Pogosto zastavljena vprašanja – 2. del

Naloga.

$$\int \frac{x\,dx}{x^4+x^2+1}$$

Rešitev. Najprej uvedemo novo spremenljivko $t = x^2$, $dt = 2x\,dx$. Po zamenjavi spremenljivke dobimo

$$\frac{1}{2}\int \frac{dt}{t^2+t+1}\ .$$

Izraz pod integralom "spominja na" $\frac{1}{x^2+1}$ (integral te funkcije pa poznamo, kajne?). Preoblikujmo zgornji integral:

$$\frac{1}{2}\int \frac{dt}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\int \frac{\frac{4}{3}\,dt}{(\frac{2}{\sqrt{3}}t+\frac{1}{\sqrt{3}})^2+1}$$

Še enkrat uvedemo novo spremenljivko: $u = \frac{2}{\sqrt{3}}t + \frac{1}{\sqrt{3}}$, $du = \frac{2}{\sqrt{3}}\,dt$:

$$\begin{aligned}\frac{1}{2}\int \frac{\frac{4}{3}\,dt}{(\frac{2}{\sqrt{3}}t+\frac{1}{\sqrt{3}})^2+1} &= \frac{1}{2}\int \frac{\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\,du}{u^2+1} = \frac{\sqrt{3}}{3}\int \frac{du}{u^2+1} \\ &= \frac{\sqrt{3}}{3}\operatorname{arctg} u + C = \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + C \\ &= \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x^2+1}{\sqrt{3}} + C\end{aligned}$$

□

Naloga.

$$\int \frac{x\,dx}{2x^2+8x+20}$$

Rešitev. Najprej izpostavimo $\frac{1}{2}$:

$$\int \frac{x\,dx}{2x^2+8x+20} = \frac{1}{2}\int \frac{x\,dx}{x^2+4x+10}$$

Radi bi uvedli novo spremenljivko $u = x^2+4x+10$, $du = (2x+4)\,dx = 2(x+2)\,dx$, toda pod integralom bi potrebovali izraz $(x+2)\,dx$, da bi ga smeli zamenjati z $\frac{du}{2}$. Upoštevajmo $x = x+2-2$ in zgornji integral razbijmo na vsoto dveh integralov:

$$\frac{1}{2}\int \frac{x+2\,dx}{x^2+4x+10} + \frac{1}{2}\int \frac{-2\,dx}{x^2+4x+10}$$

Prvega rešimo z uvedbo nove spremenljivke $u = x^2+4x+10$, $du = (2x+4)\,dx$:

$$\frac{1}{2}\int \frac{x+2\,dx}{x^2+4x+10} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln u + C_1 = \frac{1}{4}\ln(x^2+4x+10) + C_1$$

Drugi integral preoblikujmo tako, da bomo po uvedbi nove spremenljivke dobili pod integralom izraz $\frac{1}{t^2+1}$:

$$\frac{1}{2}\int \frac{-2\,dx}{x^2+4x+10} = -\int \frac{dx}{(x+2)^2+6} = -\frac{1}{6}\int \frac{dx}{(\frac{1}{\sqrt{6}}x+\frac{2}{\sqrt{6}})^2+1}$$

Za novo spremenljivko vzamemo $t = \frac{1}{\sqrt{6}}x + \frac{2}{\sqrt{6}}$, $dt = \frac{1}{\sqrt{6}}\,dx$:

$$-\frac{\sqrt{6}}{6}\int \frac{dt}{t^2+1} = -\frac{\sqrt{6}}{6}\operatorname{arctg} t + C_2 = -\frac{\sqrt{6}}{6}\operatorname{arctg}(\tfrac{1}{\sqrt{6}}x+\tfrac{2}{\sqrt{6}}) + C_2$$

Vsota obeh integralov (tj. rešitev naloge):

$$\frac{1}{4}\ln(x^2+4x+10) - \frac{\sqrt{6}}{6}\operatorname{arctg}(\tfrac{1}{\sqrt{6}}x+\tfrac{2}{\sqrt{6}}) + C$$

□

Naloga.

$$\int \frac{dx}{\sqrt{x}(\sqrt[4]{x}+\sqrt[6]{x})}$$

Rešitev. Če nastopajo v integralu izrazi $\sqrt[\alpha_1]{x}$, $\sqrt[\alpha_2]{x}$, $\sqrt[\alpha_3]{x}, \ldots$, kjer so $\alpha_1, \alpha_2, \ldots \in \mathbb{N}$, uvedemo novo spremenljivko $x = t^\alpha$, kjer je α skupni večkratnik števil $\alpha_1, \alpha_2, \ldots$ (glej zapiske z vaj). Najmanjši skupni večkratnik števil 2, 4 in 6 je 12, torej je $x = t^{12}$ in $dx = 12t^{11}\,dt$:

$$\int \frac{12t^{11}\,dt}{t^6(t^3+t^2)} = 12\int \frac{t^{11}\,dt}{t^8(t+1)} = 12\int \frac{t^3\,dt}{t+1} = 12\int (t^2 - t + 1 - \frac{1}{t+1})\,dt =$$
$$12\cdot\frac{t^3}{3} - 12\cdot\frac{t^2}{2} + 12t - \ln(t+1) + C = 4\sqrt[4]{x} - 6\sqrt[6]{x} + 12\sqrt[12]{x} - \ln(\sqrt[12]{x}+1) + C$$

□

Naloga.

$$\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx$$

Rešitev. Uvedemo novo spremenljivko $x = t^4$, $dx = 4t^3\,dt$ pa dobimo:

$$4\int \frac{1+t}{t^4+t^2}t^3\,dt = 4\int \frac{t^3(t+1)}{t^2(t^2+1)}\,dt = 4\int \frac{t^2+t}{t^2+1}\,dt = 4\int (1+\frac{t-1}{t^2+1})\,dt =$$
$$4\int dt + \int \frac{t}{t^2+1}\,dt - \int \frac{1}{t^2+1}\,dt = 4t + \frac{1}{2}\ln(t^2+1) - \operatorname{arctg} t + C =$$
$$4\sqrt[4]{x} + \frac{1}{2}\ln(\sqrt{x}+1) - \operatorname{arctg}\sqrt[4]{x} + C$$

Integral $\int \frac{t}{t^2+1}\,dt$ rešimo z uvedbo nove spremenljivke $u = t^2+1$ ($du = 2t\,dt$):

$$\int \frac{t}{t^2+1}\,dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(t^2+1)$$

□

Naloga.

$$\int_0^{\pi} \frac{dx}{9+7\sin^2 x}$$

Rešitev. Rešimo najprej nedoločeni integral. Če v integralu nastopajo sode potence funkcij sin in cos, lahko vpeljemo novo spremenljivko $t = \operatorname{tg} x$.

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \cos^2 x = \frac{1}{\operatorname{tg}^2 x + 1} = \frac{1}{t^2+1}$$

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \sin^2 x = 1 - \cos^2 x \quad \Rightarrow \quad \sin^2 x = 1 - \frac{1}{t^2+1} = \frac{t^2}{t^2+1}$$

$$dt = \frac{1}{\cos^2 x}\, dx \quad \Rightarrow \quad \cos^2 x\, dt = dx \quad \Rightarrow \quad \frac{1}{t^2+1}\, dt = dx$$

Dobimo:

$$\int \frac{dx}{9+7\sin^2 x} = \int \frac{\frac{1}{t^2+1}\, dt}{9 + \frac{7t^2}{t^2+1}} = \int \frac{dt}{9(t^2+1)+7t^2} = \int \frac{dt}{16t^2+9} =$$
$$\int \frac{\frac{1}{9}\, dt}{(\frac{4}{3}t)^2+1} \overset{(*)}{=} \int \frac{\frac{1}{12}\, du}{u^2+1} = \frac{1}{12}\operatorname{arctg} u + C = \frac{1}{12}\operatorname{arctg}(\frac{4}{3}\operatorname{tg} x) + C$$

$(*)$: $u = \frac{4}{3}t$, $du = \frac{4}{3}\, dt$

Pri integriranju smo uvedli novo spremenljivko $t = \operatorname{tg} x$, funkcija tg pa ima pol v $\frac{\pi}{2}$, zato:

$$\int_0^{\pi} \frac{dx}{9+7\sin^2 x} = \lim_{\substack{a\to\frac{\pi}{2}\\ a<\frac{\pi}{2}}} \int_0^{a} \frac{dx}{9+7\sin^2 x} + \lim_{\substack{a\to\frac{\pi}{2}\\ a>\frac{\pi}{2}}} \int_a^{\pi} \frac{dx}{9+7\sin^2 x} =$$
$$= \lim_{\substack{a\to\frac{\pi}{2}\\ a<\frac{\pi}{2}}} \tfrac{1}{12}\operatorname{arctg}(\tfrac{4}{3}\operatorname{tg} a) - \tfrac{1}{12}\operatorname{arctg}(\tfrac{4}{3}\operatorname{tg} 0) + \tfrac{1}{12}\operatorname{arctg}(\tfrac{4}{3}\operatorname{tg} \pi) - \lim_{\substack{a\to\frac{\pi}{2}\\ a>\frac{\pi}{2}}} \tfrac{1}{12}\operatorname{arctg}(\tfrac{4}{3}\operatorname{tg} a) =$$
$$= \lim_{b\to\infty} \tfrac{1}{12}\operatorname{arctg}(b) - \tfrac{1}{12}\operatorname{arctg} 0 + \tfrac{1}{12}\operatorname{arctg} 0 - \lim_{b\to-\infty} \tfrac{1}{12}\operatorname{arctg}(b) = \frac{\pi}{12}$$

□