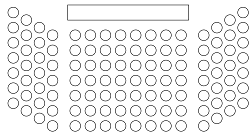
Analiza 1 (F): 2. izpit

22.5.2013

Vse odgovore je potrebno dobro utemeljiti.
Veliko uspeha!



Sedež (2.05)

1	
2	
3	
4	
Σ	

Ime in priimek

Vpisna številka

1. naloga (25 točk)

(a) Izračunaj nedoločeni integral

$$\int \frac{x^2-x+6}{x^3-x^2-x+1}\,dx.$$

Rešitev:

$$x^3-x^2-x+1=(x-1)^2(x+1)$$

$$\int \frac{x^2-x+6}{x^3-x^2-x+1}\,dx=\frac{A}{x-1}+B\ln|x-1|+C\ln|x+1|+E$$

$$\frac{x^2-x+6}{x^3-x^2-x+1}=-\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+1}$$

$$x^2-x+6=-Ax-A+Bx^2-B+Cx^2-2Cx+C$$

$$\begin{aligned} x^2 &: \quad B+C=1 \\ x &: \quad -A-2C=-1 \\ 1 &: \quad -A-B+C=6 \end{aligned}$$

$$A=-3,\ B=-1,\ C=2$$

(b) Izračunaj limito

$$\lim_{x\to 0}\frac{\cos x-1}{e^{2x}\sin x-x}.$$

Rešitev:

$$\lim_{x\to 0}\frac{\cos x-1}{e^{2x}\sin x-x}\overset{L.P.}{=}\lim_{x\to 0}\frac{-\sin x}{2e^{2x}\sin x+e^{2x}\cos x-1}\overset{L.P.}{=}$$

$$\overset{L.P.}{=}\lim_{x\to 0}\frac{-\cos x}{4e^{2x}\sin x+2e^{2x}\cos x+2e^{2x}\cos x-e^{2x}\sin x}=-\frac{1}{4}$$

2. naloga (25 točk)

Funkcija $z = z(x, y)$ je podana implicitno z enačbo

$$z \ln z + xz^2 + y^3 e^x - 1 = 0.$$

Določi Taylorjev polinom 2. stopnje funkcije z v točki $(0, 1)$ in približno izračunaj $z(0.1, 0.8)$.

Rešitev:

$$z \ln z + 0 + 1 - 1 = 0$$

$$z(0, 1) = 1$$

$$z_x \ln z + z_x + z^2 + 2xzz_x + y^2 e^x = 0$$

$$0 + z_x + 1 + 0 + 1 = 0$$

$$z_x(0, 1) = -2$$

$$z_y \ln z + z_y + 2xzz_y + 3y^2 e^x = 0$$

$$0 + z_y + 0 + 3 = 0$$

$$z_y(0, 1) = -3$$

$$z_{xx} \ln z + z_x \frac{z_x}{z} + z_{xx} + 2zz_x + 2zz_x + 2xz_x z_x + 2xzz_{xx} + y^3 e^x = 0$$

$$0 + 4 + z_{xx} - 4 - 4 + 0 + 0 + 1 = 0$$

$$z_{xx}(0, 1) = 3$$

$$z_{xy} \ln z + z_x \frac{z_y}{z} + z_{xy} + 2zz_y + 2xz_y z_x + 2xzz_{xy} + 3y^2 e^x = 0$$

$$0 + 6 + z_{xy} - 6 + 0 + 0 + 3 = 0$$

$$z_{xy}(0, 1) = -3$$

$$z_{yy} \ln z + z_y \frac{z_y}{z} + z_{yy} + 2xz_y z_y + 2xzz_{yy} + 6ye^x = 0$$

$$0 + 9 + z_{yy} + 0 + 0 + 6 = 0$$

$$z_{yy}(0, 1) = -15$$

$$\begin{aligned} T(h, k) &= z(0, 1) + z_x(0, 1)h + z_y(0, 1)k + \frac{1}{2} z_{xx}(0, 1)h^2 + z_{xy}(0, 1)hk + \frac{1}{2} z_{yy}(0, 1)k^2 \\ &= 1 - 2h - 3k + \frac{3}{2}h^2 - 3hk - \frac{15}{2}k^2 \end{aligned}$$

$$z(0.1, 0.8) \doteq T(0.1, -0.2) = 1.175$$

3. naloga (25 točk)

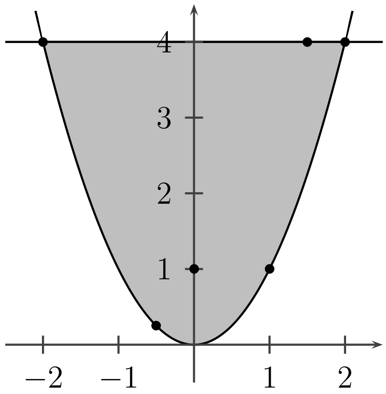
Določi globalne ekstreme funkcije

$$f(x,y) = x^2 - xy + \frac{y^2}{2} + x - y$$

na območju

$$D = \{(x,y)\ ;\ x^2 \leq y \leq 4\}.$$

Rešitev:



Stacionarne točke:

$$\begin{aligned} f_x &= 2x - y + 1 = 0 \\ f_y &= -x + y - 1 = 0 \end{aligned}$$

$$x = 0,\ y = 1$$

Rob območja $y = 4$:

$$g(x) = f(x,4) = x^2 - 4x + 8 + x - 4 = x^2 - 3x - 4$$

$$g'(x) = 2x - 3 = 0$$

$$x = \frac{3}{2},\ y = 4$$

Rob območja $y = x^2$:

$$h(x) = f(x,x^2) = x^2 - x^2 + \frac{x^4}{2} + x - x^2 = \frac{x^4}{2} - x^3 + x$$

$$h'(x) = 2x^3 - 3x^2 + 1$$

$$x_0 = -\frac{1}{2},\ y_0 = \frac{1}{4}, \quad x_{1,2} = 1,\ y_{1,2} = 1$$

Kandidati za ekstreme so $(0,1)$, $(\frac{3}{2},4)$, $(1,1)$, $(-\frac{1}{1},\frac{1}{4})$ in še robni točki $(2,4)$, $(-2,4)$.

$$f(0,1) = -\frac{1}{2},\ \ f(\frac{3}{4},4) = \frac{7}{4},\ \ f(1,1) = \frac{1}{2},\ \ f(-\frac{1}{2},\frac{1}{4}) = -\frac{11}{32},\ \ f(2,4) = 2,\ \ f(-2,4) = 14$$

Globalni maksimum je dosežen v točki $(-2,4)$, globalni minimum pa v točki $(0,1)$.

4. naloga (25 točk)

Poišči rešitev diferencialne enačbe

$$xy' + \frac{1}{2}(y-1) + (x^2+x)(y-1)^2 = 0,$$

ki ustreza začetnemu pogoju $y(1) = 0$.

Nasvet. Uvedi novo funkcijo $z(x) = \frac{1}{y(x)-1}$.

Rešitev:

$$y = \frac{1}{z} + 1, \quad y' = -\frac{z'}{z^2}$$

$$-\frac{xz'}{z^2} + \frac{1}{2z} + \frac{x^2+x}{z^2} = 0$$

$$-xz' + \frac{z}{2} + x^2 + x = 0$$

Homogeni del:

$$-xz' + \frac{z}{2}$$

$$z' = \frac{1}{2x}$$

$$\frac{dz}{z} = \frac{dx}{2x}$$

$$\ln z = \frac{1}{2}\ln x + \ln C$$

$$z = Cx^{\frac{1}{2}}$$

Variacija konstante:

$$-x(C'x^{\frac{1}{2}} + \frac{1}{2}Cx^{-\frac{1}{2}}) + \frac{1}{2}Cx^{\frac{1}{2}} + x^2 + x = 0$$

$$-C'x^{\frac{3}{2}} = -x^2 - x$$

$$C' = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$C = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + D$$

$$z = \frac{2}{3}x^2 + 2x + Dx^{\frac{1}{2}}$$

$$y = \frac{1}{\frac{2}{3}x^2 + 2x + D\sqrt{x}} + 1$$

$$y(1) = 0 \quad \Rightarrow \quad D = -\frac{11}{3}$$