MATEMATIKA 2, KEMIJA, 2. KOLOKVIJ, REŠITEV

D

1. NALOGA

$\vec r(t) = (t,\ 3\sin(t)-1,\ 3\cos(t))$

$\dot{\vec r}(t) = (1,\ 3\cos(t),\ -3\sin(t))$

$\ddot{\vec r}(t) = (0,\ -3\sin(t),\ -3\cos(t))$

$\dot{\vec r}\times\ddot{\vec r} = (-9,\ 3\cos(t),\ -3\sin(t))$

$\|\dot{\vec r}\times\ddot{\vec r}\| = 3\cdot\sqrt{10}\ ,\quad \|\dot{\vec r}\| = \sqrt{10}$

$\vec b(t) = \frac{1}{\sqrt{10}}(-3,\ \cos(t),\ -\sin(t))$

$\vec\tau(t) = \frac{1}{\sqrt{10}}(1,\ 3\cos(t),\ -3\sin(t))$

$\vec\nu(t) = \frac{1}{10}(0,\ -10\sin(t),\ -10\cos(t)) = (0,\ -\sin(t),\ -\cos(t))$

$t=\frac{\pi}{3}$: $\vec b = \frac{1}{\sqrt{10}}\left(-3,\ \frac12,\ -\frac{\sqrt3}{2}\right)$

$\vec\tau = \frac{1}{\sqrt{10}}\left(1,\ \frac32,\ -\frac{3\sqrt3}{2}\right)$

$\vec\nu = \left(0,\ -\frac{\sqrt3}{2},\ -\frac12\right)$

$\vec r\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3},\ \frac{3\sqrt3}{2}-1,\ \frac32\right)$

Pritisnjena ravnina:

$-3x + \frac12 y - \frac{\sqrt3}{2}z = -\pi + \frac{3\sqrt3}{4} - \frac12 - \frac{3\sqrt3}{4}$ oziroma

$\boxed{6x - y + z\sqrt3 = 1 + 2\pi}$

2. NALOGA

$\vec r(\varphi,t) = (\cos\varphi,\ \sin\varphi,\ (1-t)(\sin(2\varphi))^2 + 5t)$

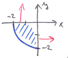
$\frac{\partial\vec r}{\partial\varphi} = (-\sin\varphi,\ \cos\varphi,\ (1-t)\,2\cdot\sin(2\varphi)\cdot\cos(2\varphi)\cdot 2)$

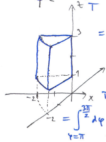
$\frac{\partial\vec r}{\partial t} = (0,\ 0,\ 5-(\sin(2\varphi))^2) = (5-\sin^2(2\varphi))\cdot(0,0,1)$

$\frac{\partial\vec r}{\partial\varphi}\times\frac{\partial\vec r}{\partial t} = (5-\sin^2(2\varphi))\cdot(\cos\varphi,\ \sin\varphi,\ 0) \Rightarrow \left\|\frac{\partial\vec r}{\partial\varphi}\times\frac{\partial\vec r}{\partial t}\right\| = 5-\sin^2(2\varphi)$

$A = \int_{\varphi=0}^{2\pi}d\varphi\int_{t=0}^{1}(5-\sin^2(2\varphi))\,dt = 5\cdot 2\pi - \int_0^{2\pi}\frac{1-\cos(4\varphi)}{2}\,d\varphi = 10\pi - \pi = 9\pi.$

3. NALOGA





$S = \{(x,y,z)\mid x^2+y^2=4,\ x\le 0,\ y\le 0,\ 1\le z\le 3\}$. Naj bo telo T četrtina valja:

$T = \{(x,y,z)\mid x^2+y^2\le 4,\ x\le 0,\ y\le 0,\ 1\le z\le 3\}$. Tedaj

$$\iiint_T(\operatorname{div}\vec F)\,dV = \underset{\substack{x=0\\ \text{normala }(1,0,0)}}{\iint}\vec F\cdot d\vec S + \underset{\substack{y=0\\ (0,1,0)}}{\iint}\vec F\cdot d\vec S + \underset{\substack{z=3\\ (0,0,1)}}{\iint}\vec F\cdot d\vec S + \underset{\substack{z=1\\ (0,0,-1)}}{\iint}\vec F\cdot d\vec S + \iint_S\vec F\cdot d\vec S =$$

$$= \int_{y=-2}^{0}dy\int_{z=1}^{3}e^y\,dz + \int_{x=-2}^{0}dx\int_{z=1}^{3}e^z\,dz + \underset{1/4\text{ kroga}}{\iint}(3+e^{x^2+y^2})\,dx\,dy + \underset{1/4\text{ kroga}}{\iint}(-1)\cdot(1+e^{x^2+y^2})\,dx\,dy + \iint_S\vec F\cdot d\vec S =$$

$$= 2(1-e^{-2}) + 2\cdot(e^3-e) + 2\cdot\tfrac14\cdot\pi\cdot 4 + \iint_S\vec F\cdot d\vec S = 2(e^3-e+1-e^{-2}) + 2\pi + \iint_S\vec F\cdot d\vec S$$

Po drugi strani $\operatorname{div}\vec F = yz + 3y^2 + 3x^2 + 1$ in $\iiint_T(\operatorname{div}\vec F)\,dV =$

$$= \int_{\varphi=\pi}^{\frac{3\pi}{2}}d\varphi\int_{r=0}^{2}dr\int_{z=1}^{3}r\cdot(3r^2 + rz\sin\varphi + 1)\,dz = \frac{\pi}{2}\cdot 2\cdot 3\frac{r^4}{4}\Big|_{r=0}^{2} + \frac{r^3}{3}\Big|_{r=0}^{2}\cdot\frac{z^2}{2}\Big|_{z=1}^{3}\cdot(-\cos\varphi)\Big|_{\pi}^{\frac{3\pi}{2}} + \frac{\pi}{2}\cdot 2\cdot\frac{r^2}{2}\Big|_0^2 =$$

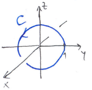
$$= 12\pi + \frac83\cdot\frac12\cdot\cancel{8}(-1) + 2\pi \Rightarrow \iint_S\vec F\cdot d\vec S = 12\pi - 2(e^3-e-e^{-2}) - \frac{34}{3}.$$

4. NALOGA

(a) Parametrizacija:

$\vec r(t) = (0,\ \cos t,\ \sin t) \Rightarrow \dot{\vec r}(t) = (0,\ -\sin t,\ \cos t)$

$\oint_C\vec A\cdot d\vec s = \int_0^{2\pi}(0,\ 3\cos t,\ 0)\cdot(0,\ -\sin t,\ \cos t)\,dt = \int_0^{2\pi}(-3\sin t\cos t)\,dt = -\frac32\int_0^{2\pi}\sin(2t)\,dt = 0.$



(b) C je robna krožnica polsfere $S = \{(x,y,z)\mid x^2+y^2+z^2=1,\ x\ge 0\}$.

Pišimo $\vec F(x,y,z) = (3x,\ 3y-2z,\ 2z+3y)$ in $f(x,y,z) = x^2+y^2+z^2$. Tedaj $\vec B = \frac{\vec F}{f}$.

Izračunamo $\operatorname{rot}\vec F = (5,0,0)$ in $\operatorname{grad} f = (2x,2y,2z) = 2(x,y,z)$.

$$\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x & 3y-2z & 2z+3y\end{vmatrix}$$

Na enotski sferi velja $d\vec S = (x,y,z)\,dS$ (kjer je dS skalarni pl. element), ker je (x,y,z) ravno „navzven" usmerjena enotska normala v $(x,y,z)\in S$.

[Pripomnimo, da je orientacija za C skladna z opisano orientacijo za S.]

Zato je $(\operatorname{rot}\vec B)\cdot d\vec S = \frac1f(\operatorname{rot}\vec F)\cdot(x,y,z)\,dS + \frac{1}{f^2}(\vec F\times 2(x,y,z))\cdot(x,y,z)\,dS = \frac1f(\operatorname{rot}\vec F)\cdot(x,y,z)\,dS =$

$= \frac{5x}{x^2+y^2+z^2}\,dS$, ker je mešani produkt enak 0.

Tako $\oint_C\vec B\cdot d\vec s = \iint_S(\operatorname{rot}\vec B)\cdot d\vec S = \underbrace{\iint_S 5x\,dS}_{\substack{\text{na sferi } S \text{ velja}\\ x^2+y^2+z^2=1}} = 5\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_{\vartheta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\cdot\cos\vartheta\cdot(\cos\vartheta\,d\vartheta) = 5\sin\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos(2\vartheta)}{2}\,d\vartheta = 5\pi.$