - \cos t\right)\right)}_{\text{D.N. Poglej si v Bronštejnu}}\Bigg|_0^\pi =$$

$$= \frac{1}{1-e^{-s2\pi}}\left(\frac{e^{-s\pi}}{s^2+1}(+1) - \frac{e^{-s\pi}}{s^2+1}(-1)\right) =$$

$$= \frac{e^{-s\pi}+1}{\underset{\uparrow}{(1-e^{-2\pi s})}(s^2+1)} = \frac{1}{(1-e^{-\pi s})(s^2+1)}$$

$$(1-e^{-\pi s})(1+e^{-\pi s})$$

- $F(s) = \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} =$

$$= \frac{A}{s-3} + \frac{B}{s+1} = \frac{(s+1)A + (s-3)B}{(s+1)(s-3)} =$$

$$= \frac{s(A+B) + A - 3B}{(s+1)(s-3)} = *$$

$A + B = 3$ $\qquad\qquad$ $A - 3B = 7$
$A = 3 - B \longrightarrow$ $\qquad\qquad$ $3 - B - 3B = 7$
$\qquad\qquad\qquad\qquad$ $-4B = 4$
$A = 4 \longleftarrow$ $\qquad\qquad$ $B = -1$

$$* = \frac{4}{s-3} - \frac{1}{s+1}$$

$$\mathcal{L}^{-1}(F(s)) = 4e^{3t} - e^{-t}$$

- $\mathcal{L}\left(\frac{s}{(s+1)^2}\right)$

gremo postopoma po potencah

$$! \quad \frac{s}{(s+1)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2} . = *$$

$$\frac{As + A + B}{\,} = s \rightarrow A = 1, \; B = -1$$

$$\boxed{\frac{\ldots}{(s+a)^n} = \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \ldots + \frac{A_n}{(s+a)^n}}$$

$$* = \frac{1}{s+1} + \frac{-1}{(s+1)^2}$$

← Vidimo, da je to premaknjena transformiranka

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^{-t} \cdot t$$

$$* \rightarrow \mathcal{L} = e^{-t} - e^{-t} \cdot t$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = e^{at}\mathcal{L}\left(\frac{1}{s^n}\right) = e^{at} \cdot \frac{t^{n-1}}{(n-1)!}} \; !$$

$$\boxed{\begin{aligned} F(s-a) &= \frac{1}{(s-a)^n} \\ F(s) &= \frac{1}{s^n} \end{aligned}}$$

- $\dfrac{6s-4}{s^2-4s+20} =$ $\qquad\qquad$ $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

to je že na parcialnih ulomkih (nerazcepno)

$$s^2-4s+20 = (s-2)^2 - 4 + 20$$
$$= (s-2)^2 - 16$$

$$\frac{6s-4}{(s-2)^2+16} = \frac{6(s-2)+8}{(s-2)^2+16}$$

prestavljena transformiranka, štimamo še števec.

$$\mathcal{L}^{-1}\left(\frac{6(s-2)+8}{(s-2)^2+16}\right) = e^{2t} \cdot \mathcal{L}^{-1}\left(\frac{6s+8}{s^2+16}\right) =$$

premik $\qquad$ premaknjena

$$= e^{2t}\mathcal{L}^{-1}\left(\frac{6s}{s^2+16} + \frac{8}{s^2+16}\right) = e^{2t}\left(6\cos(4t) + 2\sin(4t)\right)$$

Inverzna Laplaceova transformacija

- parcialni ulomki
- inverzna formula (residui)

$$\boxed{f(t) = \sum_{s_i} \operatorname{res}_{s=s_i}\left[F(s)\cdot e^{st}\right]}$$ kjer si sing. $F(s)e^{st}$

- tabela

- $F(s) = \dfrac{(s+2)}{(s+1)(s-2)(s^2+4)}$ iščemo $\mathcal{L}^{-1}$

$f(t) = \sum \operatorname{res}\left(F(s)e^{st}\right)$

singularnosti: $-1, 2, \pm 2i$

$$\operatorname{res}_{s=-1} F(s)e^{st} = \lim_{s\to -1} \frac{(s+2)e^{st}}{(s-2)(s^2+4)} = -\frac{e^{-t}}{15}$$

$$\operatorname{res}_{s=2} F(s)e^{st} = \lim_{s\to 2} \frac{(s+2)e^{st}}{(s+1)(s^2+4)} = \frac{e^{2t}}{6}$$

$$\operatorname{res}_{s=2i} F(s)e^{st} = \lim_{s\to 2i} \frac{(s+2)e^{st}}{(s+1)(s-2)(s+2i)} = \frac{(2i+2)e^{2it}}{(2i+1)(2i-2)4i} =$$

$$= \frac{1}{4}\,\frac{(1+i)e^{2it}}{1-3i} \quad \Big/ \frac{(1+3i)}{(1+3i)}$$

$$= \frac{1}{4}\,\frac{(1+3i-3+i)e^{2it}}{10} = \frac{1}{20}(-1+2i)e^{2it}$$

11.3.2009

$$\operatorname*{res}_{s=-2i} F(s)e^{st} = \cancel{\lim_{s\to -2i}} \frac{1}{20}(-1-2i)e^{-2it}$$

$$f(t) = \frac{-e^{-t}}{15} + \frac{e^{2t}}{6} + \frac{1}{20}(-1+2i)e^{2it} + \frac{1}{20}(-1-2i)e^{-2it} = *$$

$$= (-1-2i)(\cos(2t) - i\sin(2t)) + (-1+2i)(\cos(2t) + i\sin(2t))$$
$$= -\cos(2t) + i\sin(2t) - 2i\cos(2t) - 2\sin(2t)$$
$$- \cos(2t) - i\sin(2t) + 2i\cos(2t) - 2\sin(2t)$$

$$* = \frac{e^{-t}}{15} - \frac{e^{2t}}{6} - \frac{1}{10}\cos(2t) - \frac{1}{5}\sin(2t)$$

- $F(s) = \dfrac{s}{(s^2+1)^2}$ to je še najbolj na parcialnih ulomkih

$$\mathcal{L}^{-1}(F(s)) = \sum_i \operatorname*{res}_{s=s_i}\left[F(s)\cdot e^{st}\right]$$ (glej residue-ma3)

sing: $\pm i$ pola 2. stopnje

$$(s^2+i)^2 = (s+i)^2(s-i)^2$$

$$\operatorname*{res}_{s=i} F(s)e^{st} = \lim_{s\to i}\left[\frac{s}{(s^2+1)^2}e^{st}(s-i)^2\right]' =$$

$$= \lim_{s\to i}\left(\frac{s\cdot e^{st}}{(s+i)^2}\right)' = *$$

$$\left(\frac{s\cdot e^{st}}{(s+i)^2}\right)' = \frac{(e^{st} + se^{st}\cdot t)\cdot(s+i)^2 - se^{st}\cdot 2(s+i)}{(s+i)^4}$$

$*$ ← vstavimo i $= \dfrac{[e^{it} + ie^{it}\cdot t](-4) - ie^{it}\cdot 4i}{16} =$

$= \frac{-4ie^{it}\cdot t}{16} = \frac{-ie^{it}}{4}\cdot t$

ker ta funkcija realna katere residuum iščemo ~~[illegible]~~ residuum v konjugirani vrednosti kar konjugirana vrednost

$$\operatorname*{res}_{s=-i} F(s)e^{st} = \frac{ie^{-it}t}{4}$$

$$\mathcal{L}^{-1}(F(s)) = \frac{-ie^{it}\cdot t}{4} + \frac{ie^{-it}t}{4} = \frac{it(-e^{it}+e^{-it})}{4}$$

$$= \frac{(-\cancel{\cos t} - i\sin t + \cancel{\cos t} - i\sin t)it}{4} = \frac{(-2i\sin t)it}{4} =$$

$$= \boxed{\frac{t\sin t}{2}}$$

- $F(s) = \frac{s}{(s^2+1)^2}$ ugotovimo, da je to odvod, potem smo na konju.

$f(t)$	$F(s)$
$t\cdot f(t)$	$-F'(s)$

$$\frac{s}{(s^2+1)^2} = (\,?\,)' \leftarrow \left(\frac{1}{s^2+1}\right)' = \frac{-2s}{(s^2+1)^2}$$

$$\frac{s}{(s^2+1)^2} = \left(\frac{-1}{2(s^2+1)}\right)' = \longrightarrow$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = t\cdot\mathcal{L}^{-1}(*)$$

$$(*) = \frac{1}{2(s^2+1)}$$

$$= \frac{t}{2}\sin t$$

• $F(s) = \frac{s}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} =$ (s konvolucijo)

$$= \mathcal{L}(\sin t)\,\mathcal{L}(\cos t)$$

$$\mathcal{L}^{-1}(\mathcal{L}(\sin t) \cdot \mathcal{L}(\cos t)) = \sin t * \cos t$$

$$\underbrace{\int_0^t \sin u \, \cos(t-u)\, du}_{\text{Bronštejn}} =$$

$$= \frac{1}{2}\int_0^t \big(\sin(2u-t) + \sin(t)\big)\,du = \frac{1}{2}\left[\frac{-\cos(2u-t)}{2} + \sin(t)\cdot u\right]_0^t =$$

$$= \frac{1}{2}\left[\frac{-\cos(t)}{2} + \frac{\cos(t)}{2} + t\sin t\right] = \frac{t \cdot \sin t}{2}$$

cos soda

• $F(s) = \frac{e^{-s/3}}{s(s^2+1)}$ poišči inverzno Laplaceovo transformacijo

kadar je v f'ji $e^{\text{nekaj}\cdot s}$ gre za stopnico

$f(t)$	$F(s)$
$u_a(t) f(t-a)$	$e^{-as}\, F(s)$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \sum_i \operatorname*{res}_{s=s_i}\left(\frac{1}{s(s^2+1)} \cdot e^{st}\right)$$

sing: $0, i, -i$

$$\operatorname*{res}_{s=0} F(s)e^{st} = \lim_{s\to 0} \frac{e^{st}}{(s^2+1)} = e^0 = 1$$

$$\operatorname*{res}_{s=i} F(s)e^{st} = \lim_{s\to i} \frac{e^{st}}{s(s+i)} = \frac{e^{it}}{i\cdot 2i} = \frac{e^{it}}{-2}$$

ker fja $\mathbb{R}$, residuum v konjugirani singularnosti, konjugirana vrednost

$$\operatorname*{res}_{s=-i} F(s)e^{st} = \frac{e^{-it}}{-2}$$

$$\mathcal{L}^{-1}(F(s)_{delni}) = 1 + \frac{e^{it}}{-2} + \frac{e^{-it}}{-2} = 1 \cancel{+} - \frac{1}{2}\left[2\cos t\right] = \underbrace{1-\cos t}_{f(t)}$$

še stopnica:

$$\mathcal{L}^{-1}(F(s)) = \boxed{\underbrace{u_{1/3}(t)}_{a=1/3}\underbrace{(1-\cos(t-1/3))}_{f(t-a)}}$$

Poiščite rešitev enačbe

$$x(t) = t + \underbrace{\int_0^t \sin(t-u)\cdot x(u)\,du}_{\text{konvolucija } \sin(t) * x(t)}$$

$$X(s) = \frac{1}{s^2} + \frac{1}{(s^2+1)^2}\cdot X(s)$$

$\mathcal{L}$ $\mathcal{L}$

$$X(s)\left(-\frac{1}{(s^2+1)^2} + 1\right) = \frac{1}{s^2}$$

$$X(s) = \frac{1}{s^2\left(1-\frac{1}{s^2+1}\right)} = \frac{1}{s^2\left(\frac{s^2}{s^2+1}\right)} = \frac{s^2+1}{s^4}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{s^2+1}{s^4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2} + \frac{1}{s^4}\right) = t + \frac{t^3}{3!}$$

$$t^n \leftrightarrow \frac{n!}{s^{n+1}}$$

- $x'' + 2x' + x = \sin t \qquad x(0) = 0 \qquad x'(0) = -1$

$$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

$$\mathcal{L}(x(t)) = X(s)$$
$$\mathcal{L}(x'(t)) = sX(s) - x(0) = sX(s)$$
$$\mathcal{L}(x''(t)) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) + 1$$

$$s^2X(s) + 1 + 2sX(s) + X(s) = \frac{1}{s^2+1}$$
$$X(s)(s^2 + 2s + 1) = 1/(s^2+1)$$
$$X(s) = \frac{1 - s^2 - 1}{(s^2+1)(s^2+2s+1)} = \frac{-s^2}{(s^2+1)(s+1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{-s^2}{(s^2+1)(s+1)^2}\right) = *$$

nastavek

$$\frac{-s^2}{(s^2+1)(s+1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{(s+1)^2} =$$

$$= \frac{(As+B)(s+1)^2 + C(s^2+1)(s+1)^2 + D(s^2+1)}{(s^2+1)(s+1)^2} =$$

$$= \frac{As^3 + Bs^2 + 2As^2 + 2Bs + As + B + Cs^3 + Cs^2 + Cs + C + Ds^2 + D}{(s^2+1)(s+1)^2}$$

$$\begin{aligned}
s^3 &: \; 0 = A + C \\
s^2 &: \; -1 = B + 2A + C + D \\
s^1 &: \; 0 = 2B + A + C \quad \longrightarrow 2B = 0 \rightarrow B = 0 \\
s^0 &: \; 0 = B + C + D \quad \longrightarrow C = -D
\end{aligned}$$

$$-1 = 2A \rightarrow A = -1/2$$
$$C = 1/2$$
$$D = -1/2$$

$$* = \mathcal{L}^{-1}\left(-\frac{1s}{2(s^2+1)} + \frac{1}{2(s+1)} - \frac{1}{2(s+1)^2}\right)$$

$$x(t) = -\frac{1}{2}\cos t + \frac{1}{2}e^{-t} - \frac{1}{2}t\cdot e^{-t}$$

- $x''' - 2x'' + x' = 4$ $\quad x(0) = 1 \quad x'(0) = 2 \quad x''(0) = -2$

- $\mathcal{L}(x(t)) = X(s)$

$\mathcal{L}(x'(t)) = sX(s) - x(0) = sX(s) - 1$

$\mathcal{L}(x''(t)) = s^2X(s) - s\,x(0) - x'(0) = s^2X(s) - s - 2$

$\mathcal{L}(x'''(t)) = s^3X(s) - s^2x(0) - s\,x'(0) - x''(0) = s^3X(s) - s^2 - 2s + 2$

$$\underline{s^3X(s) - s^2 - 2s + 2}\ \underline{-2s^2X(s) + 2s + 4} + sX(s) - 1 = 4$$

$$X(s)(s^3 - 2s^2 + s) = \frac{4}{s} + s^2 + \cancel{2s} - \cancel{2} - \cancel{2s} - \cancel{4} + \cancel{1} - 5$$

$$X(s) = \frac{4/s + s^2 - 5}{s^3 - 2s^2 + s} = \frac{4 + s^3 - 5s}{(s^2 - 2s + 1)s^2} = \frac{4 + s^3 - 5s}{s^2(s-1)^2}$$

$$\mathcal{L}^{-1}(X(s)) = x(t)$$

$$\frac{4 + s^3 - 5s}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} =$$

$$\frac{As(s-1)^2 + B(s-1)^2 + C(s-1)s^2 + Ds^2}{s^2(s-1)^2}$$

$$As^3 - 2As^2 + As + Bs^2 - 2Bs + B + Cs^3 - Cs^2 + Ds^2 = 4 + s^3 - 5s$$

$s^3: \quad 1 = A + C$

$s^2: \quad 0 = -2A + B - C + D$

$s^1: \quad -5 = A - 2B$

$s^0: \quad 4 = B$ ✓

$\rightarrow$ $A = 3$ ✓, $C = -2$ ✓, $D = 0$ ✓

$$x(t) = \mathcal{L}^{-1}\left(\frac{3}{s} + \frac{4}{s^2} - \frac{2}{(s-1)}\right) = 3 + 4 \cancel{m} - 2e^t$$

- $x' - y' - 2x + 2y = 1 - 2t$ $\qquad x(0)=0 \quad x'(0)=0 \quad y(0)=0$

  $x'' + 2y' + x = 0$

$\mathcal{L}(x(t)) = X(s) \qquad \mathcal{L}(y(t)) = Y(s)$

$\mathcal{L}(x'(t)) = sX(s) - 0 \qquad \mathcal{L}(y'(t)) = sY(s) - 0$

$\mathcal{L}(x''(t)) = s^2X(s) - 0 - 0$

$$sX(s) - sY(s) - 2X(s) + 2Y(s) = \frac{1}{s} - \frac{2}{s^2}$$

$$s^2X(s) + 2sY(s) + X(s) = 0$$

$$X(s)(s-2) = \underbrace{\frac{1}{s} - \frac{2}{s^2}}_{\frac{s-2}{s^2}} + Y(s)(s-2)$$

$$X(s) = \frac{1}{s^2} + Y(s)$$

$$s^2\frac{1}{s^2} + s^2Y(s) + 2sY(s) + \frac{1}{s^2} + Y(s) = 0$$

$$Y(s)(s^2 + 2s + 1) = -\frac{1}{s^2} - 1$$

$$Y(s) = -\frac{1+s^2}{s^2(s^2+2s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2}$$

$$A(s(s+1)^2) + B(s+1)^2 + Cs^2(s+1) + Ds^2 = -1 - s^2$$

$$As^3 - 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2 = -1 - s^2$$

$$\left.\begin{aligned} s^3 &: A + C = 0 \\ s^2 &: 2A + B + C + D = -1 \\ s^1 &: A + 2B = 0 \\ s^0 &: \underline{B = -1} \ \checkmark \end{aligned}\right\} \quad \begin{aligned} C &= -2 \ \checkmark \\ D &= -2 \ \checkmark \\ A &= 2 \ \checkmark \end{aligned}$$

$$Y(s) = \frac{2}{s} - \frac{1}{s^2} - \frac{2}{s+1} - \frac{2}{(s+1)^2}$$

$$\mathcal{L}^{-1}(Y(s)) = 2 - t - 2e^{-t} - 2te^{-t} = y(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = \\ = e^{at}\,\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \\ = e^{at}\frac{t^{n-1}}{(n-1)!}$$

$$X(s) = \frac{1}{s^2} + Y(s) = \frac{2}{s} - \frac{2}{s-1} - \frac{2}{(s+1)^2}$$

$$\mathcal{L}^{-1}(X(s)) = 2 - 2e^{-t} - 2te^{-t} = x(t)$$

$$\underbrace{t x''}_{1} + \underbrace{(1-2t)}_{2|3} x' - \underbrace{2x}_{4} = 0 \qquad x(0)=1, \quad x'(0)=2$$

ni s konst koeficienti

$f(t)$	$F(s)$
$f^{(n)}$	$\dots$
$t \cdot f(t)$	$-F'(s)$

$$\mathcal{L}(x(t)) = X(s)$$
$$\mathcal{L}(x'(t)) = sX(s) - 1$$
$$\mathcal{L}(x''(t)) = s^2X(s) - s - 2$$

$$\mathcal{L}(tX'(t)) = -(\mathcal{L}(x'(t)))' = -(sX(s) - x(0))' =$$
$$= -(X(s) + sX'(s))$$

$$\mathcal{L}(tx''(t)) = -\mathcal{L}(s^2X(s) - s - 2)' =$$
$$= -(2sX(s) + s^2X'(s) - 1)$$

$$\underbrace{-2sX(s) - s^2X'(s) + 1}_{1} + \underbrace{sX(s) - 1}_{2} + \underbrace{2[X(s) + sX'(s)]}_{3} - \underbrace{2X(s)}_{4} = 0$$

$$X'(s)(-s^2 + 2s) + X(s)(-2s + s + 2 - 2) = 0$$

$$X'(s)\, s(2-s) = X(s) \cdot s$$

$$\frac{X'(s)}{X(s)} = \frac{1}{2-s}$$

$$\frac{dX(s)}{X(s)} = \frac{ds}{2-s} \quad / \int \qquad -\frac{ds}{s-2}$$

/ da lahko antilogaritmiramo

$$\ln X(s) = -\ln(s-2) + \ln C$$

$$X(s) = \frac{C}{s-2}$$

$$x = Ce^{2t}$$
$$x(0) = 1 \rightarrow C = 1$$
$$\boxed{x(t) = e^{2t}}$$

SPECIALNE FJE

Gama fja:

$$\boxed{\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt}$$

$$\boxed{\Gamma(x+1) = x\Gamma(x)}$$ – rekurzivna formula

če $n \in \mathbb{N}$: $\Gamma(n+1) = n!$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$$\int_0^\infty e^{-x^n} dx = \int_0^\infty e^{-t} \frac{1}{n t^{1-\frac{1}{n}}} dt = \frac{1}{n}\int_0^\infty e^{-t} t^{-1+\frac{1}{n}} dt = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$$

$t = x^n$

$dt = n x^{n-1} dx$

$$dx = \frac{dt}{n x^{n-1}} = \frac{dt}{n x^n \frac{1}{x}} = \frac{dt}{n t \frac{1}{\sqrt[n]{t}}} = \frac{dt}{n t^{1-\frac{1}{n}}}$$

Reševanje D.E. s potenčnimi vrstami

$$y = \sum_{n=0}^\infty a_n x^n$$

ker je 1. člen enak 0

$$y' = \left(\sum_{n=0}^\infty a_n x^n\right)' = \sum_{n=0}^\infty (a_n x^n)' = \sum_{n=0}^\infty a_n \cdot n \cdot x^{n-1} = \sum_{n=1}^\infty a_n n \cdot x^{n-1}$$

$$y'' = \sum_{n=0}^\infty a_n n(n-1) x^{n-2} = \sum_{n=2}^\infty a_n n(n-1) x^{n-2}$$

Rešite D.E.: (rešitev izrazite z rekurzivno formulo)

$$(x+1)y' - (2x+3)y = 0$$

$$(x+1)\sum_{n=1}^\infty a_n n x^{n-1} - (2x+3)\sum_{n=0}^\infty a_n x^n = 0$$

$$\sum_{n=1}^\infty a_n n \underline{x^n} + \sum_{n=1}^\infty a_n n \underline{x^{n-1}} - 2\sum_{n=0}^\infty a_n \underline{x^{n+1}} - 3\sum_{n=0}^\infty a_n \underline{x^n} = 0$$

$$\sum_{n=1}^\infty a_n n x^n + \sum_{n=0}^\infty a_{n+1}(n+1) x^n - 2\sum_{n=1}^\infty a_{n-1} x^n - 3\sum_{n=0}^\infty a_n x^n = 0$$

$$a_1 \cdot 1 \cdot \underset{1}{x^0} - 3a_0 \underset{1}{x^0} + \sum_{n=1}^{\infty} \left(a_n \cdot n + a_{n+1}(n+1) - 2a_{n-1} - 3a_n \right) x^n = 0$$

$n \geq 1$: $a_1 - 3a_0 = 0 \Rightarrow \boxed{a_1 = 3a_0}$ $\boxed{a_0 = \text{poljuben}}$

$$a_n n + a_{n+1}(n+1) - 2a_{n-1} - 3a_n = 0$$

$$a_{n+1}(n+1) = 2a_{n-1} + 3a_n - a_n \cdot n$$

$$\boxed{a_{n+1} = \frac{2a_{n-1} + 3a_n - a_n \cdot n}{n+1}}$$

18. 3. 2009

~~* MANJKA 1. URA~~

• $y' = 3x^2 y$ rešitev izrazimo z elementarnimi fjami

$$\sum_{n=1}^{\infty} a_n n x^{n-1} = 3x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} = 3 \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$\sum_{n=-2}^{\infty} a_{n+3} (n+3) x^{n+2} = 3 \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$a_1 + a_2 \cdot 2x + \sum_{n=0}^{\infty} \left[a_{n+3} (n+3) - 3a_n \right] x^{n+2} = 0$$

$a_1 = 0$ $n \geq 0$

$a_2 = 0$ $a_{n+3}(n+3) - 3a_n = 0$

$$\boxed{a_{n+3} = \frac{3a_n}{n+3}}$$

a_0 - poljuben

to je v obliki rekurzivne formule

To zapišimo kot formulo nja

$a_{n+3} = \frac{3}{n+3} \cdot a_n$ ko je nek a je tudi tisti 3 naprej enak $\emptyset$

$a_1 = 0 \longrightarrow a_{3k+1} = 0$

$a_2 = 0 \longrightarrow a_{3k+2} = 0$

a_{3k}: $a_{3k+3} = \left[\frac{3}{3k+3} \right] a_{3k}$

$$\boxed{a_{3(k+1)} = \frac{1}{k+1} a_{3k}}$$

$$\boxed{a_{3k}} = \frac{1}{k} a_{3(k-1)} = \frac{1}{k} \cdot \frac{1}{k-1} \cdot a_{3(k-2)} = \frac{1}{k} \cdot \frac{1}{k-1} \cdot \frac{1}{k-2} \cdot a_{3(k-3)}$$

lahko gremo do a_0

$$= \frac{1}{k} \cdot \frac{1}{k-1} \cdot \frac{1}{k-2} \cdots \frac{1}{1} a_0 = \boxed{\frac{a_0}{k!}}$$

torej: $a_{3k+1} = 0 \quad a_{3k+2} = 0 \quad a_{3k} = \frac{a_0}{k!}$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{3k} x^{3k} + \cancel{\sum_{k=0}^{\infty} a_{3k+1} x^{3k+1}} + \cancel{\sum_{n=0}^{\infty} a_{3k+2} x^{3k+2}}$$

ostanejo samo členi $3k$

$$= \sum_{k=0}^{\infty} \frac{a_0}{k!} x^{3k} = \sum_{k=0}^{\infty} a_0 \cdot \frac{(x^3)^k}{k!} =$$

! $\boxed{e^x = \sum_{k=0}^{\infty} x^k / k!}$ $\quad = \underline{a_0 e^{x^3}}$

$x \cdot y'' + 2y' + xy = 0$

dn: $y = \sum_{n=0}^{\infty} a_n x^n$

dobivamo samo tiste rešitve, ki nimajo singularnosti v 0.

RAZŠIRIMO NASTAVEK:

$$y = x^r \sum_{n=0}^{\infty} a_n x_n \quad (a_0 \neq 0)$$

$r \in \mathbb{Z}$, torej s tem nastavkom pokrijemo tudi pole.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{(n-1)+r}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

s tem gremo v enačbo

$$\underbrace{\sum_{n=0}^{\infty} a_n \cdot (n+r)(n+r-1) x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}}_{-\,||\,-} +$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$-\,||\,- + \sum_{n=2}^{\infty} a_{(n-2)} x^{n+r-1} = 0$$

$$\Big[\underbrace{a_0 r(r-1)x^{r-1}}_{n=0} + \underbrace{a_1(r+1)\cdot r\, x^{r}}_{n=1} + \underbrace{2a_0 r x^{r-1}}_{n=0} +$$

iz prve vrste — iz druge vrste

$$+\underbrace{2a_1(1+r)x^r}_{n=1} +$$

$$+\sum_{n=2}^{\infty}\Big[a_n(n+r)(n+r-1)+2a_n(n+r)+a_{n-2}\Big]x^{n+r-1}\Big]=0$$

x^{r-1} : $r\cdot(r-1)a_0 + 2a_0\cdot r = 0$

x^{r} : $a_1(r+1)r + 2a_1(r+1) = 0$

$n \geq 2$: $a_n(n+r)(n+r-1)+2a_n(n+r)+a_{n-2}=0$

x^{r-1} : $a_0(r(r-1)+2r)=0$

$a_0(r^2+r)=0$

$a_0 r(r+1)=0$

ker $a_0 \neq 0$, velja

a) $r=0$ ali b) $r=-1$

$r=0$: $2a_1=0 \rightarrow \underline{a_1=0}$

$a_n n(n-1) + 2a_n\cdot n + a_{n-2} = 0$

$a_n(n^2-n+2n) + a_{n-2} = 0$

$a_n = -\dfrac{a_{n-2}}{n(n+1)}$ rekurzivna formula

$$a_n = -\frac{a_{n-2}}{n(n+1)} = -\frac{1}{n(n+1)} \cdot \left[-\frac{a_{n-4}}{(n-2)(n-1)}\right] =$$

Če začnemo s sodim, se ustavimo pri a_0

$$= \frac{-1^{2}}{(n+1)(n)(n-1)(n-2)(n-3)(n-4)} a_{n-6} = \ldots$$

$= \ldots =$

$n = 2k+1$:

$(*)$ $a_1 = 0$, $a_1 = 0$

$n = 2k$:

$$\frac{(-1)^k \cdot a_0}{(2k+1) \cdot \ldots \cdot 3 \cdot 2} = \frac{(-1)^k a_0}{(2k+1)!}$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^n = \underbrace{\sum_{n=0}^{\infty} a_{2k+1} x^{2k+1}}_{=0} + \sum_{k=0}^{\infty} a_{2k} x^{2k} =$$

$\underline{r=0}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k+1)!} x^{2k} = \frac{a_0}{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} =$$

$$= \frac{a_0 \sin x}{x}$$

$\underline{r=-1}$:

x^r; $c = 0$

a_1 poljuben

$$n \geq 2: \quad a_n(n-1)(n-2) + 2a_n(n-1) + a_{n-2} = 0$$

$$a_n(n^2 - 3n + 2 + 2n - 2) = -a_{n-2}$$

$$a_n = -\frac{a_{n-2}}{n(n-1)}$$

$$a_n = -\frac{a_{n-2}}{n(n-1)} = \frac{a_{n-4}}{n(n-1)(n-2)(n-3)} =$$

$$= -\frac{a_{n-6}}{n(n-1)(n-2)(n-3)(n-4)(n-5)} =$$

$$= \begin{cases} n = 2k: & \dfrac{(-1)^k a_0}{2k(2k-1)\cdots 2} = \dfrac{(-1)^k a_0}{2k!} \\[2ex] n = 2k+1: & \dfrac{(-1)^k a_1}{(2k+1)\cdot 2k \cdots 3\cdot 2} = \dfrac{(-1)^k a_1}{(2k+1)!} \end{cases}$$

$$y = \underset{\substack{\wedge \\ r=-1}}{x^{-1}} \sum_{n=0}^{\infty} a_n x^n = x^{-1}\left(\sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}\right)$$

$$= x^{-1}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} a_0 x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} a_1 x^{2k+1}\right] =$$

$$= \frac{1}{x}\left(a_0 \cos x + a_1 \sin x\right) = \frac{a_0 \cos x}{x} + \frac{a_1 \sin x}{x}$$

V splošnem seštejemo. Tu je rešitev pregleda že vsebovana.

Legendrova D.E.

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0$$

$y = P_n(x)$ Legendrovi polinomi

- $(a^2 - x^2)y'' - 2xy' + 12y = 0$

uvedemo substitucijo $x = a \cdot z$

$$z = \frac{x}{a}$$

$$(x, y(x)) \rightsquigarrow (z, y(z))$$

kako se zamenjujejo odvodi $y' \leftarrow \dot{y}$

$$\frac{dz}{dx} = \frac{1}{a}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \dot{y} \cdot \frac{1}{a}$$

$$y'' = \frac{dy'}{dx} = \frac{d\left(\frac{\dot{y}}{a}\right)}{dz} \cdot \frac{dz}{dx} = \frac{\ddot{y}}{a} \cdot \frac{1}{a} = \frac{\ddot{y}}{a^2}$$

To vstavimo v d.e.

$$(a^2 - a^2 z^2)\frac{\ddot{y}}{a^2} - 2az\frac{\dot{y}}{a} + 12y = 0$$

$$\ddot{y}(1 - z^2) - 2z\dot{y} + 12y = 0 \quad \text{Legendrova D.E. za } n = 3$$

torej $y = P_3(z) = P_3\left(\frac{x}{a}\right)$

25.3.2009

$$(1-x^2)y'' - 2xy' + 2y = 0$$

ena rešitev je $P_1(x) = x$. Poiščimo še drugo rešitev. Vpeljimo substitucijo $y = z \cdot x$.

$$y' = \frac{dy}{dx} = \frac{d(zx)}{dx} = \qquad (x, y(x)) \rightsquigarrow (x, z(x))$$

Opomba: z funkcija xa

$$= z'x + z$$

$$y'' = z''x + z' + z' = z''x + 2z'$$

$$(1-x^2)(z''x + 2z') - 2x(z'x + z) + 2zx = 0$$

$$(1-x^2)xz'' + 2z' - 2x^2z' - 2x^2z' \cancel{-2xz} \cancel{+2zx} = 0$$

$$(1-x^2)xz'' + 2z' - 4x^2z' = 0$$

$$(1-x^2)xz'' + 2z'(1-2x^2) = 0$$

$z' = u$ znižamo red

$$(1-x^2)xu' + 2u(1-2x^2) = 0$$

$$\frac{u'}{u} = -\frac{2(1-2x^2)}{x(1-x^2)} \qquad \text{ločljive spremenljivke} \quad /\int$$

$$\ln u = -\int \frac{2(1-2x^2)}{x(1-x^2)}\,dx \quad \to \text{parcialni ulomki}$$

$(1+x)(1-x)$

$$\frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$A(1+x)(1-x) + Bx(1+x) + C(1-x)x$$

$$A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

$$\left.\begin{array}{ll} x^2: & -A + B - C = -4 \\ x^1: & B + C = 0 \\ x^0: & A = 2 \end{array}\right\} \to C = 1 \quad B = -1$$

$$-\int\left[\frac{2}{x}-\frac{1}{1-x}+\frac{1}{1+x}\right]dx=-2\ln x-\ln(1-x)-\ln(1+x)+\ln C$$

$$\ln\frac{c}{x^2(1-x^2)}=\ln u$$

$$u=\frac{Const}{x^2(1-x^2)}$$

$$z=\int u \qquad \text{parcialni}$$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(1-x)}+\frac{D}{(1+x)}$$

$$Ax(1-x^2)+B(1-x^2)+Cx^2(1+x)+Dx^2(1-x)=Const$$

$$Ax-Ax^3+B-Bx^2+Cx^2+Cx^3+Dx^2-Dx^3=Const$$

$$\left.\begin{array}{ll} x^3: & -A+C-D=0 \\ x^2: & -B+D+C=0 \\ x^1: & \boxed{A=0} \\ x^0: & \boxed{B=C_1} \end{array}\right\} \begin{array}{l} C=D \\ C=C_1/2 \\ D=C_1/2 \end{array}$$

$$z=C_1\int\left[\frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2}\frac{1}{1+x}\right]dx$$

$$z=C_1\left[-\frac{1}{x}-\frac{1}{2}\ln(1-x)+\frac{1}{2}\ln(1+x)+C_2\right]$$

$$z=C_1\left[-\frac{1}{x}+\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+C_2\right]$$

$$\boxed{y=zx=C_1\left(-1+\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)+xC_2\right)}$$

Akta

Ortogonalne funkcije

- f in g sta ortogonalni na [a, b], z utežjo $\varphi(x)$, če velja

$$\int_a^b f(x) \cdot g(x) \cdot \rho(x)\, dx = 0$$

Če $\{f_i\}$ zaporedje funkcij polni ortogonalni sistem, lahko poljubno fjo f lahko zapišemo

$$f = \sum_i c_i f_i \quad , \text{ kjer } c_i \text{ računamo kot}$$

$$c_i = \frac{\langle f, f_i \rangle}{\langle f_i, f_i \rangle}$$ ← skalarni produkt ~~[illegible]~~

Legendrovi polinomi so ortogonalne funkcije na intervalu [-1, 1] z utežjo $\rho(x) = 1$.

$$f = \sum_{n=0}^{\infty} c_n \cdot P_n$$

$$c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2}{2\cancel{n+1}} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

$\langle P_n, P_n \rangle = \| P_n \|^2$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

P_n – n-te stopnje
pri sodih indeksih
sode potence
če n-lih samo
lihe potence

$P_n(1) = 1$
$P_n(-1) = (-1)^n$ sledi iz Rodriguezove formule

- $P(x) = 3x^2 - 4x + 5$

$$C_0 = \frac{2\cdot 0+1}{2}\int_{-1}^{1}(3x^2-4x+5)\underset{P_0(x)}{1}\,dx = \frac{1}{2}\left(x^3\Big|_{-1}^{1} - 2x^2\Big|_{-1}^{1} + 5x\Big|_{-1}^{1}\right) = 1+5=6$$

$$C_1 = \frac{3}{2}\int_{-1}^{1}(3x^2-4x+5)\underset{P_1(x)}{x}\,dx = \frac{3}{2}\int_{-1}^{1}(3x^3-4x^2+5x)\,dx =$$

$$= \frac{3}{2}\left[\frac{3}{4}x^4\Big|_{-1}^{1} - \frac{4}{3}x^3\Big|_{-1}^{1} + \frac{5}{2}x^2\Big|_{-1}^{1}\right] =$$

$$= \frac{3}{2}\left(-\frac{4}{3}\right)\cdot 2 = -4$$

$$C_2 = \frac{5}{2}\int_{-1}^{1}(3x^2-4x+5)\underbrace{\frac{1}{2}(3x^2-1)}_{P_2(x)}\,dx = \ldots = 2$$

$C_3 = 0$ (potence zgoraj v enačbi niso ~~tretje~~ tretje stopnje)

$$\boxed{P(x) = 6P_0 - 4P_1 + 2P_2}$$

Besselove funkcije in Besselova D.E.

D.E.: $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$

rešitve so Besselove funkcije $J_\nu(x)$

Lastnosti:

- če $\nu \notin \mathbb{Z}$, potem sta J_ν in $J_{-\nu}$ linearno neodvisni in hkrati obe rešitvi D.E. To pa je splošna rešitev:

$$y = C_1 J_\nu + C_2 J_{-\nu}$$

- če $\nu \in \mathbb{Z}$, velja $J_\nu = (-1)^\nu J_{-\nu}$ in se splošna rešitev glasi

$$y = C_1 J_\nu + C_2 N_\nu$$

- $x y'' + 3 y' + x y = 0$

Podana substitucija $y = \frac{u}{x}$ Vidimo, da u nova fja

$$y' = u' \frac{1}{x} - u \frac{1}{x^2}$$

$$y'' = u'' \frac{1}{x} - u' \frac{1}{x^2} - u' \frac{1}{x^2} + 2u \frac{1}{x^3}$$

$$u'' - 2u' \cdot \frac{1}{x} + 2u \frac{1}{x^2} + 3u' \frac{1}{x} - 3u \frac{1}{x^2} + u = 0 \quad / \cdot x^2$$

$$u'' x^2 - 2u' x - u + u x^2 = 0$$

$$u'' x^2 + u' x + u (x^2 - 1) = 0 \qquad \nu = 1 \in \mathbb{Z}$$

$$u = C_1 J_1(x) + C_2 N_1(x)$$

$$\boxed{y = \frac{c_1}{x} J_1(x) + \frac{c_2}{x} N_1(x)}$$

- $y'' + (e^{2x} - \frac{1}{9}) \cdot y = 0$ substitucija $z = e^x$ nova spremenljivka

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \dot{y} e^x = \dot{y} z$$

$$y'' = \frac{dy'}{dx} = \frac{d(\dot{y} z)}{dz} \cdot \underset{e^x = z}{\frac{dz}{dx}} = (\ddot{y} z + \dot{y}) \cdot \cancel{e^x} z$$

$$(\ddot{y} z + \dot{y}) z + (z^2 - \tfrac{1}{9}) y = 0$$

$$z^2 \ddot{y} + z \dot{y} + (z^2 - \underbrace{1/9}) y = 0 \quad \nu \notin \mathbb{Z}$$

$$y = C_1 J_{1/3}(z) + C_2 J_{-1/3}(z) = \boxed{C_1 J_{1/3}(e^x) + C_2 J_{-1/3}(e^x)}$$

- $x^2 y'' - 3xy' + 4(x^4 - 3)y = 0$

substitucija: $y = x^2 u$, $x^2 = z$ $\quad (x, y(x)) \to (u, u(z))$

$$y = x^2 u =$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{d(z \cdot u)}{dz} \cdot \frac{dz}{dx} =$$

$$= (u + z\dot{u}) 2x$$

odvod produkta

$$y'' = \frac{dy'}{dx} = \frac{d((u + z\dot{u})2x)}{dx} \stackrel{\downarrow}{=} \frac{d(u + z\dot{u})}{dx} 2x + (u + z\dot{u}) \frac{d(2x)}{dx}$$

$$= \frac{d(u+z\dot{u})}{dz} \cdot \frac{dz}{dx} \cdot 2x + (u+z\dot{u}) \cdot 2 =$$

$$= (\dot{u} + \dot{u} + \ddot{u} z) \cdot 2x \cdot 2x + 2(u + z u) =$$

$$= 8 z \dot{u} + 4 z^2 \ddot{u} + 2u + 2z\dot{u} =$$

$$= 10 z \dot{u} + 4 z^2 \ddot{u} + 2u$$

$$\underbrace{10 z^2 \dot{u} + 4 z^3 \ddot{u} + 2zu}_{x^2 y''} \underbrace{- 6z(u + z\dot{u})}_{-3xy} + 4(z^2 - 3) zu = 0$$

$$10 z^2 \dot{u} + 4 z^3 \ddot{u} + 2zu - 6zu - 6z^2 u' + 4(z^2 - 3) zu = 0$$

$$4 z^3 \ddot{u} + 4 z^2 \dot{u} + zu (-4 + 4(z^2 - 3)) = 0$$

$$4 z^3 \ddot{u} + 4 z^2 \dot{u} + zu \, (\underbrace{4z^2 - 16}_{4(z^2-4)}) = 0 \qquad 1 : 4z$$

$$z^2 \ddot{u} + z\dot{u} + u (z^2 - 4) = 0$$

Bess. D.E. za $\nu = 2 \in \mathbb{Z}$

$$u = C_1 J_2(z) + C_2 N_2(z)$$

$$\boxed{y = x^2 u = x^2 \left(C_1 J_2(\underset{z}{x^2}) + C_2 N_2(x^2) \right)}$$

- $$y'' + \frac{y'}{x} + 2y = \frac{(y')^2}{2y}$$

substitucija: $y = u^2$

$$y' = 2u \cdot (u')$$

$$y'' = 2u' \cdot u' + 2u\, u''$$

$$\underbrace{2(u')^2 + 2u u''}_{} + 2u^2 - \frac{(2uu')^2}{2u^2} + \frac{2uu'}{x} \quad /:2$$

$$(u')^2 + uu'' + u^2 + \frac{uu'}{x} + u^2 - u'^2 = 0 \quad /\cdot x$$

$$xuu'' + uu' + u^2 = 0$$

$$xu'' + u' + xu = 0 \qquad \text{Besselova za } \nu = 0$$

$$x^2 u'' + xu' + (x^2 - 0^2)u = 0$$

$$u = C_1 J_0(x) + C_2 N_0(x)$$

$$y = u^2 = (C_1 J_0(x) + C_2 N_0(x))^2$$

Besselove f-je:

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \qquad (1)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x) \qquad (2)$$

Primer uporabe: dokaži (3) $\boxed{J_0' = -J_1}$

$$J_{-1} - J_1 = 2J_0'$$

$$J_0' = \frac{J_{-1} - J_1}{2} = \frac{-J_1 - J_1}{2} = -J_1$$

$$J_{-n} = (-1)^n J_n$$

• J_0'' izrazimo z J_0 in J_1

$$(J_0')' = (-J_1)' \underset{(2)}{=} -\frac{J_0 - J_2}{2} \underset{(1)}{=} -\frac{J_0 - \frac{2}{x}J_1 + J_0}{2} =$$

$$J_2 = \frac{2}{x}J_1(x) - J_0$$

$$= -\frac{2J_0 + \frac{2}{x}J_1}{2} = -J_0 + \frac{J_1}{x}$$

• $\int J_3 = f(J_0, J_1)$

$$\boxed{\int (2)} \to \int J_1 - \int J_3 = 2J_2 \to \int J_3 = \int J_1 - 2J_2 \overset{(3)}{=}$$

$$= -J_0 - 2J_2 = -J_0 - 2\left(\frac{2}{x}J_1 - J_0\right) =$$

$$= -J_0 - \frac{4}{x}J_1 + 2J_0 = \boxed{J_0 - \frac{4}{x}J_1}$$

1.4.2009

$\boxed{\text{Priprave } 1-k}$

- $y'' - xy' + 4y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$y(0) = 3$
$y'(0) = 0$

vemo, da $y(0) = c_0$, $y'(0) = c_1$, $y''(0) = 2c_2$ $\quad y^{(n)}(0) = n!\, c_n$

$y'' = \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2}$

$\sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} - x \sum_{n=1}^{\infty} a_n (n) x^{\cancel{n-1}} + 4\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} - \sum_{n=1}^{\infty} a_n (n) x^{n} + 4\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^{n} - \sum_{n=1}^{\infty} a_n n x^{n} + 4\sum_{n=0}^{\infty} a_n x^n = 0$

$2a_2 + 4a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n n + 4a_n \right] x^n = 0$

$2a_2 + 4a_0 = 0 \longrightarrow 2c_2 = -12$, ker $c_0 = 3 \rightarrow a_2 = -6$

$a_{n+2} = \dfrac{a_n (n-4)}{(n+2)(n+1)} \longrightarrow a_3 = \dfrac{-3a_1}{3 \cdot 2} = 0$

$a_4 = \dfrac{-2a_2}{4 \cdot 3} \qquad a_5 = \dfrac{-c_3}{5 \cdot 4} = 0 \qquad c_6 = 0$

$\hookrightarrow$ sledi $c_{2k} = 0$, kjer $k \geq 3$.

$y = 3 - 6x^2 + x^4$

$(1-x^2)y'' - 4xy' - 2y = 0$

$y = \sum_0^\infty c_n x^n \qquad y' = \sum_1^\infty c_n n x^{n-1} \qquad y'' = \sum_2^\infty c_n n(n-1) x^{n-1}$

$\sum_2^\infty c_n n(n-1) x^{n-2} - x^2 \sum_2^\infty c_n n x^{n-2}(n-1) - 4x \sum_1^\infty c_n n x^{n-1} -$

$- 2\sum_0^\infty c_n x^n$

$\sum_0^\infty c_{n+2}(n+2)(n+1)x^n - \sum_2^\infty c_n n(n-1)x^n - 4\sum_1^\infty c_n n x^n - 2\sum_0^\infty c_n x^n = 0$

$2c_2 + 6c_3 x + \sum_2^\infty c_{n+2}(n+2)(n+1)x^n - \sum_2^\infty c_n n(n-1)x^n -$

$- 4c_1 x - 4\sum_2^\infty c_n n x^n - 2c_0 - 2c_1 x - 2\sum_2^\infty c_n x^n = 0$

$2c_2 + 6c_3 x - 4c_1 x - 2c_0 - 2c_1 x + \sum_2^\infty x^n \big(c_{n+2}(n+2)(n-1) - c_n n(n-1) -$

$- 4nc_n - 2c_n\big) = 0$

$x^0: 2c_2 - 2c_0 = 0$

$x: 6c_3 - 4c_1 - 2c_1 = 0.$
$\quad 6c_3 - 6c_1 = 0$

$x^n: c_{n+2}(n+2)(n-1) - c_n n(n-1) - 4nc_n - 2c_n = 0$

$$c_{n+2} = \frac{c_n n(n-1) + 4nc_n + 2c_n}{(n+2)(n+1)} =$$

$$= \frac{c_n[n(n-1) + 4n + 2]}{(n+2)(n+1)} = \frac{c_n(n^2+3n+2)}{(n+2)(n+1)} = c_n$$

$c_2 = c_0$
$c_3 = c_1$
$n \geq 2: c_{n+2} = c_n \longmapsto c_{2k} = c_0$

$\qquad c_{2k+1} = c_1$

$$\sum_{n=0}^\infty c_n x^n = \sum_{k=0}^\infty \underset{\substack{\| \\ 0}}{c_{2k}} x^{2k} + \sum_{k=0}^\infty \underset{\substack{\| \\ c_1}}{c_{2k+1}} x^{2k+1} = c_0 \sum_{k=0} x^{2k} + c_1 \sum_{k=0}^\infty x^{2k+1}$$

$$= C_0 \frac{1}{1-x^2} + C_1 \frac{x}{1-x^2} = \frac{C_0 + C_1 x}{1-x^2}$$

- $f(x) = 2x - 1$
  $g(x) = 15\sqrt{x} - 2$

  ortogonalni na $k(x) = x$ na intervalu $(0, p)$ brez uteži.

  $$\int_0^p (2x-1)\,x\,dx = 0 \qquad \frac{2x^3}{3} - \frac{x^2}{2}\Big|_0^p$$

  $$\frac{2p^3}{3} - \frac{p^2}{2} = 0 \qquad p^2\left(\frac{2p}{3} - \frac{1}{2}\right) = 0$$

  enako tudi za drugo

- $y'' - \mathrm{tg}\,x\; y' + 2y = 0$

  $t = \sin x$

  $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \dot{y}\cos x$$

  $$y'' = \frac{dy'}{dx} = \frac{d(\dot{y}\cos x)}{dx} = \frac{d\dot{y}}{dx}\cos x + \dot{y}\frac{d(\cos x)}{dx}$$

  $$= \frac{d\dot{y}}{dt}\frac{dt}{dx}\cdot\cos x + \dot{y}(-\sin x) =$$

  $$= \ddot{y}\underbrace{\cos^2 x}_{1-\sin^2 x} - \dot{y}\sin x = \ddot{y}(1-t^2) - \dot{y}t$$

  vstavimo $\ddot{y}(1-t^2) - \dot{y}t - \dot{y}t + 2y = 0$

  $\ddot{y}(1-t^2) - \dot{y}(t+t) + 2y = 0$

  $\ddot{y}(1-t^2) - 2t\,\dot{y} + 2y = 0$, $\quad 2 \to n(n+1)$

  $y = P_1(t) \qquad y = P_1(\sin x)$

$x y'' - y' + xy = 0 \qquad y = x J_r(x) \quad$ pridemo do $y = x u$

$$y' = \frac{dy}{dx} \frac{d(xu)}{dx} = u + u'x$$

$$y'' = \frac{dy'}{dx} = \frac{d(u+u'x)}{dx} = \frac{du}{dx} + \frac{d(u'x)}{dx} = u' + u''x + u' = u''x + 2u'$$

$$x(u''x + 2u') - (u + u'x) + x(xu) = 0$$

$$x^2u'' + 2xu' - u - u'x + x^2u = 0$$

$$u''x^2 + u'(2x - x) + u(-1 + x^2) = 0$$

$$u''x^2 + u'x + u(x^2 - 1) = 0$$

$$r = \pm 1$$

UPORABNO PRI POSREDNEM ODV.

FAA DI BRUNO

$$\frac{d(f\circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{d^2(f\circ g)}{dx^2} = \frac{d^2f}{dg^2}\left(\frac{dg}{dx}\right)^2 + \frac{df}{dg}\frac{d^2g}{dx^2}$$

$$\frac{d^3(f\circ g)}{dx^3} = \frac{d^3f}{dg^3}\left(\frac{dg}{dx}\right)^3 + 3\frac{d^2f}{dg^2}\frac{dg}{dx}\frac{d^2g}{dx^2} + \frac{df}{dg}\frac{d^3g}{dx^3}$$

- z neodvisno spremenljivko $\quad z = x^2$

$$\left(x^2 - \frac{1}{x^2}\right)\left(y'' - \frac{y'}{x}\right) + 4xy' - 48y = 0$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \dot{y}\,2x$$

$$y'' = \frac{d(\dot{y}2x)}{dx} = \cancel{\frac{d(\dot{y}2x)}{dz}\cdot\frac{dz}{dx}} = \frac{\dot{y}}{dx}2x + \dot{y}2 =$$

$$= \frac{\dot{y}}{dz}\frac{dz}{dx}2x + \dot{y}2 = \ddot{y}2x2x + \dot{y}2 = \ddot{y}4x^2 + \dot{y}2 =$$

$$= 4\ddot{y}z + \dot{y}2$$

$$\left(z - \frac{1}{z}\right)\left(4\ddot{y}z + 2\dot{y} - 2\dot{y}\right) + 8z\dot{y} - 48y = 0$$

$$4\ddot{y}z^2 - 4\ddot{y} + 8\dot{y} - 48y = 0 \qquad /:4$$

$$(z^2 - 1)\ddot{y} + 2\dot{y}z - \underbrace{12}_{n(n+1)}y = 0 \qquad n(n+1) \to n = 3$$

$$y = P_3(z) = P_3(x^2)$$

PARCIALNE D.E.
8.4.2009

- $u_{xx} + u = 0$ $\qquad u(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + u = 0$$

Ker imamo parcialne odvode samo po X se delamo kot da je u odvisen samo od x.

$$u'' + u = 0$$

$$\lambda^2 + 1 = 0 \longmapsto \lambda = \pm i$$

glej MA2!

$$u = C_1 \cos 1x + C_2 \sin 1x$$

kar je pred i

to bi bila rešitev, če bi bil u odvisen samo od x, ker pa odvisen tudi od y so C_1 in C_2 funkcije y.

$$\boxed{u = C_1(y)\cos x + C_2(y)\sin x}$$

- $u_{xy} = 2y \cdot u_x$

$v = u_x$ vzamemo novo spremenljivko

$$v_y = 2y \cdot v$$

$$\frac{dv}{dy} = 2y\, v$$

$$\frac{dv}{v} = 2y\, dy \;/\int$$

$$\ln v = \cancel{2}\frac{y^2}{\cancel{2}} + \ln C$$

$$v = C \cdot e^{y^2} \Rightarrow v = C(x) \cdot e^{y^2}$$

$$u_x = c(x)\, e^{y^2}$$

$$u = \int c(x) e^{y^2} dx = e^{y^2} \overbrace{\int C(x)\,dx}^{E(x)} + D(y)$$

$$= \boxed{E(x)\, e^{y^2} + D(y)}$$

- $u_{xy} + u_x + x + y = \emptyset$
- $u_x = v$

$v_y + v + x + y = 0$

$v' + v + x + y = 0$ rešimo homogeni del. Tisti, ki ima funkcijo. Vedno ločljive spremenljivke

HOMOGENI: $v' + v = \emptyset$

$\frac{dv}{dy} = -v \longmapsto \frac{dv}{v} = -dy$

$\ln v = -y + \ln C$

$v = C \cdot e^{-y}$ ker v v resnici odvisen tudi od x, dobimo, da je C odvisen od x.

$v = C(x) \cdot e^{-y}$

NEHOMOGENI DEL: variacija konstante

$v = C(x,y)e^{-y}$

$v_y = C_y(x,y)e^{-y} - C(x,y)e^{-y}$

$C_y(x,y)e^{-y} - \cancel{C(x,y)e^{-y}} + \cancel{C(x,y)e^{-y}} + x + y = 0$ členi, ki imajo neodvajano konstanto, se vedno pokrajšajo.

$C_y(x,y) = (-x-y)e^{y} / \int$ per partes (produkt

$C(x,y) = \int (-xe^{y} - ye^{y})\,dy$

$= -xe^{y} - \left(ye^{y} - \int e^{y} dy\right) =$ $\boxed{\begin{array}{ll} \text{vedno polinom} & \\ y = u & e^{y}dy = dv \\ dy = du & e^{y} = v \end{array}}$

$= -xe^{y} - ye^{y} + e^{y} + D(x)$ D odvisen od x zato, ker je bil parcialni odvod po y.

$v = -x - y + 1 + e^{-y} \cdot D(x)$

$\parallel$

u_x

$$u=\int\left[-x-y+1+e^{y}D(x)\right]dx$$

$$u=-\frac{x^2}{2}-xy+x+e^{-y}\underbrace{\int D(x)\,dx}_{E(x)}+F(y)$$

$$u=-\frac{x^2}{2}-xy+x+e^{-y}E(x)+F(y)$$

- $u_x=2xy\cdot u \qquad u(\emptyset,y)=y$

$$u'=2xy\,u$$

$$\frac{du}{u}=2xy\,dx \quad /\int$$

$$\ln u=yx^2+\ln C$$

$$u=e^{yx^2}\cdot C \longmapsto u=e^{yx^2}C(y)$$

$$y=C(y)\longrightarrow u=e^{x^2y}y$$

- $u_x+Xu_{xx}=y$

pogoji: $u(1,y)=3y$
$u_x(1,y)=2y$

$$u'+Xu''=y \qquad \text{znižamo red} \qquad u'=v$$

$$v+Xv'=y \qquad \text{je nehomogena.}$$

HOMOGENI DEL: $v+xv'=\emptyset$

$$v+x\frac{dv}{dx}=\emptyset$$

$$x\frac{dv}{dx}=-v \longmapsto \frac{dv}{v}=-\frac{dx}{x} \quad /\int$$

$$\ln v=-\ln x+\ln C \longmapsto v=\frac{C}{x}$$

NEHOMOGENA z variacijo konstante

$\cdot v = \frac{C(x)}{x}$

$v' = \frac{C'(x)x - C(x)}{x^2}$ vstavimo v začetno formulo

$\cancel{\frac{C(x)}{x}} + \frac{C'(x)x - \cancel{C(x)}}{x} = y$

$\uparrow$
v

$\frac{C'(x)\cancel{x}}{\cancel{x}} = y$

$C'(x) = y \quad / \int dx$

$C(x) = yx + D$

ker $v = \frac{C(x)}{x}$: $v = \frac{xy + D}{x} \longmapsto v = y + \frac{D}{x}$

ker $v = u'$: $u = \int (y + \frac{D}{x}) dx$

$u = yx + D \ln x + E$

ker v resnici u odvisen od y:

$u = yx + D(y) \ln x + E(y)$ to je zdaj splošna rešitev.

zadostimo še začetnim pogojem

$u(1, y) = 3y$:

$3y = y + E(y) \mapsto E(y) = 2y$

$u_x(1, y) = 2y$

opazimo $u_x = v$

$u_x = y + \frac{D(y)}{x}$

$y + D(y) = 2y \longmapsto D(y) = y$

$\boxed{u = yx + y \ln x + 2y}$

- $u_{xx} + 2u_{xy} + u_{yy} = 0$
- Rešite tako, da uvedete $t = x \quad z = x - y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\underbrace{\frac{\partial t}{\partial x}}_{=1} + \frac{\partial u}{\partial z}\cdot\underbrace{\frac{\partial z}{\partial x}}_{=1} = u_t + u_z \cdot 1$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial x} = \frac{\partial(u_t + u_z)}{\partial x} = \frac{\partial u_t + u_z}{\partial t}\cdot\underbrace{\frac{\partial t}{\partial x}}_{=1} + \frac{\partial(u_t + u_z)}{\partial z}\underbrace{\frac{\partial t}{\partial x}}_{=1} =$$

$$= u_{tt} + u_{tz} + u_{zt} + u_{zz} = \boxed{u_{tt} + 2u_{tz} + u_{zz}}$$

$$u_{xy} = \frac{\partial(u_t + u_z)}{\partial y} = \frac{\partial(u_t + u_z)}{\partial t}\underbrace{\frac{\partial t}{\partial y}}_{=0} + \frac{\partial(u_t + u_z)}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{=-1} =$$

$$= \boxed{-u_{tz} - u_{zz}}$$

$$u_y = \frac{\partial u}{\partial t}\underbrace{\frac{\partial t}{\partial y}}_{=0} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{=-1} = \boxed{-u_z}$$

$$u_{yy} = \frac{\partial u_y}{\partial t}\underbrace{\frac{\partial t}{\partial y}}_{=0} + u_{zz} = \boxed{u_{zz}}$$

VSTAVIMO

$$u_{tt} + 2u_{tz} + u_{zz} - 2(u_{tz} + u_{zz}) + u_{zz} = 0$$

$$u_{tt} + u_{zz} - u_{zz} = 0$$

$$u_{tt} = 0 \quad / \int dt$$

$$u_t = C(z) \quad / \int dt$$

$u = C(z)\cdot t + D(z)$ gremo nazaj na x in y

$$u = C(x-y)\cdot x + D(x-y)$$

- $x \cdot u_x - y u_y = 2u$ s substitucijo $t = x^2$, $z = xy$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = u_t 2x + u_z y$

$u_y = \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = u_t \cdot 0 + u_z x = u_z x$

$2u_t \underset{t}{\underset{\parallel}{x^2}} + \cancel{u_z x y} - \cancel{u_z x y} = 2u$

$u_t \cdot t = u \longmapsto u' t = u$

$\frac{du}{du} = \frac{dt}{t} /\int$

$\ln u = \ln t + \ln C$

$u = C \cdot t = C(z) \cdot t$

nadomestimo

$\boxed{u = C(xy)x^2}$

- $x \cdot z_x - y z_y = 2x^2 + y$; $u = xy$, $v = x^2 - y$

iščemo rešitev, ki zadošča pogoju $z(x,x) = -x$

$z_x = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$z_x = z_u \cdot y + z_v 2x$

$z_y = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$

$z_y = z_u \cdot x - z_v$

$xy z_u + 2x^2 z_v - xy z_u + y z_v = 2x^2 + y$

$z_v(2x^2 + y) = 2x^2 + y \longmapsto z_v = 1$

$z = \int dv = v + C(u)$

$z = x^2 - y + C(xy)$ to je splošna rešitev

$x^2 - x + C(x^2) = -x$

$C(x^2) = -x^2$

$C(xy) = ? = -x \cdot y$

$\boxed{z = x^2 - y - xy}$

- $u_x + u_y = \varnothing$

$u(x,y) = F(x)G(y)$

$u_x(x,y) = F'(x)G(y)$

$u_y(x,y) = F(x)G'(y)$

$F'(x)G(y) + F(x)G'(y) = \varnothing$

$F'(x)G(y) = -F(x)G'(y)$

$\frac{F'(x)}{F(x)} = -\frac{G'(y)}{G(y)}$

I $\frac{F'(x)}{F(x)} = C$

$F'(x) = F(x)C \quad /\int$

$\frac{dF(x)}{dx} = F(x)C$

$\frac{dF(x)}{F(x)} = C\,dx$

$\ln F(x) = Cx + \ln D$

$F(x) = De^{cx}$

II $-\frac{G'(y)}{G(y)} = C$

$\frac{dG(y)}{G(y)} = -C\,dy \quad /\int$

$\ln G(y) = -Cy + \ln E$

$G(y) = e^{-Cy} E$

$u(x,y) = De^{Cx} E e^{-Cy} = DE e^{Cx-Cy} = F e^{C(x-y)}$

- $u_{xy} - u = 0$

$u(x,y) = F(x)\, G(y)$

$u_{xy} = F'(x) \cdot G'(y)$

$F'(x) G'(y) - F(x) G(y) - 0$

$\frac{F'(x)}{F(x)} = \frac{G(y)}{G'(y)} = C$

$\frac{F'(x)}{F(x)} = C \quad / \int$

$\ln F(x) = Cx + \ln D$

$F(x) = e^{cx} \cdot D$

$\frac{G(y)}{G'(y)} = C$

$G(y) = C\, G'(y)$

$dG(y)/dy$

$\frac{dy}{C} = \frac{dG(y)}{G(y)} \quad / \int$

$\frac{y}{C} + \ln E = \ln G(y)$

$G(y) = \cancel{\ } e^{y/c} \cdot E$

$u = F(x) \cdot G(y) = D e^{cx} \cdot e^{y/c} \cdot E = H e^{cx + y/c}$

- $u_x + u_y = 2x \cdot u$

$u(x,y) = F(x)G(y)$

$u_x = F'(x)G(y)$
$u_y = F(x)\,G'(y)$

$F'(x)G(y) + F(x)G'(y) = 2x\,F(x)G(y) \quad / F(x)G(y)$

$$\frac{F'(x)}{F(x)} + \frac{G'(x)}{G(x)} = 2x$$

$$\frac{F'(x)}{F(x)} - 2x = -\frac{G'(y)}{G(y)} = C$$

(I) $\dfrac{F'(x)}{F(x)} - 2x = C$

$$\frac{dF(x)}{F(x)} = (2x + C)dx \quad / \int$$

$\ln F(x) = x^2 + Cx + \ln D \quad / e^x$

$F(x) = e^{x^2 + Cx} \cdot D$

(II) $-\dfrac{G'(y)}{G(y)} = C$

$$\frac{dG(y)}{G(y)} = -C\,dy$$

$\ln G = -Cy + \ln E \quad / e^x$

$G = e^{-Cy} \cdot E$

$$\boxed{u = F(x)G(x) = e^{-Cy + x^2 + Cx} \cdot H}$$

• $x^2 u_{xy} + 3y^2 u = \emptyset$

$\cdot u(x,y) = F(x)G(y)$

$u_{xy} = F'(x)G'(y)$

$x^2 F'(x)G'(y) + 3y^2 F(x)G(y) = \emptyset \quad / : F(x)G(y)$

$3y^2 F(x)G(y) = -x^2 F'(x)G'(y)$

$$\frac{3F(x)}{x^2 F'(x)} = -\frac{G'(y)}{G(y)\,y^2} = C$$

(I) $\dfrac{3F(x)}{x^2F'(x)} = C \mapsto \dfrac{3F(x)}{x^2} = C\dfrac{dF(x)}{dx} \mapsto \dfrac{3\,dx}{Cx^2} = \dfrac{dF(x)}{F(x)}$

$\ln F(x) = \frac{3}{C}\left(-x^{-1}\right) + \ln D \quad / e^x$

$\boxed{F(x) = e^{-3/xC} D}$

(II) $\dfrac{-G'(y)}{G(y)\,y^2} = C \longmapsto \dfrac{-dG(y)}{G(y)} = y^2 dy$

$\ln G(y) = \frac{-Cy^3}{3} + \ln E \quad / e^x$

$G(y) = e^{-cy^3/3} \cdot E$

$u = F(x)G(x) = e^{-cy^3/3} \cdot E \cdot e^{-3/xc} D =$

$= \boxed{e^{-\left(\frac{cy^3}{3} + \frac{3}{xc}\right)} \cdot H}$

15.4.2009

Fourierjeva separacija

Primer:

- $u_{tt} = u_{xx}$ kjer $u(x,t)$, $0<x<\pi$, $t>0$

$u(0,t)=0$ } robni pogoji (homogeni: vrednost fje je $\emptyset$)
$u(\pi,t)=0$

$u(x,0)=\sin 2x$ } začetni pogoji
$u_t(x,0)=\sin 3x$

$u(x,t) = F(x)\,G(t)$

$F(x)\,G''(t) = F''(x)\,G(t)$

$\frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)}$ to lahko enako samo, kadar je to konstanta

1) $\frac{F''(x)}{F(x)} = k$

$F'' - kF = \emptyset$ z nastavkom e^{λ} (glej MA2-avditorne vaje)

$\lambda^2 - k = \emptyset$ $\lambda_{1,2} = \pm\sqrt{k}$

1.1) $k>\emptyset \longrightarrow k = l^2 \ (l>\emptyset)$

$\lambda_{1,2} = \pm l$

$F_l(x) = A_l e^{lx} + B_l e^{-lx}$

<u>Upoštevamo robne pogoje:</u>

$F_l(0)=0$: torej $\emptyset = A_l + B_l$

$F_l(\pi)=0$: $\emptyset = A_l e^{l\pi} + B_l e^{-l\pi}$

$A_l = -B_l$
$A_l(e^{l\pi} - e^{-l\pi}) = \emptyset$

možnosti: • $A_l=0$, torej tudi $B_l=0$. Sledi $F_l(x)=\emptyset$
to je trivialna rešitev, ki nas ne zanimajo

• $e^{l\pi} = e^{-l\pi} \rightarrow l\pi = -l\pi \rightarrow 2l\pi = \emptyset \rightarrow l = \emptyset$ // ker $l > 0$

1.2.) $k = 0$ (glej D.E. [illegible], ker bo drugače p[illegible]ja)

$\lambda_{1,2} = 0$

$F_0(x) = A_0 + B_0 x$

$$\left.\begin{array}{l} F_0(0) = 0 : A_0 = 0 \\ F_0(\pi) = 0 : A_0 + B_0\pi = 0 \end{array}\right\} B_0\pi = 0 \rightarrow B_0 = 0$$

$\rightarrow F_0(x) = 0$ spet trivialna rešitev

1.3.) $k < 0 \rightarrow k = -l^2 \quad (l > 0)$

$\lambda_{1,2} = \pm\sqrt{-l^2} = \pm i l$

$F_l(x) = A_l \cos lx + B_l \sin lx$

$$\left.\begin{array}{l} F_l(0) = 0 : \quad 0 = A_l \\ F_l(\pi) = 0 : \quad 0 = A_l \cos l\pi + B_l \sin l\pi \end{array}\right\} B_l \sin l\pi = 0$$

• $B_l = 0 \underset{A_l = \emptyset}{\uparrow}\!\!\rightarrow F_l(x) = 0$

• $\sin l\pi = \emptyset$

$l \in \mathbb{Z} \xrightarrow[l > 0]{} l \in \mathbb{N}$

$\boxed{F_l(x) = B_l \sin lx, \quad l \in \mathbb{N}}$ imamo netrivialno rešitev

Če imamo homogene robne pogoje, pride prav samo del s sinusi in cosinusi (konstanta je negativna)

2) $\frac{G''(t)}{G(t)} = k$ $k>0$ in $k=0$ ni treba, ker smo dobili le $F_k = 0$

2.3) $k = -\ell^2$: $G''(t) + \ell^2 G(t) = 0$

$\lambda_{1,2} = \pm i\ell$

$$\boxed{G_\ell(t) = C_\ell \cos \ell t + D_\ell \sin \ell t}$$

3) $u(x,t) = \sum_k F_k(x)\, G_k(t) = \sum_{\ell=1}^{\infty} F_\ell(x) G_\ell(t) =$

$= \sum_{\ell=1}^{\infty} B_\ell \sin \ell x \,(C_\ell \cos(\ell t) + D_\ell \sin(\ell t))$

PIŠIMO $B_\ell \cdot C_\ell = E_\ell$ $\quad B_\ell D_\ell = H_\ell$

$$\boxed{u(x,t) = \sum_{\ell=1}^{\infty} \sin \ell x \,(E_\ell \cos \ell t + H_\ell \sin \ell t)}$$

4) začetni pogoj vstavimo

- $u(x,0) = \sin 2x$

$\sin 2x = \sum_{\ell=1}^{\infty} \sin \ell x \cdot E_\ell$ (ker $t=0$, členi odpadejo)

$\longrightarrow E_2 = 1$ & $E_\ell = 0$ $(\ell \neq 2)$

- $u_t(x,0) = \sin 3x$

$u_t(x,t) = \sum_{\ell=1}^{\infty} \sin \ell x \,(-\ell E_\ell \sin \ell t + \ell H_\ell \cos \ell t)$

$\sin 3x = \sum_{\ell=1}^{\infty} \sin \ell x \,(\ell \cdot H_\ell)$

$1 = 3 \cdot H_3$ $\quad \ell \cdot H_\ell = 0$ za $\ell \neq 3$

$H_3 = 1/3$ $\quad H_\ell = 0$ za $\ell \neq 3$

5) Rešitev

- $u(x,t) = E_2 \sin 2x \cos 2t + \frac{1}{3} \sin 3x \sin 3t$

$\uparrow E_2 = 1$ $\quad$ $\nwarrow H_3 = 1/3$

- $u_{xx} = u_{tt} + 2u_t$

$0 < x < \pi$
$t > 0$

$u(0,t) = 0$
$u(\pi,t) = 0$
$u(x,0) = \sin x$
$u_t(x,t) = 0$

$u(x,t) = F(x)G(x)$

$F''(x)G(t) = F(x)G''(t) + 2F(x)G'(t) \qquad / : F(x)G(t)$

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{G(t)} + 2\frac{G'(t)}{G(t)} = k$$

1) $\frac{F''(x)}{F(x)} = k$

imamo homogene robne pogoje. zato dobimo le $k = -\ell^2$ $(\ell > 0)$ oziroma

$F_\ell(x) = A_\ell \cos \ell x + B_\ell \sin \ell x$

$F_\ell(0) = 0 \longrightarrow A_\ell = 0$

$F_\ell(\pi) = 0 \longrightarrow B_\ell \cdot \sin \ell\pi = 0 \longrightarrow \ell \in \mathbb{N}$

$\boxed{F_\ell(x) = B_\ell \sin \ell x, \ \ell \in \mathbb{N}}$

2) $\frac{G''(t)}{G(t)} + 2\frac{G'(t)}{G(t)} = -\ell^2$ (ostalo lahko spustimo)

$$G''(t) + 2G'(t) + \ell^2 G(t) = 0$$

$$\lambda^2 + 2\lambda + \ell^2 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-4\ell}}{2} = -1 \pm \sqrt{1-\ell^2}$$

- $\ell = 1$: $\lambda_{1,2} = -1$

$$G_\ell(t) = C_\ell e^{-t} + D_\ell \cdot t \cdot e^{-t}$$

- $\ell > 1$: $\lambda_{1,2} = -1 \pm i\sqrt{\ell^2-1}$

$$G_\ell(t) = e^{-t}\left(C_\ell \cos(\sqrt{\ell^2-1}\,t) + D_\ell \sin(\sqrt{\ell^2-1}\,t)\right)$$

3) $u(x,t) = \sum_k F_k(x) G_k(t)$

$$= \sum_{\ell=1}^{\infty} F_\ell(x) G_\ell(t) = F_1(x) G_1(t) + \sum_{\ell=2}^{\infty} F_\ell(x) G_\ell(t)$$

$$= B_1 \sin x \cdot (C_1 e^{-t} + D_1 t e^{-t}) +$$

$$+ \sum_{\ell=2}^{\infty} B_\ell \sin \ell t \left(C_\ell \cos(\sqrt{\ell^2-1}\,t) + D_\ell \sin(\sqrt{\ell^2-1}\,t)\right)$$

e^{-t}

$B_i C_i = E_i \quad B_i D_i = H_i \quad i = 1, 2, 3 \ldots$

$$u(x,t) = \sin x \left(E_1 e^{-t} + H_1 t e^{-t}\right) + \sum_{\ell=2}^{\infty} \sin \ell x \; e^{-t}\left(E_\ell \cos(\sqrt{\ell^2-1}\,t) + \right.$$

$$\left. + H_\ell \sin(\sqrt{\ell^2-1}\,t)\right)$$

4) začetni pogoji

- $u(x,0) = \sin x$

$$\sin x = \sin x (E_1) + \sum_{\ell=2} \sin \ell x \cdot E_\ell$$

sledi $E_1 = 1$, $E_\ell = 0$ za $\ell \geq 2$

- $u_t(x,\emptyset)=\emptyset$;

$$u_t(x,t) = \sin x\left(-E_1 e^{-t} + H_1 e^{-t} - H_1 t e^{-t}\right) +$$
$$+ \sum_{l=2}^{\infty}\Big[-\sin lx\; e^{-t}\left(E_l \cos(\sqrt{l^2-1}\,t) + H_l \sin(\sqrt{l^2-1}\,t)\right) +$$
$$+ \sin lx\, e^{-t}\left(-\sqrt{l^2-1}\,E_l \sin(\sqrt{l^2-1}\,t) +\right.$$
$$\left.+\sqrt{l^2-1}\,H_l \cos(\sqrt{l^2-1}\,t)\right)\Big]$$

$$0 = u_t(x,0) = \sin x(-E_1+H_1) +$$
$$\sum_{l=2}^{\infty}\left[-\sin lx \cdot E_l + \sin lx \sqrt{l^2-1}\, H_l\right]$$

sledi

- $-E_1+H_1=0$
  $H_1 = E_1 = 1$

- $-E_l + \sqrt{l^2-1}\, H_l = 0$

$$\sqrt{l^2-1}\, H_l = 0 \longrightarrow H_l = 0 \text{ za } l \geq 2$$

za $E_l=\emptyset$ za $l \geq 2$ $\qquad l\geq 2$

5) $u(x,t) = \sin x(\underbrace{e^{-t}}_{E_1=1} + \underbrace{t}_{H_1=1} \cdot e^{-t})$

- $u_{tt} = c^2 u_{xx}$
- $0 < x < a$
  $t > 0$

$u_x(0,t) = 0$
$u_x(a,t) = 0$

$u(x,0) = \sin^2 \frac{\pi x}{a}$
$u_t(x,0) = 0$

$u(x,t) = F(x)\, G(t)$

$F(x)\, G''(t) = c^2 F''(x)\, G(t)$

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

1) $\frac{F''(x)}{F(x)} = k$ $\qquad \lambda_{1,2} = \pm\sqrt{k}$

1.1.) $k > 0 \qquad (k = \ell^2,\ \ell > 0)$

$F_\ell(x) = A_\ell e^{\ell x} + B_\ell e^{-\ell x}$

$F'_\ell(0) = 0$;
$F'_\ell(a) = 0$; $\qquad$ D.N.: $A_\ell = B_\ell = 0$
$\longrightarrow F_\ell(x) = 0$

1.2.) $k = 0$
$\lambda_{1,2} = 0$

$F_0(x) = A_0 + B_0 x \longrightarrow F'_0(x) = B_0$

$F'_0(0) = 0 \longrightarrow B_0 = 0$
$F'_0(a) = 0 \longrightarrow B_0 = 0$

$\longrightarrow F_0(x) = A_0$ poljuben

1.3) $k<0 \rightarrow k=-\ell^2 \quad \ell>0$

$$F_\ell(x)=A_\ell \cos \ell x + B_\ell \sin \ell x$$

$$F_\ell'(x)=-\ell A_\ell \sin \ell x + \ell B_\ell \cos \ell x$$

$$F_\ell'(0)=0 \Rightarrow \ell \cdot B_\ell = 0 \longrightarrow B_\ell = 0, \text{ ker } \ell > \emptyset$$

$$F_\ell'(a)=0 \Rightarrow -\ell \cdot A_\ell \cdot \sin \ell a = \emptyset$$

- $\cancel{\ell = 0}$, ker $\ell > 0$
- $A_\ell = \emptyset$ ker pa $B_\ell = 0$, bi dobili le trivialno rešitev $F_\ell(x)=0$
- $\sin \ell a = 0$; $\ell a = i\pi$, kjer $i \in \mathbb{Z}$

oziroma ker $\ell > 0 \mapsto i \in \mathbb{N}$

$$\boxed{\ell = \frac{n\pi}{a}, \quad n \in \mathbb{N}}$$

$$F_n(x) = A_n \cos \underbrace{\frac{n\pi}{a}}_{\ell} \cdot x, \quad n \in \mathbb{N}$$

2) $k<0$ ni potrebno (vidimo od prej)

2.2) $k=0$:

$$\frac{G''(t)}{c^2 G(t)} = 0 \longrightarrow G''(t) = \emptyset$$

$$G_0 = C_0 + D_0 \cdot t$$

2.3) $k = -\ell^2 \quad (\ell > 0)$

$$k = -\left(\frac{n\pi}{a}\right)^2 \quad n \in \mathbb{N}$$

$$G''(t) + \frac{c^2 n^2 \pi^2}{a^2} G(t) = \emptyset$$

$$\lambda_{1,2} = \pm i \frac{n\pi c}{a}$$

$$\boxed{G_n(t) = C_n \cos\left(\frac{n\pi c}{a} t\right) + D_n \sin\left(\frac{n\pi c}{a} t\right)}$$

3) $u(x,t) = F_0(x) G_0(t) + \sum_{n=1}^{\infty} F_n(x) G_n(t) =$

$$= A_0 (C_0 + D_0 t) + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\left(C_n \cos\frac{n\pi c t}{a} + D_n \sin\frac{n\pi c t}{a}\right)$$

$A_i C_i = E_i \qquad A_i \cdot D_i = H_i \qquad i = 0, 1, 2 \ldots$

$$u(x,t) = E_0 + H_0 t + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a}\left(E_n \cos\frac{n\pi c t}{a} + H_n \sin\frac{n\pi c t}{a}\right)$$

4) • $u(x,0) = \sin^2\frac{\pi x}{a}$

$$\sin^2\left(\frac{\pi x}{a}\right) = E_0 + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a} \cdot E_n$$

razvoj $\sin^2\left(\frac{\pi x}{a}\right)$ v Fourierjevo vrsto $(0,a)$

finta: $\sin^2\varphi = \frac{1 - \cos 2\varphi}{2}$

$$\left.\begin{array}{l} \sin^2\varphi + \cos^2\varphi = 1 \\ \cos^2\varphi - \sin^2\varphi = \cos 2\varphi \end{array}\right\}$$

$$\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right) = E_0 + \sum_{n=1} \cos\frac{n\pi x}{a} \cdot E_n \qquad \text{PRIMERJAMO}$$

sledi: $E_0 = 1/2$

$-\frac{1}{2} = E_2$ (kadar $n = 2$)

$E_n = 0$; $n \neq 0, 2$

- $u_t(x, \varnothing) = \varnothing$

$$u_t(x,t) = H_0 + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a}\left(-\frac{n\pi c}{a}\cdot E_n \sin\frac{n\pi ct}{a} + \right.$$

$$\left. + \frac{n\pi c}{a}\cdot H_n \cdot \cos\frac{n\pi ct}{a}\right)$$

$$\varnothing = H_0 + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a}\cdot\frac{n\pi c}{a}\cdot H_n$$

$$H_0 = 0 \quad \& \quad \frac{n\pi c}{a}\cdot H_n = \varnothing \implies \begin{matrix} n \in \mathbb{N} \\ \pi \neq \varnothing \\ c > 0 \end{matrix} \implies H_n = 0$$

5) $$\boxed{u(x,t) = \underset{\substack{\uparrow \\ E_0}}{\frac{1}{2}} - \underset{\substack{\uparrow \\ E_2}}{\frac{1}{2}}\cdot\cos\frac{2\pi x}{a}\cdot\cos\frac{2\pi ct}{a}}$$

- $u_t + u = u_{xx}$

$u(-\pi, t) = u(\pi, t)$
$u_x(-\pi, t) = u_x(\pi, t)$
$u(x, 0) = 2 + x$

$$F(x)G'(t) + F(x)G(t) = F''(x)G(t) \qquad /: F(x)G(t)$$

$$\frac{G'(t)}{G(t)} + 1 = \frac{F''(x)}{F(x)} = k$$

1) $\dfrac{F''(x)}{F(x)} = k$

1.1.) $k > 0$, $\lambda_{1,2} = \pm l$

$k = l^2 \ (l > 0)$

$F_l(x) = A_l e^{lx} + B_l e^{-lx}$

pogoj: $F_l(-\pi) = F_l(\pi)$ pogoj: $F_l'(-\pi) = F_l'(\pi)$

D.N. $A_\ell = B_\ell = \emptyset \longrightarrow F_\ell(x) = 0$

1.2) $k = \emptyset \quad \lambda_{1,2} = 0$

$F_0(x) = A_0 + B_0 x$

robni: $F_0(-\pi) = F_0(\pi)$: $A_0 - B_0\pi = A_0 + B_0\pi$

$2B_0\pi = 0$

$B_0 = 0$

$F_0'(-\pi) = F_0(\pi) \qquad F_0'(x) = B_0$

$B_0 = B_0$ ✓

$F_0(x) = A_0$

1.3.) $k < 0$, $k = -\ell^2 \; (\ell > 0)$

$\lambda_{1,2} = \pm i\ell$

$F_\ell(x) = A_\ell \cos \ell x + B_\ell \sin \ell x$

robni: • $F_\ell(-\pi) = F_\ell(\pi)$

$A_\ell \cos \ell\pi - B_\ell \sin \ell\pi = A_\ell \cos \ell\pi + B_\ell \sin \ell\pi$

$2B_\ell \sin \ell\pi = 0$

• $B_\ell = 0$

• $\sin \ell\pi = 0 \underset{\ell>0}{\longrightarrow} \ell \in \mathbb{N}$

• $F_\ell'(-\pi) = F_\ell'(\pi)$

$F_\ell'(x) = -\ell A_\ell \sin \ell x + \ell B_\ell \cos \ell x$

$\ell A_\ell \sin \ell\pi + B_\ell \ell \cos \ell\pi = -\ell A_\ell \sin \ell + B_\ell \ell \cos \ell\pi$

$2\underset{\ell>0}{\ell} A_\ell \sin \ell\pi = 0$

• $A_\ell = 0$

• $\sin \ell\pi = 0 \underset{\ell>0}{\longrightarrow} \ell \in \mathbb{N}$

Da zadostimo obema robnima pogojema, imamo

- $B_\ell = 0$ & $A_\ell = 0 \Rightarrow F_\ell(x) = 0$
- $B_\ell = 0$ & $\ell \in \mathbb{N}$
- $\ell \in \mathbb{N}$ & $A_\ell = 0$
- $\ell \in \mathbb{N}$ & $\ell \in \mathbb{N} \Rightarrow \ell \in \mathbb{N}$

$F_\ell(x) = A_\ell \cos \ell x + B_\ell \sin \ell x;\ \ell \in \mathbb{N}$ zgolj $\ell \in \mathbb{N}$ je res nujno (ker se pojavi v vseh)

2. $\dfrac{G'(t)}{G(t)} + 1 = k$

2.1) $k > 0$ ni treba

2.2) $k = 0$ $\quad G'(t) + G(t) = 0$

$$\frac{dG}{G} = -dt$$

$$\log G = -t + \log C$$

$$\boxed{G_0(t) = C_0 \cdot e^{-t}}$$

2.3) $k < 0$

$$\frac{G'(t)}{G(t)} = -\ell^2 - 1$$

$$\log G = (-\ell^2 - 1) \cdot t + \log C$$

$$\boxed{G_\ell(t) = C_\ell \cdot e^{(-\ell^2-1)\cdot t}}$$

3.) $u(x,t) = F_0(x)\, G_0(t) + \sum_{\ell=1}^{\infty} F_\ell(x) \cdot G_\ell(t) =$

$$= A_0 \cdot C_0\, e^{-t} + \sum_{\ell=1}^{\infty} (A_\ell \cos \ell x + B_\ell \sin \ell x) \cdot C_\ell \cdot e^{(-\ell^2-1)t}$$

$$A_i \cdot C_i = D_i \qquad B_i \cdot C_i = E_i \qquad i = 0, 1, 2$$

$$u(x,t) = D_0\, e^{-t} + \sum_{\ell=1}^{\infty} (D_\ell \cos \ell x + E_\ell \cdot \sin \ell x) \cdot e^{(-\ell^2-1)t}$$

22.04.2009

4.) začetni pogoj

$u(x,0)=2+x$

$$2+x=D_0+\sum_{\ell=1}^{\infty}(D_\ell\cos\ell x+E_\ell\sin\ell x)$$

Ponovitev razvoja v F.v.

$[-a,a]$

$$a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{a}+b_n\sin\frac{n\pi x}{a}\right)$$

$$a_0=\frac{1}{2a}\int_{-a}^{a}f(x)\,dx\qquad a_n=\frac{1}{a}\int_{-a}^{a}f(x)\cos\frac{n\pi x}{a}\,dx$$

$$b_n=\frac{1}{a}\int_{-a}^{a}f(x)\sin\frac{n\pi x}{a}\,dx$$

$a=\pi$

$$D_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}(2+x)\,dx=\frac{1}{2\pi}\left(2x+\frac{x^2}{2}\right)\Big|_{-\pi}^{\pi}=\frac{1}{2\pi}\left(2\pi+2\pi+\frac{\pi^2}{2}-\frac{\pi^2}{2}\right)=\underline{\underline{2}}$$

$$D_\ell=\frac{1}{\pi}\int_{-\pi}^{\pi}(2+x)\cdot\cos\ell x\,dx=\frac{1}{\pi}\left((2+x)\frac{\sin\ell x}{\ell}\Big|_{-\pi}^{\pi}-\underbrace{\int_{-\pi}^{\pi}\frac{\sin\ell x}{\ell}\,dx}_{0\text{, ker liha}}\right)$$

$u=2+x \qquad \cos\ell x\,dx=dv$

$du=dx \qquad v=\frac{\sin\ell x}{\ell}$

$$=\frac{1}{\pi}\left(\frac{(2+\pi)\sin\ell\pi}{\ell}-(2-\pi)\frac{\sin(\ell(-\pi))}{\ell}\right)=0$$

$$E_\ell=\frac{1}{\pi}\int_{-\pi}^{\pi}(2+x)\sin\ell x\,dx=\frac{1}{\pi}\left(-(2+x)\frac{\cos\ell x}{\ell}\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{\cos\ell x}{\ell}\,dx\right)=$$

$2+x=u$

$dx=du$

$\sin\ell x\,dx=dv$

$-\frac{\cos\ell x}{\ell}=v$

$$=\frac{1}{\pi}\left[\frac{-(2+\pi)\cos\ell\pi}{\ell}+\frac{(2-\pi)\cos(-\ell\pi)}{\ell}+\underbrace{\frac{\sin\ell x}{\ell^2}\Big|_{-\pi}^{\pi}}_{0}\right]$$

$$=\frac{1}{\pi}\left[\frac{-(2+\pi)(-1)^\ell}{\ell}+\frac{(2-\pi)(-1)^\ell}{\ell}\right]=\frac{(-1)^\ell}{\ell\pi}\left(\cancel{-2}-\pi+\cancel{2}-\pi\right)=$$

$$=\frac{(-1)^\ell(-2\pi)}{\ell\pi}=\frac{2\cdot(-1)^{\ell+1}}{\ell}$$

5.1 Rešitev

$$u(x,t)=2\cdot e^{-t}+\sum_{l=1}^{\infty}\frac{2\cdot(-1)^{l+1}}{l}\sin lx\cdot e^{-(l^2+1)t}$$

[redacted] $\Delta u=0$ Dirichletov problem

$0<x<\pi$

$0<y<\pi$

$u(x,y)=?$

$u(0,y)=0$

$u(\pi,y)=0$

$u(x,0)=\sin x$

$u(x,\pi)=\sin 3x$

$$\to u_{xx}+u_{yy}=0$$

$$F''(x)G(y)+F(x)G''(y)=0$$

$$\frac{F''(x)}{F(x)}=-\frac{G''(y)}{G(y)}=k$$

1.) $\frac{F''(x)}{F(x)}=k$

ker imamo homogene robne pogoje, moramo gledati le $k=-l^2\ (l>0)$

$$F_l(x)=A_l\cos lx+B_l\sin lx$$

$F_l(0)=0\Rightarrow A_l=0$

$F_l(\pi)=0\Rightarrow B_l\cdot\sin l\pi=0$

- $B_l=0\Rightarrow F_l=0$
- $\sin l\pi=0\Rightarrow l\in\mathbb{N}$

$F_l(x)=B_l\sin lx\quad l\in\mathbb{N}$

2.) $-\frac{G''(y)}{G(y)}=-l^2\qquad k=-l^2$

$\lambda_{1,2}=l^2$

$\lambda_{1,2}=\pm l$

$$G_l=C_l e^{ly}+D_l e^{-ly}$$

$$u(x,y)=\sum_{l=1}^{\infty} F_l(x)\,G_l(y)=\sum_{l=1}^{\infty} B_l \sin lx\left(C_l e^{ly}+D_l e^{-ly}\right)$$

$$B_l\cdot C_l=E_l \qquad B_l\cdot D_l=H_l$$

$u(x,0)=\sin x$:

$$\sin x=\sum_{l=1}^{\infty} B_l \sin lx\,(C_l+D_l)=\sum_{l=1}^{\infty} \sin lx\,(E_l+H_l)$$

$$\boxed{1=E_1+H_1} \qquad \boxed{E_l+H_l=0 \text{ za } l\neq 1}$$

$u(x,\pi)=\sin 3x$:

$$\sin 3x=\sum_{l=1}^{\infty} \sin lx\left(E_l\cdot e^{l\pi}+H_l\, e^{-l\pi}\right)$$

$$\boxed{1=E_3\, e^{3\pi}+H_3\, e^{-3\pi}}$$

$$\boxed{E_l\cdot e^{l\pi}+H_l\cdot e^{-l\pi}=0 \quad \text{za } l\neq 3}$$

l_1: $H_1=1-E_1$

$$E_1 e^{\pi}+H_1 e^{-\pi}=0$$

$$\Rightarrow E_1\cdot e^{\pi}+e^{-\pi}-E_1\cdot e^{-\pi}=0$$

$$\Rightarrow \boxed{E_1=\frac{-e^{-\pi}}{e^{\pi}-e^{-\pi}}} \Rightarrow H_1=1-\frac{-e^{-\pi}}{e^{\pi}-e^{-\pi}}=\boxed{\frac{e^{\pi}}{e^{\pi}-e^{-\pi}}=H_1}$$

l_3: $E_3+H_3=0 \Rightarrow H_3=-E_3$

$$1=E_3 e^{3\pi}+H_3 e^{-3\pi}$$

$$1=E_3 e^{3\pi}-E_3\, e^{-3\pi}$$

$$\boxed{E_3=\frac{1}{e^{3\pi}-e^{-3\pi}}} \qquad \boxed{H_3=-\frac{1}{e^{3\pi}-e^{-3\pi}}}$$

$\ell \neq 1, 3$:

$E_\ell + M_\ell = 0 \Rightarrow H_\ell = -E_\ell$

$E_\ell \cdot e^{\ell\pi} + H_\ell \cdot e^{-\ell\pi} = 0$

$E_\ell (e^{\ell\pi} - e^{-\ell\pi}) = 0$

$e^{\ell\pi} - e^{-\ell\pi} = 0 \Rightarrow \ell\pi = -\ell\pi \Rightarrow \ell = 0$ // ker $\ell \in \mathbb{N}$

$\boxed{E_\ell = 0}$ $\ell \neq 1, 3$

$\boxed{H_\ell = 0}$ $\ell \neq 1, 3$

Rešitev

$u(x,y) = \sin x \cdot \left(-\frac{e^{-\pi}}{e^{\pi} - e^{-\pi}} \cdot e^{y} + \frac{e^{\pi}}{e^{\pi} - e^{-\pi}} e^{-y}\right) + \sin 3x \cdot \left(\frac{1}{e^{3\pi} - e^{-3\pi}} e^{3y} + \frac{1}{e^{3\pi} - e^{-3\pi}} e^{-3y}\right.$

$u(r, \varphi)$

$\Delta u = 0$

$1 < r < 3$

$u(1, \varphi) = \cos^2 \varphi$

$u(3, \varphi) = \cos 2\varphi$

$u(r, \varphi) = F(r) \cdot G(\varphi)$

D.N. $u_{xx} + u_{yy} = 0$ $\quad x = r\cos\varphi$, $y = r\sin\varphi$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0$

$F''(r) \cdot G(\varphi) + \frac{1}{r} F'(r) G(\varphi) + \frac{1}{r^2} F(r) G''(\varphi) = 0 \quad / : (F(r)G(\varphi))$

$\frac{F''(r)}{F(r)} + \frac{1}{r} \frac{F'(r)}{F(r)} + \frac{1}{r^2} \frac{G''(\varphi)}{G(\varphi)} = 0 \quad / \cdot r^2$

$\frac{G''(\varphi)}{G(\varphi)} = -\frac{r^2 F''(r)}{F(r)} - \frac{r F'(r)}{F(r)} = k$

$\boxed{u(r, \varphi + 2\pi) = u(r, \varphi)}$

$\boxed{G(\varphi + 2\pi) = G(\varphi)}$

1.1 $\frac{G''(\varphi)}{G(\varphi)} = k$

1.1. $k = \ell^2 \quad (\ell > 0)$

$\lambda_{1,2} = \pm \ell$

$G_\ell(\varphi) = A_\ell \cdot e^{\ell\varphi} + B_\ell \cdot e^{-\ell\varphi}$

pogoj: $G_\ell(\varphi) = G_\ell(\varphi + 2\pi)$

$\Rightarrow \quad A_\ell = B_\ell = 0$

D.N.

1.2. $k = 0 \quad \lambda_{1,2} = 0$

$G_0(\varphi) = A_0 + B_0\varphi$

$G_0(\varphi + 2\pi) = G_0(\varphi)$:

$A_0 + B_0(\varphi + 2\pi) = A_0 + B_0\varphi$

$2\pi B_0 = 0$

$\boxed{B_0 = 0} \quad \Rightarrow \quad \boxed{G_0(\varphi) = A_0}$

1.3 $k = -\ell^2 \quad (\ell > 0)$

$\lambda_{1,2} = \pm i\ell$

$G_\ell(\varphi) = A_\ell \cos \ell\varphi + B_\ell \sin \ell\varphi$

$G_\ell(\varphi) = G_\ell(\varphi + 2\pi)$:

$A_\ell \cos \ell\varphi + B_\ell \sin \ell\varphi = A_\ell \cos(\ell\varphi + 2\pi\ell) + B_\ell \sin(\ell\varphi + 2\pi\ell)$

$\ell \in \mathbb{Z} \underset{\ell > 0}{\Rightarrow} \ell \in \mathbb{N}$

$G_\ell(\varphi) = A_\ell \cos \ell\varphi + B_\ell \sin \ell\varphi, \; \ell \in \mathbb{N}$

2.1 $-\frac{r^2 F''(r)}{F(r)} - \frac{r F'(r)}{F(r)} = h$

$$r^2 F''(r) + r F'(r) + h F(r) = 0$$

2.1 $h = \ell^2$ ni treba

2.2 $h = 0$:

$$r^2 F''(r) + r F'(r) = 0$$

$$\lambda(\lambda-1) + \lambda = 0$$

$$\lambda^2 - \lambda + \lambda = 0$$
$$\lambda^2 = 0$$
$$\lambda_{1,2} = 0$$

$$\boxed{F_0(r) = C_0 + D_0 \log r}$$

2.3 $h = -\ell^2$ $(\ell > 0)$

$$\lambda \cdot (\lambda - 1) + \lambda - \ell^2 = 0$$

$$\lambda^2 = \ell^2$$
$$\lambda_{1,2} = \pm \ell$$

$$F_\ell(r) = C_\ell r^\ell + D_\ell \cdot r^{-\ell}$$

3.) $u(r, \varphi) = F_0(r) G_0(\varphi) + \sum_{\ell=1}^{\infty} F_\ell(r) G_\ell(\varphi) =$

$$= (C_0 + D_0 \log r) \cdot A_0 + \sum_{\ell=1}^{\infty} (C_\ell r^\ell + D_\ell r^{-\ell})(A_\ell \cos \ell\varphi + B_\ell \sin \ell\varphi)$$

$u(1, \varphi) = \cos^2 \varphi$:

$$\underset{\frac{1+\cos 2\varphi}{2}}{\overset{}{\cos^2 \varphi}} = A_0 \cdot C_0 + \sum_{\ell=1}^{\infty} (C_\ell + D_\ell)(A_\ell \cos \ell\varphi + B_\ell \sin \ell\varphi)$$

$\boxed{A_0 C_0 = \frac{1}{2}}$ $\boxed{\frac{1}{2} = A_2 (C_2 + D_2)}$ $\boxed{\ell \neq 2 : A_\ell (C_\ell + D_\ell) = 0}$

$\boxed{(C_\ell + D_\ell) B_\ell = 0}$

$u(3,\varphi)=\cos 2\varphi$:

$$\cos 2\varphi = A_0C_0 + A_0D_0\log 3 + \sum_{\ell=1}^{\infty}(C_\ell 3^\ell + D_\ell\cdot 3^{-\ell})\cdot(A_\ell\cos\ell\varphi + B_\ell\sin\ell\varphi)$$

$\boxed{A_0\cdot C_0 + A_0D_0\log 3 = 0}$ $\quad 1=(C_2\cdot 3^2 + D_2\cdot 3^{-2})\cdot A_2$

$\ell\neq 2: (C_\ell\cdot 3^\ell + D_\ell\cdot 3^{-\ell})\cdot A_\ell = 0$

$B_\ell\cdot(C_\ell\cdot 3^\ell + D_\ell\cdot 3^{-\ell}) = 0$

$$A_0D_0 = \frac{-A_0C_0}{\log 3} = -\frac{1}{2\cdot\log 3}$$

$\frac{1}{2} = A_2C_2 + A_2D_2$ & $1 = 9A_2C_2 + \frac{1}{9}A_2D_2$

$\Rightarrow A_2D_2 = \frac{1}{2} - A_2C_2$

$\Rightarrow 1 = 9A_2C_2 + \frac{1}{9}(\frac{1}{2} - A_2C_2)$

$\frac{17}{18} = \frac{80A_2C_2}{9}$

$\boxed{A_2C_2 = \frac{17}{160}}$ $\quad A_2D_2 = \frac{1}{2} - \frac{17}{160} = \boxed{\frac{63}{160} = A_2D_2}$

$\ell\neq 2$: $A_\ell C_\ell + D_\ell A_\ell = 0$ & $A_\ell\cdot C_\ell\cdot 3^\ell + A_\ell\cdot D_\ell\cdot 3^{-\ell} = 0$

$\cancel{A_\ell}C_\ell = -A_\ell C_\ell \rightarrow A_\ell C_\ell(3^\ell - 3^{-\ell}) = 0$

$\neq 0$ ker $\ell\neq 0$

$A_\ell C_\ell = 0$ & $D_\ell A_\ell = 0$

$B_\ell C_\ell + B_\ell D_\ell = 0$ & $B_\ell C_\ell 3^\ell + B_\ell D_\ell\cdot 3^{-\ell} = 0$

$\vdots$

$B_\ell C_\ell = 0$ $\quad B_\ell D_\ell = 0$

Rešitev

$$\boxed{u(r,\varphi)=\underset{\substack{\uparrow\\ A_0C_0}}{\frac{1}{2}}-\underset{\substack{\uparrow\\ A_0D_0}}{\frac{1}{2\cdot\log 3}}\log r+\underset{\substack{\uparrow\\ A_2C_2}}{\frac{17}{760}}r^{2}\cdot\cos 2\varphi+\underset{\substack{\uparrow\\ A_2D_2}}{\frac{63}{760}}r^{-2}\sin 2\varphi}$$

II $\Delta u=0 \quad r<1$

$u(1,\varphi)=4\cdot\sin^3\varphi$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}=0$$

$$F''(r)G(\varphi)+\frac{1}{r}F'(r)G(\varphi)+\frac{1}{r^2}F(r)G''(\varphi)=0 \Big/ :(F(r)G(\varphi)/r^2)$$

$$\frac{G''(\varphi)}{G(\varphi)}=-\frac{r^2F''(r)}{F(r)}-\frac{rF'(r)}{F(r)}=k$$

$$G(\varphi)=G(\varphi+2\pi)$$

1.1 $k=l^2 \quad (l>0)$

$G_l(\varphi)=A_l e^{l\varphi}+B_l\cdot e^{l\varphi}$; periodično $\Rightarrow G_l(\varphi)=0$

1.2 $k=0 \qquad G_0(\varphi)=A_0+B_0\varphi$

$G(\varphi)=G(\varphi+2\pi) \Rightarrow A_0+B_0\varphi=A_0+B_0(\varphi+2\pi)$

$2\pi B_0=0 \quad B_0=0$

$G_0(\varphi)=A_0$

1.3 $k=-l^2 \ (l>0)$

$\lambda_{1,2}=\pm il$

$$G_l(\varphi)=A_l\cos l\varphi+B_l\sin l\varphi$$

periodičnost $l\in\mathbb{N}$

Aleta

$$u(r,\varphi)=F_0(r)G_0(\varphi)+\sum_{l=1}^{\infty}F_l(r)G_l(\varphi)$$

2.) $r^2\cdot F''+rF'+kF=0$

2.1. //

2.2. $k=0$ $\quad z_{1,2}=0$ $\quad F_0(r)=C_0+D_0\log r$

2.3. $k=-l^2$ $\quad z_{1,2}=\pm l$ $\quad \boxed{F_l(r)=C_l r^l+D_l r^{-l}}$

$$u(r,\varphi)=(C_0+D_0\log r)\cdot A_0+\sum_{l=1}^{\infty}(C_l r^l+D_l r^{-l})(A_l\cos l\varphi+B_l\sin l\varphi)$$

$u(1,\varphi)=4\cdot\sin^3\varphi$:

ker želimo definiranost za $r<1$ (tudi za $r=0$),

mora veljati

A_0 ~~$A_0=0$~~ & $D_l=0$

tako dobimo:

$$u(r,\varphi)=A_0C_0+\sum_{l=1}^{\infty}C_l\cdot r^l(A_l\cos l\varphi+B_l\sin l\varphi)$$

$A\cdot C=E$

$B\cdot C=H$

$$u(r,\varphi)=E_0+\sum_{l=1}^{\infty}r^l(E_l\cos l\varphi+H_l\sin l\varphi)$$

$$4\cdot\sin^3\varphi=E_0+\sum_{l=1}^{\infty}(E_l\cos l\varphi+H_l\sin l\varphi)$$

$\quad ||$

$3\sin\varphi-\sin 3\varphi$

$E_0=0$

$E_l=0$ | $3=H_1$ | $H_3=-1$

$$u(r,\varphi)=3r\cdot\sin\varphi-r^3\cos 3\varphi$$

6.5.2009

MOLIA

Poissonova enačba

- $\Delta u = f$ v prostoru

$u(r, \varphi, \vartheta)$

- $\Delta u = r + 1$ za $r < 1$ in za katere velja
$u(1, \varphi, \vartheta) = 0$

Poiščimo tiste rešitve, ki so odvisne samo od r.

D.N. V enačbo $u_{xx} + u_{yy} + u_{zz}$ uvedite substitucije
$x = r\cos\varphi\cos\vartheta$
$y = r\sin\varphi\cos\vartheta$
$z = r\sin\vartheta$

Uvedite tudi polarne koordinate.

$$\boxed{\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin^2\vartheta}u_{\varphi\varphi} + \frac{1}{r^2}u_{\vartheta\vartheta} + \frac{1}{r\,tg\,\vartheta}u_{\vartheta}}$$

iščemo (r, φ, ϑ) odvisen samo od (r)
$u(r, \varphi, \vartheta) = u(r)$

Enačba se nam poenostavi na

$u_{rr} + \frac{2}{r}u_r = r + 1$ To pa je pravi odvod, zato pišemo:

$u'' + \frac{2}{r}u' = r + 1$

$u' = v$ (znižamo stopnjo)

$v' + \frac{2}{r}v = r + 1$

- homogeni del (vedno tipa ločljivih spremenljivk)

$v' + \frac{2}{r}v = 0$

$\frac{dv}{v} = -\frac{2}{r}dr$

$\log v = -2\log r + \log C$

$$\boxed{v = \frac{C}{r^2}}$$

$$v = \frac{C(r)}{r^2}$$

$$v' = \frac{C'(r)r^2 - C(r)r\,2}{r^4}$$

$$\underbrace{\frac{C'(r)}{r^2} - \cancel{\frac{2C(r)}{r^3}}}_{r'} + \cancel{\frac{2C(r)}{r^3}} = r + 1 \quad / r^2$$

$$C'(r) = r^3 + r^2 \quad / \int$$

$$C(r) = \frac{r^4}{4} + \frac{r^3}{3} + D$$

$$v = \frac{C(r)}{r^2} = \frac{r^2}{4} + \frac{r}{3} + \frac{D}{r^2}$$

$v = u'$!:

$$u = \int\left(\frac{r^2}{4} + \frac{r}{3} + \frac{D}{r^2}\right)dr = \underline{\underline{\frac{r^3}{12} + \frac{r^2}{6} - \frac{D}{r} + E}}$$

POGOJ: $u(1, \varphi, \vartheta) = 0$
$r < 1$
ker iščemo rešitve, definirane za vse $r < 1$ (torej tudi $r = 0$), velja da D/r odpade.
Sledi $D = 0$.

$$u(1) = 0 \rightarrow \frac{1}{12} + \frac{1}{6} + E = 0 \mapsto E = -1/4$$

Iz tega dobimo rešitev:

$$\boxed{u = \frac{r^3}{12} + \frac{r^2}{6} - \frac{1}{4}}$$

$\mathcal{L}(u(x,t)) = U(x,s)$

$\mathcal{L}(u_t(x,t)) = sU(x,s) - u(x,\emptyset)$

• $\mathcal{L}(u_x(x,t)) =$

$= \int_0^\infty u(x,t)\cdot e^{-st}\,dt$

$= \int_0^\infty \frac{\partial(u(x,t)e^{-st})}{\partial x}\,dt =$ menjamo vrstni red

$= \frac{\partial}{\partial x}\underbrace{\int_0^\infty u(x,t)\,e^{-st}\,dt}_{U} = U_x$

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - F(0)$
c	$\frac{c}{s}$

• $u_x + 2xu_t = 2x$ za $t > 0$

pri pogoju $u(x,\emptyset) = 1$, $u(\emptyset,t) = 1$

$U_x(x,s) + 2x(sU(x,s) - u(x,0)) = \frac{2x}{s}$ ← $2x$ je konstanta

$U_x(x,s) + 2xsU(x,s) + 2x = \frac{2x}{s}$

ker imamo parcialne odvode samo po eni spremenljivki, si lahko za trenutek mislimo, da je U odvisen le od x. To pa bi bil potem pravi odvod.

$U' + 2xsU = 2x + \frac{2x}{s}$

• $U' + 2xsU = 0$

$\frac{dU}{U} = -2xs\,dx \quad /\int$

$\log U = -x^2 s + \log C$

$U = C\cdot e^{-x^2 s}$

• $U = C(x)e^{-x^2 s}$

$U' = C'(x)e^{-x^2 s} - C(x)e^{-x^2 s}\cdot 2xs$

$C'(x)e^{-x^2 s} - \cancel{C(x)e^{-x^2 s}2xs} + \cancel{2xsC(x)e^{-x^2 s}} = 2x + \frac{2x}{s}$

$C'(x)e^{-x^2 s} = 2x + \frac{2x}{s} \quad / e^{x^2 s}$

$C'(x) = 2x\left(1+\frac{1}{s}\right)e^{x^2 s} \quad / \int$

$x^2 s = v \quad ; \quad 2xs\,dx = dv$

$C(x) = \int\left(1+\frac{1}{s}\right)\cdot\frac{e^v dv}{s} = \left(\frac{1}{s}+\frac{1}{s^2}\right)e^v + D =$

$= \left(\frac{1}{s}+\frac{1}{s^2}\right)e^{x^2 s} + D$

ker U v resnici odvisen tudi od s:

$U = C(x)e^{-x^2 s} = \frac{1}{s} + \frac{1}{s^2} + De^{-x^2 s}$

$U(x,s) = \frac{1}{s} + \frac{1}{s^2} + D(s)e^{-x^2 s}$

Pogoj $u(0,t) = 1$ transformiramo:

$U(0,s) = \frac{1}{s}$. Torej

$\frac{1}{s} = \frac{1}{s} + \frac{1}{s^2} + D(s)e^{-0^2 s} \longmapsto D(s) = -\frac{1}{s^2}$

$U(x,s) = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-x^2 s}}{s^2}$

$f(t)$	$F(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

$\mathcal{L}^{-1}\left(\frac{e^{-x^2 s}}{s^2}\right) = u_{x^2}(t)\cdot(t-x^2)$

$a = x^2$
$F(s) = 1/s^2 \rightarrow f(t) = t$

$$u(x,t) = 1+t - u_{x^2}(t)(t-x^2)$$

$$u(x,t) = \begin{cases} 1+t & ; t < x^2 \\ 1+t - \cancel{4}t + x^2 & ; t > x^2 \end{cases}$$

VARIACIJSKI RAČUN

Funkcional imenujemo preslikavo, ki funkcijam priredi števila.

Oglejmo si oblike:

$$\boxed{F(y) = \int_a^b f(x,y,y')\,dx}$$

Primer:

$$F(y) = \int_0^\pi y y'\,dx$$

$$F(x^2) = \int_0^\pi x^2\, 2x\,dx = 2\frac{x^4}{4}\Big|_0^\pi = \frac{\pi^4}{2}$$

$$F(\sin x) = \int_0^\pi \sin x \cos x\,dx = \int_0^\pi \frac{\sin 2x}{2}\,dx$$ dvojni kot

$$= \frac{-\cos 2x}{4}\Big|_0^\pi = -\frac{1}{4} + \frac{1}{4} = \emptyset$$

Zanimale nas bodo ekstremale funkcionalov.
Eulerjev pogoj za nastop ekstremale v funkcionalu

$F(y) = \int_a^b f(x,y,y')\,dx$ je $\boxed{\dfrac{\partial f}{\partial y} - \dfrac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = 0}$

Posebni primeri:

- $f = f(x, y')$: $\boxed{\dfrac{\partial f}{\partial y'} = C}$
- $f = f(y, y')$: $\boxed{f - y' \dfrac{\partial f}{\partial y'} = C}$
- $f = f(x, y)$: $\boxed{\dfrac{\partial f}{\partial y} = \varnothing}$

- $F(y) = \int_a^b \dfrac{\sqrt{1 + y'^2}}{x}\, dx$

$$\frac{\partial f}{\partial y} = \varnothing$$

$$\frac{\partial f}{\partial y'} = \frac{2y'}{2x\sqrt{1+y'^2}} \longmapsto \frac{2y'}{2x\sqrt{1+y'^2}} = C \quad /^2$$

$$y'^2 = C^2 x^2 (1 + y'^2)$$
$$y'^2 = c^2 x^2 + c^2 x^2 y'^2$$
$$y'^2 (1 - c^2 x^2) = c^2 x^2$$
$$y'^2 = \frac{c^2 x^2}{1 - c^2 x^2} \qquad y' = \sqrt{\frac{c^2 x^2}{1 - c^2 x^2}} = \frac{cx}{\sqrt{1 - c^2 x^2}} \quad / \int$$

$$1 - c^2 x^2 = t$$
$$-2c^2 x\, dx = dt$$

$$y = \int \frac{dt}{-2\sqrt{t}\, c} = \frac{\sqrt{t}}{-2c \cdot \frac{1}{2}} + D = \frac{\sqrt{1 - c^2 x^2}}{-c} + D$$

$$-Cy + CD = \sqrt{1 - c^2 x^2} \quad /^2$$

$$c^2 (D - y)^2 = 1 - c^2 x^2$$
$$c^2 x^2 + c^2 (y - D)^2 = 1 \quad / : c^2$$
$$x^2 + (y - D)^2 = 1 / c^2$$

premaknjena krožnica.
središče na y osi.

- $F(y) = \int_a^b (y^2 + 2xyy')dx$

$\frac{\partial f}{\partial y} = 2y + 2xy'$

$\frac{\partial f}{\partial y'} = 2xy$

produkt

$\frac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = (2xy)' \overset{\downarrow}{=} 2y + 2xy'$

pogoj: $(2y + 2xy') - (2y + 2xy') = 0$

$0 = 0$

Vsaka funkcija je ekstremala tega funkcionala.

- $F(y) = \int_0^1 (y^2 + xy - 2y^2y')\,dx$

$y(0) = 1 \qquad y(1) = 2$

$\frac{\partial f}{\partial y} = \boxed{2y + x - 4yy'}$

$\frac{\partial f}{\partial y'} = -2y^2$

$\frac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = (-2y^2)' = \boxed{-4yy'}$

pogoj: $(2y + x - 4yy') - (-4yy') = 0$

$2y + x = 0$

$y = -x/2$

$y(0) = 1 : \quad 1 = 0 \quad X$

taka ekstremala ne obstaja

- $F(y) = \int_{-1}^{1} \left(x^2 y'^2 + 12 y^2\right) dx$

$y(-1) = -1 \qquad y(1) = 1$

$\dfrac{\partial f}{\partial y} = 24y$

$\dfrac{\partial f}{\partial y'} = 2x^2 y'$

$\dfrac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = 4xy' + 2x^2 y''$

$24y - (4xy' + 2x^2y'') = 0 \qquad /:(-2)$

$x^2y'' + 2xy - 12y = 0$

nastavek $y = x^\lambda$

$\lambda(\lambda - 1) + 2\lambda - 12 = 0$

$\lambda^2 + \lambda - 12 = 0$

$(\lambda + 4)(\lambda - 3) = 0 \qquad \lambda_1 = -4, \quad \lambda_2 = 3$

$\underline{y = C_1 x^{-4} + C_2 x^3}$ če ni pogojev, je to konec

pogoji: • $y(-1) = -1$:

$-1 = C_1 - C_2$

• $y(1) = 1$:

$1 = C_1 + C_2$

$+ \rightarrow 0 = 2C_1 \rightarrow C_1 = 0$

$C_2 = 1$

$$\boxed{y = x^3}$$

- $F(y) = \int_0^1 \frac{\sqrt{1+y'^2}}{y}\, dx \qquad y(0)=1, \quad y(1)=2$

$$\frac{\partial F}{\partial y} = -\frac{\sqrt{1+y'^2}}{y^2}$$

$$\frac{\partial f}{\partial y'} = \frac{2y'}{2y\sqrt{1+y'^2}} = \frac{y'}{y\sqrt{1+y'^2}}$$

$$\frac{d\left[\frac{\partial f}{\partial y'}\right]}{dx} = \frac{y''y\sqrt{1+y'^2} - y'\left[y'\sqrt{1+y'^2} + y\cdot\frac{2y'y''}{2\sqrt{1+y'^2}}\right]}{y^2(1+y'^2)}$$

pogoj:

$$-\frac{\sqrt{1+y'^2}}{y^2} - \left[\frac{y''y\sqrt{1+y'^2} - y'^2\sqrt{1+y'^2} - \frac{yy'^2y''}{\sqrt{1+y'^2}}}{y^2(1+y'^2)}\right] = 0 \quad /*$$

$*:\ (-y^2(1+y'^2)^{3/2})$

$$(1+y'^2)^2 + yy''(1+y'^2) - y'^2(1+y'^2) - yy'^2y'' = 0$$

$$1+2y'^2+y'^4+yy''+yy''y'^2 - y'^2 - y'^4 - yy'^2y'' = 0$$

$$1+y'^2+yy'' = 0$$

$$y'^2+yy'' = -1$$

$$y'y'+yy'' = -1 = (yy')' \ ! \quad /\int$$

$$yy' = -x + c$$

$$y\,dy = (-x+c)\,dx \quad /\int$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + cx + D$$

14.5.2009

$$\frac{y^2}{2} + \frac{x^2}{2} - cx = D$$

upoštevamo konstante: $y(0)=1$: $\frac{1}{2} + 0 - 0 = D \rightarrow D = \frac{1}{2}$

$y(1)=2$: $2 + \frac{1}{2} - c = D \rightarrow c = 2$

Torej: $\frac{y^2}{2} + \frac{x^2}{2} - 2x = \frac{1}{2} \quad / \cdot 2$

$$x^2 - 4x + y^2 = 1$$
$$(x-2)^2 - 4 + y^2 = 1$$
$$(x-2)^2 + y^2 = 5$$

ta krožnica je ekstremala

Lahko bi rešili tudi z $F - y' \frac{\partial F}{\partial y'} = C$.
To pa zato, ker nimamo x v F.

$$\frac{\sqrt{1+y'^2}}{y} - y' \cdot \frac{y'}{y\sqrt{1+y'^2}} = C \quad / \cdot y\sqrt{1+y'^2}$$

In tako dalje.

- Poišči ekstremalo funkcionala $\int_0^\pi y'^2 \, dx$, pogoji $y(0)=0$, $y(\pi)=0$, $\int_0^\pi y^2 dx = 1$

$$H(y) = \int_0^\pi \Big(\underbrace{y'^2}_{\text{star funkcional}} + \underbrace{\lambda}_{\text{lambda}} \cdot \underbrace{y^2}_{\text{vez}} \Big) dx$$

$$\frac{\partial f}{\partial y} - \frac{d(\partial f/\partial y')}{dx} = 0$$

$$\frac{\partial f}{\partial y} = 2\lambda y \qquad \frac{\partial f}{\partial y'} = 2y' \qquad \frac{d(\partial f/\partial y')}{dx} = 2y''$$

$$2\lambda y - 2y'' = 0 \quad /:-2$$
$$y'' - \lambda y = 0$$

nastavek: e^{kx}

$$k^2 - \lambda = 0$$
$$k_{1/2} = \pm\sqrt{\lambda}$$

ločimo tri primere:

a) $\lambda > 0$: $\lambda = \ell^2 \ (\ell \neq \emptyset)$

$$k_{1/2} = +/- \ell$$
$$y = C_1 e^{\ell x} + C_2 e^{-\ell x}$$

poglejmo pogoje • $y(0) = 0$:
$$0 = C_1 + C_2 \rightarrow C_2 = -C_1$$

• $y(\pi) = 0$:
$$0 = C_1 e^{\ell\pi} + C_2 e^{-\ell\pi}$$
$$0 = C_1(e^{\ell\pi} - e^{-\ell\pi})$$

① $C_1 = 0 \rightarrow C_2 = 0 \rightarrow y = 0$
to pa ne zadosti pogoju $\int_0^\pi y^2 dx = 1$, Torej to ni rešitev

② $e^{\ell\pi} - e^{-\ell\pi} = 0 \rightarrow \ell\pi = -\ell\pi$
$\ell = 0$ to pa ne gre
(dve realni rešitvi)

b) $\lambda = 0$

$k_{1/2} = 0$

$y = C_1 + C_2 x$ spet gledamo pogoje

① $y(0) = 0$: $0 = C_1$

② $y(\pi) = 0$: $0 = C_1 + C_2\pi \to C_2 = 0 \to y = 0$. To pa spet ne zadosti zadnjemu pogoju.

c) $\lambda < 0$, $\lambda = -\ell^2$ $(\ell \neq 0)$

$k_{1/2} = \pm\sqrt{-\ell^2} = \pm i\ell$

$y = C_1 \cos \ell x + C_2 \sin \ell x$

① $y(0) = 0$: $C_1 = 0$

② $y(\pi) = 0$: $C_1 \cos \ell\pi + C_2 \sin \ell\pi = 0 \to C_2 \sin \ell\pi = 0$

- $C_2 = 0 \to y = 0$. To rešitev spet ne zadosti tretjemu pogoju
  $\uparrow$
  $C_1 = 0$

- $\sin \ell\pi = 0 \to \ell \in \mathbb{Z} \setminus \{\emptyset\}$
  $\uparrow$
  $\ell \neq \emptyset$

$y = C_2 \cdot \sin \ell x$, $\ell \in \mathbb{Z} \setminus \{\emptyset\}$ preverimo še, če zadostimo zadnjemu pogoju $\int_0^\pi y^2 dx = 1$

$$\int_0^\pi C_2^2 \sin^2 \ell x \, dx = 1$$

$$C_2^2 \int_0^\pi \frac{1 - \cos 2\ell x}{2} dx = 1$$ dvojni koti

$$C_2^2 \left(\frac{1}{2} x - \frac{\sin 2\ell x}{4\ell} \right) \Big|_0^\pi = 1$$

$$C_2^2 \left(\frac{1}{2}\pi - 0 - 0 + 0 \right) = 1$$

$$C_2^2 = \frac{2}{\pi} \to C_2 = \sqrt{2/\pi}$$

Rešitev: $y = \sqrt{\frac{2}{\pi}} \sin lx \qquad l \in \mathbb{Z} \setminus \{0\}$

isto: $y = \pm\sqrt{\frac{2}{\pi}} \sin lx, \quad l \in \mathbb{N}$

Kombinatorika in verjetnost

- **Permutacije:**

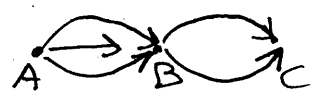
n elementov postavljamo na n mest, vrstni red je važen

$\underbrace{\ }_{n}\ \underbrace{\ }_{(n-1)}\ \underbrace{\ }_{(n-2)}\ \underbrace{\ }_{(n-3)} \dots \underbrace{\ }_{1}$

$n \cdot (n-1) \cdot (n-2) \cdot \ldots 1 = n!$

$\boxed{P_n = n!}$

Osnovni izrek kombinatorike:



m mest n cest

$m \times n$ načinov

- **Permutacije s ponavljanjem:**

enako, kot permutacije, le da so nekateri elementi med njimi lahko enaki

(k_1 - enaki, k_2 - enaki)

$$P_n^{k_1, k_2, \ldots} = \frac{n!}{k_1!\, k_2! \cdot \ldots}$$

- **Variacije:** (n elementov postavljamo na r mest, kjer je $r < n$), vrstni red pomemben

$\underbrace{\ }_{1}^{n}\ \underbrace{\ }_{2}^{(n-1)}\ \underbrace{\ }_{3}^{(n-2)} \dots \underbrace{\ }_{r}^{(n-r+1)}$

$$V_n^r = n \cdot (n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

- **variacije s ponavljanjem:** n elementov postavljamo na r mest, kjer se elementi lahko ponavljajo. Vrstni red je važen.

$$\begin{array}{cccccc} n & n & n & n & & n \\ \sqcup & \sqcup & \sqcup & \sqcup & \cdots & \sqcup \\ 1 & 2 & 3 & 4 & & r \end{array} \qquad V_n^r = n^r$$

- **Kombinacije:** izmed n elementov izberemo r elementov – vrstni red ni pomem-ben,

$$C_n^r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Kombinacije s ponavljanjem: izmed n elementov jih izberemo r, kjer se elementi lahko večkrat ponovijo. Vrstni red ni pomemben.

$$^{(p)}C_n^r = \binom{n+r-1}{r}$$

Verjetnost

$\{H_i\}$ je popoln sistem dogodkov, če velja

(i) $P(H_i \cap H_j) = \emptyset$ nezdružljiva dogodka za vse i, j, $i \neq j$

(ii) $P(H_1 \cup H_2 \cup H_3 \cup \ldots) = 1$

Če naše dogajanje opišemo s popolnim sistemom enakoverjetnih dogodkov, velja, da je

$P(H_i) = P(H_j)$ za vse i, j

verjetnost $\boxed{P(A) = \dfrac{\text{št. ugodnih elementarnih dogodkov}}{\text{št. vseh elementarnih dogodkov}}}$

• Vržemo dve kocki. Zanima nas verjetnost naslednjih dogodkov:

a) enako število na obeh kockah
b) vsota je enaka osem
c) produkt je enak osem
d) vsota je večja od produkta

a) $P(A) = \frac{6}{\underbrace{6\cdot 6}_{\text{možni meti}}} = \frac{1}{6}$ ← na obeh kockah enako

b) $P(B) = \frac{5}{6\cdot 6} = \frac{5}{36}$

$2+6$
$3+5$
$4+4$
$6+2$
$5+3$
$4+4$

6 dogodkov $-1\ (4\times 4)$

c) $P(C) = \frac{2}{36}$ ← 2×4 ali 4×2

d) $P(D) = \frac{11}{36}$ → $1\cdot 1$, $1\cdot 2$, $\vdots$ + obrnemo − 1×1 ne smemo dvakrat šteti

	1	2	3	4	5	6
1	x	x	x	x	x	x
2	x					
3	x					
4	x					
5	x					
6	x					

• Šest listkov: H,R,U,Š,K,A. Pomešamo. Kakšna je verjetnost, da bomo dobili spet isto besedo?

$$P(A) = \frac{1}{6!} = \frac{1}{720}$$

• Listki: B, A, N, A, N, A. Kakšna je zdaj verjetnost

$$P(A) = \frac{1}{\frac{6!}{3!\,2!}} = \frac{3!\,2!}{6!} = \frac{1}{60}$$

• 32 kart
3 karte

A) 3 karte as
B) vsaj 1 as
C) 3 karte iste barve

A) $P(A) = \dfrac{\binom{4}{3}}{\binom{32}{3}}$ ← izmed štirih asov potegnemo 3. ← izmed 32 elementov izberemo 3 elemente

$$= \frac{\frac{4!}{3!\,1!}}{\frac{32!}{3!\,29!}} = \boxed{\frac{1}{1240}}$$

B) $P(B) = \dfrac{\binom{4}{1}\binom{28}{2} + \binom{4}{2}\binom{28}{1} + \binom{4}{3}}{\binom{32}{3}}$

↑ točno en as ↑ točno 2 asa ↑ točno en as

možno je to narediti tudi z negacijo dogodka (nimamo nobenega asa)

$$P(B) = 1 - P(\text{noben as})$$

$$P(\text{noben as}) = \frac{\binom{28}{3}}{\binom{32}{3}} =$$

c) $P(C) = \dfrac{4\binom{8}{3}}{\binom{32}{3}}$ osem kart vsake barve

katera barva

- 8 kart = 4 rdeče + 4 črne

postavimo v vrsto

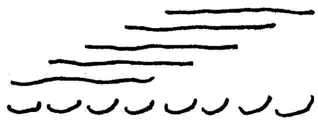
a) rdeče karte so skupaj in črne skupaj

$P(A) = \dfrac{2 \cdot 4! \cdot 4!}{8!}$ mešanje črnih in mešanje rdečih

črne, rd.
rd, č.

vse možne razporeditve

b) rdeče karte so skupaj



$P(B) = \dfrac{5 \cdot 4! \cdot 4!}{8!}$ mešanje

pet položajev

c) barve se prepletajo

$P(C) = \dfrac{2 \cdot 4! 4!}{8}$

možnosti, s čim začnemo

d) niz se začne in konča z rdečo barvo

$P(D) = \dfrac{\binom{4}{2} \cdot 2! \cdot 6!}{8!}$ ostale

izbira robnih

katera je prva in katera zadnja

podobno razmišljanje: $4\ \overbrace{\smile\smile\smile\smile\smile\smile}^{6!}\ 3$

• Na slepo izberemo naravno število

- a) kvadrat se konča z 1
- b) četrta potenca se konča z 1
- c) tretja potenca se konča z 11

a)

št	kvadrat
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	255

enice kvadrata se ponavljajo odvisno s čim se končajo

Na enice kvadrata vplivajo le enice orig. števila.

Dokaz: $(10a+b)^2 = 100a^2 + 20ab + b^2$

enice

ne vpliva na enice, ker deljivo z 10.

enice št.	enice kv.
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
0	0

Torej: $\boxed{P(A) = \frac{2}{10} = \frac{1}{5}}$

deset možnosti za enice

b) Opazimo (lahko bi delali enako), da je četrta potenca kvadrat kvadrata. Torej po prejšnjem velja, da bo četrta potenca imela enice enake ena, ko bo kvadrat imel enice [redacted] 1 ali 9.

1, 3, 7, 9 → $\boxed{P(B) = \frac{4}{10} = \frac{2}{5}}$

c) najprej poglejmo zgolj enice $(10a+b)^3 =$
$= \underbrace{1000a^3 + 300a^2b + 30ab^2}_{\text{deljivo z deset}} + b^3$

enice št..	enice tretje potence
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9
0	0

→ edini primer je, da velja $b=1$. S tem so enice trihtane.

$b=1$: $(10a+b^3)=\underbrace{1000a^3+300a^2}_{\text{deljivo s 100, ne vpliva na zadnji dve števki}}+30a+1$

Želimo: $30a+1$ se konča z 11
sledi $30a$ se mora končati z 10
sledi $3a$ se mora končati z 1

$a=10c+d \rightarrow 3a=\underbrace{30c}_{\text{ne vpliva na enice}}+3d$

enice a	enice 3a
1	3
2	6
3	9
4	2
5	5
6	8
7	1
8	4
9	7
0	0

→ edina možnost: enice a znaša 7.

Torej: Naše število se mora končati z 71.

$$\boxed{P(C)=\frac{1}{100}}$$

20.5.2009

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Če $P(A \cap B) = \emptyset$ (nezdružljiva), velja $P(A \cup B) = P(A) + P(B)$
- $P(A \cap B) = P(A) \cdot P(B/A)$ Verjetnost, da se zgodi dogodek B, če se zgodi dogodek A.

Če $P(B/A) = P(B)$ (B neodvisen od dogodka A), je verjetnost, da se zgodita oba $P(A \cap B) = P(A)P(B)$

- Škatla: dve beli, tri črne. Dva igralca izmenično vlečeta, ne vračata, zmaga tisti, ki prvi potegne belo. Kakšna je verjetnost, da zmaga tisti, ki igro začne.

prvi potegne belo

A = Bela ∪ Črna Črna B ∪ ČČČČB → nimamo dovolj črnih, ta možnost odpade

drugi potegne črno

takoj potegne belo; potegne črno

$$P(A) = P(B \cup ČČB) = \underset{\text{nezdružljiva}}{P(B) + P(ČČB)} = \frac{\overset{\text{ugodni}}{2}}{\underset{\text{vsi}}{5}} + \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} = \frac{3}{5}$$

nezdružljiva dogodka, kar pomeni, da je verjetnost vsota verjetnosti dogodkov

Enako kot prej, le da kroglice vračata.

A = Bela ∪ Črna Črna Bela ∪ Črna Črna Črna Črna Bela ∪
∪ ČČČČČČB .....

$$P(A) = P(B \cup ČČB \cup ČČČČB \cup \ldots) \underset{\text{nezdružljivi}}{=} P(B) + P(ČČB) + P(ČČČČB) + \ldots$$

geom vrsta

$$= \frac{2}{5} + \underbrace{\overset{\text{prvi}}{\frac{3}{5}} \cdot \overset{\text{drugi}}{\frac{3}{5}}}\cdot\frac{2}{5} + \underbrace{\frac{3}{5}\cdot\frac{3}{5}}\cdot\underbrace{\frac{3}{5}\cdot\frac{3}{5}}\cdot\frac{2}{5} + \ldots =$$

$q = 9/25$

$$= \frac{\frac{2}{5}}{1 - \frac{9}{25}} = \frac{5}{8} \qquad \Sigma = \frac{a}{1-q}$$

Zmaga tisti, ki igre ne začne (brez vračanja): ČB ∪ ČČČB

• 52 kart, izberemo dve. Eno obrnemo in vidimo, da je dama. Ti dve karti med seboj premešamo, spet eno pogledamo. Verjetnost, da se pokaže AS.

← druga karta je as.

$$\frac{4}{51} \cdot \frac{1}{2} = \frac{2}{51}$$

mešanje

za eno že vemo, da je dama

Enako kot prej, le da se prikaže dama.

H_1 - imamo eno damo
H_2 - imamo dve dami

$\{H_i\}$ popoln sistem hipotez : $P(H_i \cap H_j) = \emptyset$; $i \neq j$
$P(H_i \cup H_j) = 1$

$$\boxed{P(A) = \sum_i P(H_i)\, P(A/H_i)}$$ polna verjetnost

$P(H_1) = \frac{48}{51}$ to so ugodne karte, ki so ostale za to hipotezo. da imamo samo eno damo

$P(H_2) = \frac{3}{51}$ da imamo obe dami, imamo še tri v setu.

preverimo $\sum_i P(H_i) = 1$

Kakšna je verjetnost, da bomo v drugo potegnili damo, če vemo, da že imamo eno damo

$P(A/H_1) = 1/2$

Verjetnost, da izmed dveh potegnemo damo, če imamo obe dami?

$P(A/H_2) = 1$

$$P(A) = P(H_1)P(A/H_1) + P(H_2)P(A/H_2) = \frac{48}{51} \cdot \frac{1}{2} + \frac{3}{51} \cdot 1 = \frac{9}{17}$$

• Štiri podjetja dobavljajo trgovini enak izdelek v razmerju 1:2:3:4
verjetnost, da je izdelek z napako je v podjetju
- 1 enaka 0,1
- 2 —"— 0,2
- 3 —"— 0,15
- 4 —"— 0,05

Kupimo izdelek, kakšna je verjetnost, da ima napako?

H_i – hipoteza, da je izdelek iz i-tega podjetja, kjer $i = 1, 2, 3, 4$

$P(H_1) = 1/10 = 1/1+2+3+4$
$P(H_2) = 2/10$
$P(H_3) = 3/10$
$P(H_4) = 4/10$

$P(A/H_1) = 0{,}1 \quad P(A/H_2) = 0{,}2 \quad P(A/H_3) = 0{,}15 \quad P(A/H_4) = 0{,}05$

$$P(A) = \sum_i P(H_i)P(A/H_i) =$$
$$= 0{,}1 \cdot \frac{1}{10} + 0{,}2 \cdot \frac{2}{10} + \frac{3}{10} \cdot 0{,}15 + \frac{4}{10} \cdot 0{,}05 = 0{,}115$$

Kakšna je verjetnost hipoteze pri dogodku

$$\boxed{P(H_i/A) = \frac{P(H_i)P(A/H_i)}{\underbrace{P(A)}_{\sum_j P(H_j)P(A/H_j)}}}$$

BAYESOVA FORMULA

Kakšna je verjetnost, da je izdelek, ki ima napako, iz tretjega podjetja?

$$P(H_3/A) = \frac{P(H_3)\ P(A/H_3)}{P(A)} = \frac{0{,}045}{0{,}115} = 39\%$$

• Dva strelca ustrelita v tarčo. Prvi zadane z verjetnostjo 0,5, drugi pa zadane z verjetnostjo 0,9.

Ob pogledu na tarčo vidimo, da je en strelec zadel. Kakšna je verjetnost, da je zadel prvi strelec?

H_{00} oba nezadaneta
H_{10} prvi ne zadane, drugi ne
H_{01} drugi zadane, prvi ne
H_{11} oba zadaneta

A - en strel zadane

$P(H_{10}/A) = ?$

verjetnost, da prvi zadane, če vemo, da je en zadel.

$P(H_{00}) = 0,5 \cdot 0,1 = 0,05$
$P(H_{10}) = 0,5 \cdot 0,1 = 0,05$
$P(H_{01}) = 0,5 \cdot 0,9 = 0,45$
$P(H_{11}) = 0,5 \cdot 0,9 = 0,45$

$P(A/H_{00}) = 0$ ← verjetnost, da en strel zadane, če vemo, da sta oba zgrešila
$P(A/H_{10}) = 1$
$P(A/H_{01}) = 1$
$P(A/H_{11}) = 0$

$$P(H_{10}/A) = \frac{P(H_{10}) \cdot P(A/H_{10})}{P(H_{00})P(A/H_{00}) + P(H_{10})P(A/H_{10}) + \dots}$$

$$= \frac{0,05 \cdot 1}{0,05 \cdot 0 + 0,05 \cdot 1 + 0,45 \cdot 1 + 0,45 \cdot 0} = \frac{0,05}{0,5} = \frac{1}{10}$$

96

• V škatli imamo štiri kroglice. 5x izvlečemo in vsakič vrnemo. Kakšna je verjetnost, da so v škatli same bele kroglice, če smo vsakič izvlekli belo?

H_0 - $\emptyset$ belih, H_1 - 1-bela, H_2, H_3, H_4,

A - vedno potegnemo belo (5x zapored)

$P(H_4/A) = ?$

vsakič je verjetnost 1/4

$P(H_0) = 1/5$ $\qquad P(A/H_0) = \emptyset$

$P(H_1) = 1/5$ $\qquad P(A/H_1) = 1/4 \cdot 1/4 \cdot 1/4 \cdot 1/4 \cdot 1/4 = 1/4^5$

$P(H_2) = 1/5$ $\qquad P(A/H_2) = 2/4 \cdot 2/4 \cdot 2/4 \cdot 2/4 \cdot 2/4 = 2^5/4^5$

$P(H_3) = 1/5$ $\qquad P(A/H_3) = 3/4 \cdot 3/4 \cdot 3/4 \cdot 3/4 \cdot 3/4 = 3^5/4^5$

$P(H_4) = 1/5$ $\qquad P(A/H_4) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

↑

ker so dogodki enako verjetni

$$P(H_4/A) = \frac{P(H_4)P(A/H_4)}{P(H_0)P(A/H_0) + P(H_1)\,P(A/H_1) + \ldots} =$$

$$= \frac{1/5}{\frac{1}{5}\cdot 0 + \frac{1}{5}\cdot\frac{1}{4^5} + \frac{1}{5}\cdot\frac{2^5}{4^5} + \frac{1}{5}\,\frac{3^5}{4^5} + \frac{1}{5}} = 256/325$$

Diskretne slučajne spremenljivke

X - slučajna spremenljivka

x_k - vrednosti, ki jih X lahko zavzame

p_k - verjetnost, da ta X zavzame ravno neki x_k

$p_k = P(X = x_k)$

$$X : \begin{pmatrix} x_1 & x_2 & x_3 & \ldots \\ p_1 & p_2 & p_3 & \ldots \end{pmatrix}$$

$\sum p_k = 1$

$E(X) = \sum_k x_k p_k$ matematično upanje

NONA

- Strelec ima 4 naboje. Strelja dokler ne porabi nabojev oz. do prvega zadetka. Verjetnost posameznega zadetka je 0,8. Sluč. spr. X je število porabljenih nabojev. Iščemo pu, E(X).

$$X: \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0{,}8 & 0{,}16 & 0{,}032 & 0{,}008 \end{pmatrix} \rightarrow \Sigma = 1$$

toliko je možnosti, koliko nabojev porabi

ne zadane

zadane

lahko zgreši ali zadane

$P(X=1)=0{,}8 \qquad P(X=2)=0{,}2\cdot 0{,}8=0{,}16$

$P(X=3)=0{,}2\cdot 0{,}2\cdot 0{,}8=0{,}032 \qquad P(X=4)=0{,}2\cdot 0{,}2\cdot 0{,}2\cdot 1=0{,}008$

$E(X)=1\cdot 0{,}8+2\cdot 0{,}16+3\cdot 0{,}032+4\cdot 0{,}008=1{,}248$

- proti cilju ustrelimo 4x. Verjetnost posameznega zadetka naj bo 0,8. X naj bo število zadetkov.

$$X: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0{,}0016 & 0{,}0256 & 0{,}1536 & 0{,}4096 & 0{,}4096 \end{pmatrix} \rightarrow \Sigma = 1$$

$P(X=0)=0{,}2\cdot 0{,}2\cdot 0{,}2\cdot 0{,}2 = 0{,}0016$ (štirikrat zgrešimo)

$P(x=1) = 0{,}2\cdot 0{,}2\cdot 0{,}2\cdot 0{,}8\cdot 4 = 0{,}0256$ ← odvisno kdaj zadane, saj so to štirje dogodki

$P(x=2)= 0{,}2\cdot 0{,}2\cdot 0{,}8\cdot 0{,}8\cdot \binom{4}{2} = 0{,}1536$; $\binom{4}{2}=6$

$P(x=3)= 0{,}2\cdot 0{,}8\cdot 0{,}8\cdot 0{,}8\binom{4}{3}=0{,}4096$

$P(x=4)= 0{,}8\cdot 0{,}8\cdot 0{,}8\cdot 0{,}8 = 0{,}4096$

$E(X)= 0\cdot 0{,}0016+1\cdot 0{,}0256+2\cdot 0{,}1536+3\cdot 0{,}4096+4\cdot 0{,}4096 = \underline{3{,}2}$

27.5.2009

- Študent sme delati izpit največ štirikrat. Verjetnost, da izpit opravi, je vsakič 0,8.
 X – število opravljanj.
 a) $E(X) = ?$
 b) kakšna je verjetnost, da kandidat izpit opravi

$$X : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0{,}8 & 0{,}16 & 0{,}032 & 0{,}008 \end{pmatrix}$$

$P(X=1) = 0{,}8$
$P(X=2) = 0{,}2 \cdot 0{,}8 = 0{,}16$
$P(X=3) = 0{,}2 \cdot 0{,}2 \cdot 0{,}8 = 0{,}032$
$P(X=4) = 0{,}2 \cdot 0{,}2 \cdot 0{,}2 \cdot 1 = 0{,}008$

a) $E(X) = 1 \cdot 0{,}8 + 2 \cdot 0{,}16 + 3 \cdot 0{,}032 + 4 \cdot 0{,}008 = 1{,}248$

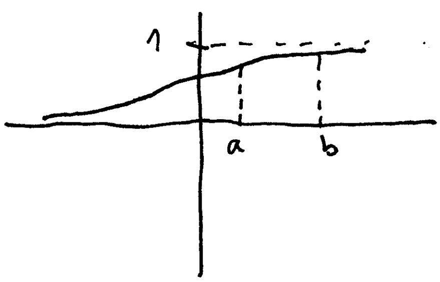
b) $0{,}8 + 0{,}16 + 0{,}032 + 0{,}2 \cdot 0{,}2 \cdot 0{,}2 \cdot 0{,}8$
ker mora opravit

lahko tudi $P(A) = 1 - P(\text{ni opravil})$

$P(A) = 1 - 0{,}2 \cdot 0{,}2 \cdot 0{,}2 \cdot 0{,}2 = 1 - 0{,}0016 = 0{,}9984$

ZVEZNE SLUČAJNE SPREMENLJIVKE

- X - slučajne spr. z vrednostmi v $\mathbb{R}$
- $P(X < x) = F(X)$ porazdelitvena funkcija
- lastnosti:
  - $\lim\limits_{x \to -\infty} F(X) = 0$
  - $\lim\limits_{x \to \infty} F(x) = 1$
  - nepadajoča, torej ne nujno strogo naraščajoča



verjetnost, da je spr. na intervalu je razlika višin

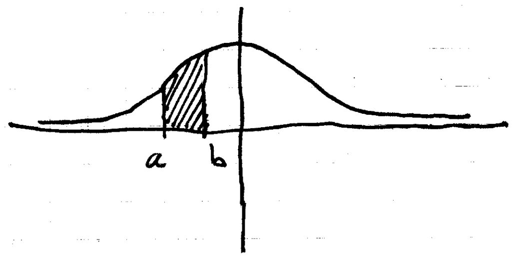
- $P(a < X < b) = F(b) - F(a)$

- $p(x) = F'(x)$ gostota verjetnosti

$$P(X < x) = \int_{-\infty}^{\infty} p(t)\,dt$$

$$\boxed{P(a < X < b) = \int_{a}^{b} p(x)\,dx}$$

lastnosti:
- nenegativna
- $\int_{-\infty}^{\infty} p(x)\,dx = 1$



- $E(X)$ matematično upanje (pričakovana vrednost)

$$\boxed{E(x) = \int_{-\infty}^{\infty} x \cdot p(x)\,dx}$$

- $D(X)$ disperzija   merilo, kako dobra ocena je matematično upanje

$$\boxed{\begin{aligned} D(X) &= E((X - E(X))^2) = \\ &= E(x^2) - E(x)^2 \end{aligned}}$$

$$E(f(X)) = \int_{-\infty}^{\infty} f(x) \cdot p_X(x)\,dx$$

- $\sigma(X)$ standardni odklon

$$\sigma(X) = \sqrt{D(X)}$$

• $F(X) = \begin{cases} 0 & ; x \leq -2 \\ \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{x}{2} & ; -2 < x \leq 2 \\ 1 & ; 2 < X \end{cases}$

a) $P(-1 < X < 1) = ?$

$= F(1) - F(-1) = \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{\pi}\left(-\frac{1}{2}\right)\right) =$

$= \frac{1}{\pi} \frac{\pi}{6} - \frac{1}{\pi}\left(-\frac{\pi}{6}\right) = \frac{1}{3}$

b) $P(1 < X < 4) = ?$

$= F(4) - F(1) = 1 - \left(\frac{1}{2} + \frac{1}{\pi}\frac{\pi}{6}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

c) $P(X > 0) = \underbrace{1 - P(X < 0)}_{\text{negacija}} = 1 - F(0) =$

$= 1 - \left(\frac{1}{2} + \frac{1}{\pi} \arcsin 0\right) = 1 - \frac{1}{2} = 1/2$

def: $F(x) = P(X < x)$

• $p(x) = \begin{cases} x e^{-x^2/2} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$ zanima nas $P(x < 1)$, $E(x)$, $D(x)$

a) $P(x < 1) = \int_{-\infty}^{1} p(x)\, dx = \int_0^1 x \cdot e^{-x^2/2}\, dx \quad \left(\frac{x^2}{2} = t, \ x dx = dt\right)$

$= \int_0^{1/2} e^{-t}\, dt = \left[-e^{-t}\right]_0^{1/2} = -e^{-1/2} + 1 = 1 - \frac{1}{\sqrt{e}}$

(do nič ne integriramo, ker taka funkcija)

b) $E(X) = \int_{-\infty}^{\infty} x \cdot p(x)\, dx = \int_0^{\infty} x^2 e^{-x^2/2}\, dx$

$\frac{x^2}{2} = t \searrow \quad x dx = dt$

$x = \sqrt{2t}$

$$= \int_0^\infty \sqrt{2t}\, e^{-t}\, dt = \sqrt{2} \int_0^\infty t^{1/2} e^{-t}\, dt = \sqrt{2}\, \Gamma\left(\tfrac{1}{2}+1\right) =$$

$$= \sqrt{2}\, \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}\sqrt{\pi} = \sqrt{\pi/2}$$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(x+1) = x\,\Gamma(x)$

c) $D(X) = E(X^2) - E(X)^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx = \int_0^\infty x^3 e^{-x^2/2}\, dx$$

$x^2/2 = t \qquad x\,dx = dt$

$$= \int_0^\infty 2t\, e^{-t}\, dt = 2\, \underbrace{\Gamma(2)}_{1} = 2 \qquad \Gamma(n+1) = n!\,,\ n \in \mathbb{N}$$

$$D(X) = E(X^2) - E(X)^2 = 2 - \pi/2$$

- $p(x) = \dfrac{k}{1+x^2}$

a) določi k, da bo $p(x)$ gostota verjetnosti X

b) $P(-1 < X < 1) = ?$

a) • $p(x) \geq 0 \rightarrow k \geq 0$

• $\displaystyle\int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1$

$$k \cdot \operatorname{arctg} x \Big|_{-\infty}^{\infty} = k \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$k\pi = 1$

$\boxed{k = 1/\pi}$

b) $P(-1 < X < 1) = \int_{-1}^{1} p(x)dx = \int_{-1}^{1} \frac{1}{\pi} \frac{1}{1+x^2} dx =$

$= \frac{1}{\pi} \operatorname{arctg} x \Big|_{-1}^{1} = \frac{1}{\pi}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \boxed{\frac{1}{2}}$

- Znotraj kroga s polmerom 1 slučajno izberemo točko. Verjetnost, da je točka v nekem delu kroga, je sorazmerna ploščini tega ~~dela~~ kroga.

X - oddaljenost točke od središča, zanima nas $E(X)$.

$$F(X) = P(X < x) = \begin{cases} \emptyset & ; x < \emptyset \\ k \cdot \pi x^2 & \emptyset < x < 1 \\ 1 & ; x \geq 1 \end{cases}$$

$\lim_{x \to 1} F(x) = 1$ ZVEZNA NA INTERVALU

$\lim_{x \to 1} k\pi x^2 = 1$

$k\pi = 1 \rightarrow K = 1/\pi$

dobimo

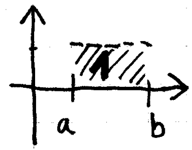
$$F(x) = \begin{cases} \emptyset & ; & x \leq \emptyset \\ x^2 & ; & \emptyset < x < 1 \\ 1 & ; & x \geq 1 \end{cases}$$

$$p(x) = F'(x) = \begin{cases} 2x & ; \emptyset < x < 1 \\ \emptyset & ; \text{sicer} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_{0}^{1} x\, 2x\, dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

• X je porazdeljena enakomerno na (a,b), če je njena gostota verjetnosti na (a,b) konstantna, drugje pa je enaka $\emptyset$.

$$p(x) = \begin{cases} \frac{1}{b-a} ; & a<x<b \\ 0 ; & \text{sicer} \end{cases}$$



$$F(x) = \begin{cases} \emptyset & x<a \\ \bigcirc & \\ 1 & x>b \end{cases}$$

$\frac{1}{b-a} x + C$

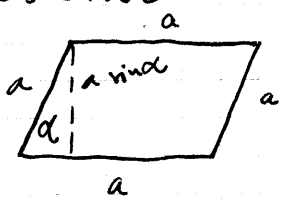
$\lim\limits_{x \to a} (\ ) = \emptyset$ $\qquad$ $\lim\limits_{x \to b} (\ ) = 1$

$\frac{a}{b-a} = -C$ $\qquad$ $\frac{1}{b-a} x - \frac{a}{b-a} = \frac{x-a}{b-a}$

to je to

• Romb s stranico dolgo ena, kot med stranicama α - porazdeljena enakomerno na $(\emptyset, \pi/2)$. To pomeni, da so vsi koti enako verjetni. Z S označimo ploščino romba. S - sluč. spr.

$p_S(x) = ?$



$S = a \cdot a \cdot \sin\alpha$

$a = 1 \rightarrow S = \sin\alpha$

$\boxed{F_S(x)} = P(S < x)$

$$= P(\sin\alpha < x) = \begin{cases} 0 ; & x<0 \\ P(\alpha < \arcsin x) ; & 0<x<1 \\ 1 ; & x \geq 1 \end{cases} =$$

to je porazdelitvena fja

$$= \begin{cases} \emptyset & ; x \leq \emptyset \\ F_\alpha(\arcsin x) & ; \emptyset \leq x < 1 \\ 1 & ; x \geq 1 \end{cases} = \ast$$

$$F_\alpha(x) = \begin{cases} \emptyset & ; x < \emptyset \\ 2x/\pi & ; \emptyset < x < \pi/2 \\ 1 & ; x \geq \pi/2 \end{cases}$$

$$\ast = \boxed{\begin{cases} \emptyset & ; x \leq \emptyset \\ 2\arcsin x/\pi & ; \emptyset < x < 1 \\ 1 & ; x \geq 1 \end{cases}}$$

vstavimo asin namesto x.

$$F_\alpha(\arcsin x) = \begin{cases} 0 & ; \arcsin x < \emptyset \\ 2\arcsin x/\pi & ; 0<\arcsin x<\pi/2 \\ 1 & ; \arcsin x \geq \frac{\pi}{2} \end{cases}$$

zamenjamo

$$\begin{cases} 0 & ; x<0 \\ 2\arcsin x/\pi & ; 0<x<1 \\ 1 & ; x>1 \end{cases}$$

104

$$p_S(x) = F_S'(x) = \begin{cases} \dfrac{2}{\pi\sqrt{1-x^2}} & ; 0<x<1 \\ 0 & ; \text{sicer} \end{cases}$$

- Polmer kroga - R je izmerjen približno, tako, da je porazdeljen enakomerno na (a,b)

S - ploščina kroga. Zanima nas $E(S)=?$

Lahko bi:

$$F_S(x) = P(S<x) = P(\pi R^2 < x)$$

$$p_S(x) = F_S'(x)$$
$$E(S) = \int_{-\infty}^{\infty} x\, p_S(x)$$

krajše:

$$S = \pi R^2$$
$$E(S) = E(\underbrace{\pi R^2}_{f(R)}) = \int_{-\infty}^{\infty} \pi x^2\, p_R(x)\,dx = *$$

$$p_R(x) = \begin{cases} \dfrac{1}{b-a} & ; a<x<b \\ \emptyset & ; \text{sicer} \end{cases}$$

$$* = \int_a^b \pi x^2 \frac{1}{b-a}\,dx = \frac{\pi}{b-a}\frac{x^3}{3}\Big|_a^b = \frac{\pi}{3(b-a)}(b^3-a^3) =$$

$$= \frac{\pi}{3}(b^2+ab+a^2)$$

• pravokotni trikotnik, hipotenuza = 2, kot ob kateti.

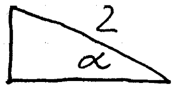
· $p_\alpha(x) = \begin{cases} \sin x; & 0 < x < \pi/2 \\ 0; & \text{sicer} \end{cases}$

Obseg – O

a) $E(O)$

b) $P(O > 3+\sqrt{3})$

$O = a+b+c = 2\sin\alpha + 2\cos\alpha + 2$



a) $E(O) = E(2\sin\alpha + 2\cos\alpha + 2) =$

$$= \int_{-\infty}^{\infty} (2\sin x + 2\cos x + 2)\, p_\alpha(x)\,dx =$$

$$= \int_0^{\pi/2} (2\sin x + 2\cos x + 2) \sin x \, dx$$

$$\int_0^{\pi/2} \Big(\underbrace{(1-\cos 2x)}_{2\sin^2 x} + \sin 2x + 2\sin x\Big) dx$$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$2\sin x\cos x = \sin 2x$

$$= \left[x - \frac{\sin 2x}{2} - \frac{\cos 2x}{2} - 2\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0 - 0 + 0 + \frac{1}{2} + \frac{1}{2} - 0 + 2 = 3 + \pi/2$$

b) $P(O > 3+\sqrt{3}) = P(2\sin\alpha + 2\cos\alpha + 2 > 3+\sqrt{3}) =$

$= P(2\sin\alpha + 2\cos\alpha > 1+\sqrt{3}) =$

$= P\left(\sin\alpha + \cos\alpha > \frac{1+\sqrt{3}}{2}\right)$ $\quad \frac{1}{2} + \frac{\sqrt{3}}{2}$ 2 kota

$= P\left(\frac{\pi}{6} < \alpha < \frac{\pi}{3}\right) = \int_{\pi/6}^{\pi/3} p_\alpha(x)\,dx =$

$= \int_{\pi/6}^{\pi/3} \sin x \, dx = -\cos x \Big|_{\pi/6}^{\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

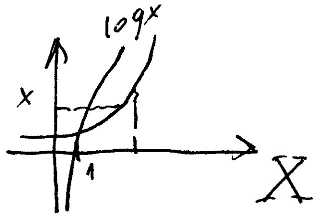
• X - enakomeren na $(0,1)$

$Y = e^X$

$p_Y(x) = ?$

$$F_Y(x) = P(Y<x) = P(e^X < x) = \begin{cases} X & ; x \leq 0 \\ P(X < \log x) & ; x > 0 \end{cases} =$$

$$= \begin{cases} \emptyset & ; x \leq \emptyset \\ F_X(\log x) & ; x > \emptyset \end{cases} = *$$



$$F_X(x) = \begin{cases} \emptyset & ; x \leq \emptyset \\ x/(b-a) & ; 0 < x < 1 \\ 1 & ; x \geq 1 \end{cases}$$

↑ X enakomerno na $(\emptyset, 1)$

$$F_X(\log x) = \begin{cases} \emptyset & ; \log x \leq \emptyset \\ \log x & ; \emptyset < \log x < 1 \\ 1 & ; \log x \geq 1 \end{cases} = \begin{cases} 0 & ; x \leq 1 \\ \log x & ; 1 < x < e \\ 1 & ; x > e \end{cases}$$

$$* = \begin{cases} \emptyset & ; x \leq 1 \\ \log x & ; 1 < x < e \\ 1 & ; x \geq e \end{cases}$$

$$p_Y(x) = F_Y'(x) = \begin{cases} \frac{1}{x} & ; 1 < x < e \\ 0 & ; \text{sicer} \end{cases}$$