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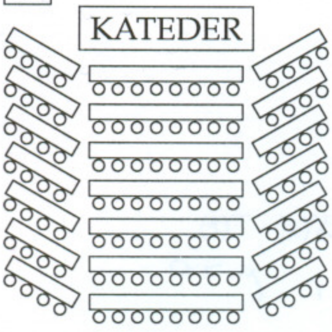

Matematika 2: 4. kolokvij

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Čas reševanja: 90 minut

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Položaj v 2.05

Ime in priimek

Vpisna številka

1. NALOGA (25 točk)

Podana je funkcija
$$f(x) = \sin\left(\tfrac{x}{6}\right).$$

a. Razvij funkcijo f v klasično Fourierevo vrsto na intervalu $[-\pi, \pi]$.

REŠITEV: $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi} \cdot \frac{k}{k^2 - \frac{1}{36}} \cdot \sin(kx)$

b. S pomočjo razvoja zapiši $\sin\left(\frac{\pi}{12}\right)$ z vsoto številske vrste.

REŠITEV: $\sin\left(\frac{\pi}{12}\right) = f\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi} \cdot \frac{k}{k^2 - \frac{1}{36}} \sin\left(k\frac{\pi}{2}\right) = \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)}{\pi\left((2i+1)^2 - \frac{1}{36}\right)}$

c. Zapiši enakost, ki jo dobiš s pomočjo razvoja za $x = \pi$.

REŠITEV: $\frac{1}{2}\left(f(\pi) + f(-\pi)\right) = 0$: Periodizacija je nezvezna v $x = \pi$.

Pomožni računi:

a. f je liha funkcija $\Rightarrow a_k = 0$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{6}\right)\sin(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos\left(\frac{x}{6} - kx\right) - \cos\left(\frac{x}{6} + kx\right)\right)dx =$$

[soda fn. na sim. intervalu]

$$= \frac{1}{\pi}\left(\frac{\sin\left(\frac{1}{6} - k\right)x}{\frac{1}{6} - k} - \frac{\sin\left(\frac{1}{6} + k\right)x}{\frac{1}{6} + k}\right)\Bigg|_0^{\pi} =$$

$$= \frac{1}{\pi}\left(\frac{\sin\left(\frac{\pi}{6} - k\pi\right)}{\frac{1}{6} - k} - \frac{\sin\left(\frac{\pi}{6} + k\pi\right)}{\frac{1}{6} - k}\right) = \frac{1}{\pi}\left(\frac{\sin\left(\frac{\pi}{6}\right)\cdot(-1)^k}{+\frac{1}{6} - k} - \frac{\sin\left(\frac{\pi}{6}\right)\cdot(-1)^k}{\frac{1}{6} + k}\right) =$$

[upoštevamo $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ in $\cos(-k\pi) = \cos(k\pi) = (-1)^k$]

$$= \frac{1}{2\pi}(-1)^k\left(\frac{\frac{1}{6} + k - \left(\frac{1}{6} - k\right)}{\frac{1}{36} - k^2}\right) = \frac{(-1)^{k-1}}{\pi} \cdot \frac{k}{k^2 - \frac{1}{36}}$$

b. $\sin\left(\frac{k\pi}{2}\right)$: je enako 0, če je k sodo število, in $(-1)^{\lfloor k/2 \rfloor}$, če je k liho.

Podana je parcialna diferencialna enačba

$$(*) \qquad \frac{\partial^2 u}{\partial x^2} + (4-2t^2)u - t^3\frac{\partial u}{\partial t} = 0$$

za funkcijo $u = u(x,t)$, kjer $(x,t) \in [0,1] \times [0,\infty)$.

a. [(**45 točk**)] Izračunaj vse rešitve enačbe $(*)$ oblike $u(x,t) = X(x) \cdot T(t)$.

$$u(x,t) = \begin{cases} (Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}) \cdot C t^{-2} \cdot e^{-\frac{1}{2}(\mu+4)t^{-2}}, & \mu > 0 \\ (Ax+B)\, C \cdot t^{-2} e^{-2t^{-2}} & \\ (A\cos(\sqrt{|\mu|}\,x) + B\sin(\sqrt{|\mu|}\,x)) \cdot \frac{C}{t^2} \cdot e^{-\frac{(\mu+4)}{2t^2}}, & \mu < 0 \end{cases}$$

Opomba: konstanto C lahko izpustimo.

REŠITEV: ______

b. [(bonus **10 točk**)] Katere od dobljenih rešitev $u = X \cdot T$ zadoščajo dodatnima pogojema

$$X(0) = 0 \text{ in } X'(1) = 0?$$

REŠITEV: $u(x,t) = B\sin((k+\frac{1}{2})\pi x) \cdot t^{-2} e^{-\frac{1}{2}(4-(k+\frac{1}{2})^2\pi^2)t^{-2}}$

Pomožni računi: Vstavimo $u = X \cdot T$ v $(*)$ in dobimo:

$$X''T + (4-2t^2)XT - t^3 X\dot{T} = 0$$

Delimo z XT in dobimo: $\frac{X''}{X} + 4 - 2t^2 - t^3\frac{\dot{T}}{T} = 0$

Preuredimo: $\frac{X''}{X} = t^3\frac{\dot{T}}{T} - 4 + 2t^2$

Leva stran je odvisna le od spremenljivke x, desna le od t, zato sta obe enaki (isti) konstanti, recimo μ.

Za levo stran: $X''/X = \mu \Leftrightarrow X'' - \mu X = 0$

(i) $\mu > 0 \Rightarrow X(x) = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}$

(ii) $\mu = 0 \Rightarrow X(x) = A + Bx$

(iii) $\mu < 0 \Rightarrow X(x) = A\cos(\sqrt{|\mu|}\,x) + B\sin(\sqrt{|\mu|}\,x)$

Za desno stran:

$t^3\dot{T}/T - 4 + 2t^2 = \mu$

$\frac{\dot{T}}{T} = \frac{\mu+4-2t^2}{t^3} = \frac{\mu}{t^3} + \frac{4}{t^3} - \frac{2}{t}$

$\log T = \log C + (\mu+4)\frac{t^{-2}}{-2} - 2\log(t)$

$T = \frac{C}{t^2} \cdot e^{\frac{(\mu+4)}{-2} \cdot t^{-2}}$

b. Obravnavamo možnosti (i), (ii), (iii):

(i) $X = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}$, $X' = \sqrt{\mu}(Ae^{\sqrt{\mu}x} - Be^{-\sqrt{\mu}x})$

$X(0) = 0 \Rightarrow A + B = 0$ $\quad X'(1) = 0 \Rightarrow Ae^{\sqrt{\mu}} - Be^{-\sqrt{\mu}} = 0$ oziroma $A(e^{\sqrt{\mu}} + e^{-\sqrt{\mu}}) = 0$

$e^{\sqrt{\mu}}$ in $e^{-\sqrt{\mu}}$ sta pozitivni števili, torej mora biti $A = 0$ in zato $B = 0$.

(ii) $X = A + Bx$, $X' = B$: $X(0) = 0 \Rightarrow A = 0$, zato $X'(1) = B = 0$.

(iii) $X = A\cos(\sqrt{|\mu|}\,x) + B\sin(\sqrt{|\mu|}\,x)$. Najprej: $X(0) = 0 \Rightarrow A = 0$

zato $X = B\sin(\sqrt{|\mu|}\,x)$ in $X'(x) = B\sqrt{|\mu|}\cos(\sqrt{|\mu|}\,x)$

$X'(1) = B\sqrt{|\mu|}\cos(\sqrt{|\mu|}) = 0 \Rightarrow \sqrt{|\mu|} = \frac{2k+1}{2}\pi$; $k = 0, \pm1, \pm2, \dots$

zato $\mu_k = -(k+\frac{1}{2})^2\pi^2$ in $X_k = B\sin((k+\frac{1}{2})\pi x)$. Zaradi konstante B zadošča vzeti $k = 0, 1, 2, \dots$

3. NALOGA (30 točk)

Obravnavaj homogeno toplotno enačbo

$$(\dagger) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

za funkcijo $u = u(x,t)$, kjer $x \in [0,b]$ in $t \in [0,\infty)$.

a. Zapiši rešitev enačbe (∗) pri pogojih $u(0,t) = u(b,t) = 0$ za vsak čas t in pri začetni porazdelitvi $u(x,0) = x(b^2 - x^2)$.

REŠITEV: $u_H(x,t) = \sum_{n=1}^{\infty} \frac{12b^3}{(n\pi)^3}(-1)^{n-1} e^{-2\left(\frac{n\pi}{b}\right)^2 t} \sin\left(\frac{n\pi}{b}x\right)$

b. Za nehomogeno toplotno enačbo

$$(\ddagger) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t} + \sum_{n=1}^{\infty}\frac{e^{-t}}{2n^2}\sin\left(\tfrac{n\pi}{b}x\right)$$

(pri pogojih $u(0,t) = u(b,t) = 0$) uporabi nastavek $u = \sum_{n=1}^{\infty} T_n(t)\cdot X_n(x)$, kjer so X_n iste funkcije kot v prejšnji točki in izračunaj rešitev.

REŠITEV: $u(x,t) = u_H(x,t) + \sum_{n=1}^{\infty}\frac{1}{n^2\left(2\left(\frac{n\pi}{b}\right)^2-1\right)}\left(e^{-2\left(\frac{n\pi}{b}\right)^2 t} - e^{-t}\right)\sin\left(\frac{n\pi}{b}x\right)$,

kjer je u_H rešitev homogene enačbe kot v $(\dagger)$.

Pomožni računi: Vemo, da velja nastavek $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\cdot X_n(x)$, kjer $X_n(x) = \sin\left(\frac{n\pi}{b}x\right)$ in $X_n''(x) = -\left(\frac{n\pi}{b}\right)^2 X_n(x)$. Zato $(\dagger)$ pove $\sum_{n=1}^{\infty} -\left(\frac{n\pi}{b}\right)^2 T_n\cdot X_n = \sum_{n=1}^{\infty}\frac{1}{2}\cdot\dot{T}_n X_n$ oziroma

$\sum_{n=1}^{\infty}\left(\dot{T}_n + 2\left(\frac{n\pi}{b}\right)^2 T_n\right)X_n = 0$ in zato $\dot{T}_n + 2\left(\frac{n\pi}{b}\right)^2 T_n = 0$.

To pomeni $T_n(t) = A_n\cdot e^{-2\left(\frac{n\pi}{b}\right)^2 t}$ in $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-2\left(\frac{n\pi}{b}\right)^2 t}\sin\left(\frac{n\pi}{b}x\right)$

Konstante A_n so določene z začetno porazdelitvijo $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b}x\right)$

Vemo: $A_n = \frac{2}{b}\int_0^b x(b^2-x^2)\sin\left(\frac{n\pi}{b}x\right)dx = 2b\left(-\frac{b}{n\pi}x\cos\left(\frac{n\pi}{b}x\right) + \left(\frac{b}{n\pi}\right)^2\sin\left(\frac{n\pi}{b}x\right)\right)\Big|_0^b$

$-\frac{2}{b}\left(-\frac{b}{n\pi}x^3\cos\left(\frac{n\pi}{b}x\right) + \left(\frac{b}{n\pi}\right)^2\cdot 3x^2\sin\left(\frac{n\pi}{b}x\right) + \left(\frac{b}{n\pi}\right)^3\cdot 6x\cos\left(\frac{n\pi}{b}x\right) - \left(\frac{b}{n\pi}\right)^4\sin\left(\frac{n\pi}{b}x\right)\right)\Big|_0^b$

$= 2b\cdot\frac{-b^2}{n\pi}\cdot(-1)^n - \frac{2}{b}\left(\frac{-b^4}{n\pi}(-1)^n + \frac{b^3\cdot 6b}{(n\pi)^3}\cdot(-1)^n\right) = \frac{12b^3}{(n\pi)^3}\cdot(-1)^{n-1}$

Za b. nam nastavek in $(\ddagger)$ pove $\sum_{n=1}^{\infty} -\left(\frac{n\pi}{b}\right)^2 T_n X_n = \sum_{n=1}^{\infty}\frac{1}{2}\dot{T}_n X_n + \sum_{n=1}^{\infty}\frac{1}{2}\frac{e^{-t}}{n^2}X_n$

oziroma $\sum_{n=1}^{\infty}\left(\dot{T}_n + 2\left(\frac{n\pi}{b}\right)^2 T_n + \frac{1}{n^2}e^{-t}\right)X_n = 0$, iz česar zaključimo, da mora biti $\dot{T}_n + 2\left(\frac{n\pi}{b}\right)^2 T_n + \frac{1}{n^2}e^{-t} = 0$ za vsako naravno število n.

Ta lin. d.e. ima enako homogeno enačbo in z nastavkom ali pa variacijo konstante izračunamo $T_n(t) = D_n e^{-2\left(\frac{n\pi}{b}\right)^2 t} - \frac{1}{n^2\left(2\left(\frac{n\pi}{b}\right)^2-1\right)}e^{-t}$

Zdaj velja $T_n(0) = D_n - \frac{1}{n^2\left(2\left(\frac{n\pi}{b}\right)^2-1\right)}$. Zaradi začetnih pogojev velja

$T_n(0) = A_n$ od prej in zato $D_n = \frac{1}{n^2\left(2\left(\frac{n\pi}{b}\right)^2-1\right)} + \frac{12b^3}{(n\pi)^3}(-1)^{n-1}$