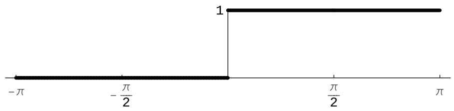
Vaje za ponavljanje pred 4. kolokvijem, Matematika 2 za kemike
Prvi del: razvoj v Fourierevo vrsto

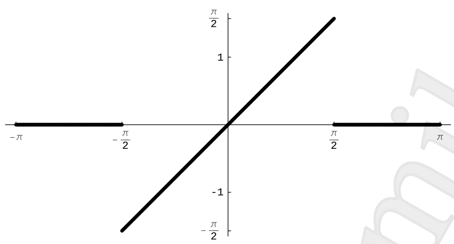
(1) Naslednje funkcije, ki so periodične s periodo 2π, razvij v standardno Fourierevo vrsto in z njeno pomočjo izračunaj vsoto podane vrste:

(i) Graf funkcije je



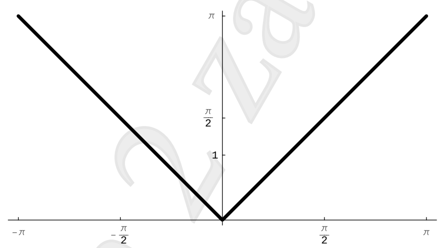
Zahtevana vrsta je $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.
[$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right)$, vstavi $x = \frac{\pi}{2}$]

(ii) Graf funkcije je



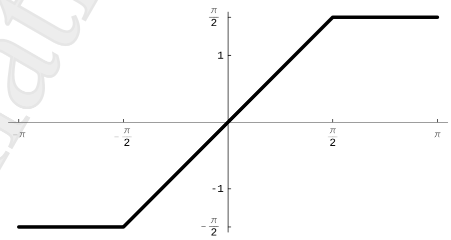
Zahtevana vrsta je $1 + \frac{1}{9} + \frac{1}{25} + \dots$.
[$f(x) = \frac{2}{\pi}\sin(x) + \frac{1}{2}\sin(2x) - \frac{2}{9\pi}\sin(3x) - \frac{1}{4}\sin(4x) + \frac{2}{25\pi}\sin(5x) + \frac{1}{6}\sin(6x) + \dots$, vstavi $x = \frac{\pi}{2}$]

(iii) Graf funkcije je



Zahtevana vrsta je $1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{36} + \frac{1}{49} - \frac{1}{64} - \frac{1}{81} + \dots$.
[$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \dots\right)$, vstavi $x = \frac{\pi}{4}$]

(iv) Graf funkcije je



Zahtevana vrsta je $2 + \pi + \frac{2-3\pi}{9} + \frac{2+5\pi}{25} + \frac{2-7\pi}{49} + \dots$ [$f(x) = \frac{2+\pi}{\pi}\sin(x) - \frac{1}{2}\sin(2x) + \frac{-2+3\pi}{9\pi}\sin(3x) - \frac{1}{4}\sin(4x) + \dots$, vstavi $x = \frac{\pi}{2}$]

(2) Če je funkcija $f\colon \mathbb{R} \to \mathbb{R}$ periodična s periodo $2L$ (in ima še kako drugo primerno lastnost), ima Fourierev razvoj oblike

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\tfrac{k\pi}{L} x\right) + b_k \sin\left(\tfrac{k\pi}{L} x\right) \right) \qquad (*)$$

Integriraj $\int_{-L}^{L}(\text{leva stran od } *) \cos\left(\frac{k\pi}{L}x\right)\, \mathrm{d}x$ in $\int_{-L}^{L}(\text{desna stran od } *) \cos\left(\frac{k\pi}{L}x\right)\, \mathrm{d}x$, da se prepričaš, da velja:

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\tfrac{k\pi}{L}x\right)\, \mathrm{d}x$$

Podobno se prepričaj, da velja

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\tfrac{k\pi}{L}x\right)\, \mathrm{d}x$$

(3) Naj bo $k > 0$. Za funkcijo

$$f(x) := \begin{cases} 0, & \text{če } -2 < x < -1, \\ k, & \text{če } -1 < x < 1, \\ 0, & \text{če } 1 < x < 2, \end{cases}$$

privzemi, da je periodična s periodo 4, in jo razvij v Fourierevo vrsto na intervalu $[-2, 2]$.
[$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos(\frac{\pi}{2}x) - \frac{1}{3}\cos(\frac{3\pi}{2}x) + \frac{1}{5}\cos(\frac{5\pi}{2}x) - \dots\right)$]

(4) Funkcijo $f(x) = \frac{\pi}{2}x^3$ dopolni v periodično funkcijo s periodo 2 in jo razvij v Fourierevo vrsto na intervalu $[-1, 1]$.
[$f(x) = \left(1 - \frac{6}{\pi^2}\right)\sin(\pi x) - \left(\frac{1}{2} - \frac{6}{2^3\pi^2}\right)\sin(2\pi x) + \left(\frac{1}{3} - \frac{6}{3^3\pi^2}\right)\sin(3\pi x) - \dots$]

(5) Funkcijo $f(x) = \begin{cases} 1 + x, & \text{če } -1 < x < 0, \\ 1 - x, & \text{če } 0 < x < 1 \end{cases}$ dopolni v periodično funkcijo s periodo 2 in jo razvij v Fourierevo vrsto na intervalu $[-1, 1]$.
[$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \dots\right)$]

(6) Funkcijo $f(x) = e^x$ $(x \in (-\pi, \pi))$ razširi do periodične funkcije s periodo 2π in jo razvij v Fourierevo vrsto na intervalu $[-\pi, \pi]$ (tj. klasično Fourierevo vrsto). Potem izrazi $\operatorname{cth}(\pi) = \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}$ z vrsto.
[$f(x) = \frac{e^\pi - e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{1+k^2}\right)$, vstavi $x = \pi$]

(7) Funkcijo, ki je podana na intervalu $[0, L]$, dopolni do sode oziroma do lihe funkcije in jo razvij v kosinusno oziroma sinusno Fourierevo vrsto na intervalu $[-L, L]$:

(i) $f(x) = x$ za $0 < x < \frac{1}{2}$, $L = \frac{1}{2}$.
[kosinusna vrsta: $f(x) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos(2\pi x) + \frac{1}{9}\cos(6\pi x) + \frac{1}{25}\cos(10\pi x) + \dots\right)$,]
[sinusna vrsta: $f(x) = \frac{1}{\pi}\left(\sin(2\pi x) - \frac{1}{2}\sin(4\pi x) + \frac{1}{3}\sin(6\pi x) - \frac{1}{4}\sin(8\pi x) + \dots\right)$.]

(ii) $f(x) = \begin{cases} 0, & \text{če } 0 < x < 2, \\ 1, & \text{če } 2 < x < 4 \end{cases}$, $L = 4$.
[kosinusna vrsta: $f(x) = \frac{1}{2} - \frac{2}{\pi}\left(\cos\left(\frac{\pi x}{4}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{4}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{4}\right) - \dots\right)$,]
[sinusna vrsta:

$$f(x) = \frac{2}{\pi}\Big(\sin\left(\tfrac{\pi x}{4}\right) - \sin\left(\tfrac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\tfrac{3\pi x}{4}\right) + \frac{1}{5}\sin\left(\tfrac{5\pi x}{4}\right)$$
$$-\frac{1}{3}\sin\left(\tfrac{3\pi x}{2}\right) + \frac{1}{7}\sin\left(\tfrac{7\pi x}{4}\right) + \frac{1}{9}\sin\left(\tfrac{9\pi x}{4}\right) - \frac{1}{5}\sin\left(\tfrac{5\pi x}{2}\right) + \dots\Big).$$

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(iii) $f(x) = x^2$ za $0 < x < L$, $L = L$.
[kosinusna vrsta: $f(x) = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos(\frac{\pi x}{L}) - \frac{1}{4}\cos(\frac{2\pi x}{L}) + \frac{1}{9}\cos(\frac{3\pi x}{L}) - \dots\right)$,]
[sinusna vrsta:

$$f(x) = \frac{2L^2}{\pi}\Big(\left(1 - \frac{4}{\pi^2}\right)\sin(\tfrac{\pi x}{L}) - \frac{1}{2}\sin(\tfrac{2\pi x}{L})$$
$$+ \left(\frac{1}{3} - \frac{4}{3^3\pi^2}\right)\sin(\tfrac{3\pi x}{L}) - \frac{1}{4}\sin(\tfrac{4\pi x}{L}) + \dots\Big).$$

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