Prvi kolokvij iz Analize 2
8. april 2010

Priimek in ime: .... Vpisna št.: | | | | | | | | |

Vrsta/sedež: | | |

1. Natanko za katera realna števila p obstaja integral

$$\int_1^\infty \frac{x^p}{\sqrt{x^4-1}}dx\ ?$$

Svoje sklepe o konvergenci in divergenci dobro utemelji.

∞: $$\int_1^\infty \frac{x^p}{x^2\sqrt{1-\frac{1}{x^4}}}dx = \int_1^\infty \frac{\frac{1}{\sqrt{1-1/x^4}}}{x^{2-p}}dx \qquad \varphi(x) = \frac{1}{\sqrt{1-\frac{1}{x^4}}}$$

$$\alpha = 2-p > 1 \qquad \lim_{x\to\infty} \varphi(x) = 1 \neq 0, \neq \infty$$
$$1 > p$$

konv. $p < 1$,

1: $$\int_1^\infty \frac{x^p}{\sqrt{(x-1)(x+1)(x^2+1)}}dx = \int_1^\infty \frac{\frac{x^p}{\sqrt{(x+1)(x^2+1)}}}{(x-1)^{1/2}}dx$$

$$\alpha = \frac{1}{2}\ \checkmark \qquad \varphi(x) = \frac{x^p}{\sqrt{(x+1)(x^2+1)}}$$ zvezna v 1 ne glede na p

$p < 1$

2. Za dani realni števili a in b, $ab > 0$, naj bo

integrand soda f.

$$I(a,b) = \int_a^b \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}} dx\,.$$

(a) Pokaži, da velja: $I(a,b) = I(-b,-a)$, $I(a,b) = I(\frac{1}{a},\frac{1}{b})$ in $I(a,\frac{1}{a}) = 0$.

(b) Uvedi novo spremenljivko (da je substitucija monotona, ni treba preverjati)

$$t = x + \frac{1}{x}$$

in izračunaj integral $I(a,b)$ v primeru, ko sta $a \geq 1$ in $b \geq 1$.

$t = \frac{1}{x}$ $\quad dt = -\frac{1}{x^2}dx$ $\quad dx = -x^2 dt = -\frac{dt}{t^2}$

(a) $$\int_{1/a}^{1/b} \frac{1-\left(\frac{1}{t}\right)^2}{\left(1+\left(\frac{1}{t}\right)^2\right)\sqrt{1+\left(\frac{1}{t}\right)^4}}\left(-\frac{dt}{t^2}\right)\frac{t^2}{t^2} = -\int_{1/a}^{1/b} \frac{t^2-1}{(t^2+1)\sqrt{t^4+1}}\,dt$$

$$I\left(a,\tfrac{1}{a}\right) = I\left(\tfrac{1}{a},a\right) = -I\left(a,\tfrac{1}{a}\right) \Rightarrow I\left(a,\tfrac{1}{a}\right) = 0$$

(b) $dt = \left(1-\frac{1}{x^2}\right)dx$ $\quad dx = \frac{dt}{1-\frac{1}{x^2}} = \frac{x^2\,dt}{x^2-1}$

$$\int_{a+\frac{1}{a}}^{b+\frac{1}{b}} \frac{\cancel{(1-x^2)}}{(1+x^2)\sqrt{1+x^4}}\,\frac{x^2\,dt}{\cancel{x^2-1}} = -\int_{a+1/a}^{b+1/b} \frac{dt}{\left(\frac{1}{x}+x\right)\sqrt{\frac{1}{x^2}+x^2}} =$$

$$x^2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$$

$$= -\int_{a+1/a}^{b+1/b} \frac{dt}{t\sqrt{t^2-2}} = -\int_{\sqrt{a^2+1/a^2}}^{\sqrt{b^2+1/b^2}} \frac{\cancel{u}\,du}{(u^2+2)\cancel{u}} = -\frac{1}{2}\int_{\sqrt{a^2+1/a^2}}^{\sqrt{b^2+1/b^2}} \frac{du}{\left(\frac{u}{\sqrt{2}}\right)^2+1} =$$

$u^2 = t^2 - 2$

$\cancel{2}u\,du = \cancel{2}t\,dt \;|:t^2$

$(a+\frac{1}{a})^2 - 2 = a^2 + 1/a^2$

$$\frac{u\,du}{u^2+2} = \frac{u\,du}{t^2} = \frac{dt}{t}$$

$$= -\frac{1}{2}\operatorname{arctg}\frac{u}{\sqrt{2}}\cdot\sqrt{2}\,\Bigg|_{\sqrt{a^2+1/a^2}}^{\sqrt{b^2+1/b^2}}$$

3. (a) Naj bosta $f, g\colon [a,b] \to \mathbb{R}$ zvezni realni funkciji. Izberi interval $[a,b] \subset \mathbb{R}$ in takšni elementarni funkciji f in g, da ploščina območja med funkcijama f in g na intervalu $[a,b]$ ne bo enaka

$$\left| \int_a^b \big(f(x) - g(x)\big)\, dx \right|.$$

(b) Poišči presečišča grafov funkcij

$$f(x) = \frac{\ln x}{4x} \quad \text{in} \quad g(x) = x \ln x$$

in izračunaj ploščino omejenega območja, ki ga omejujeta.

(b)

$x > 0$

$$\frac{\cancel{\ln x}}{4x} = x \cancel{\ln x} \qquad \underline{x = 1}$$

$$1 = 4x^2$$

$$\underline{\frac{1}{2} = x}$$

$u = \ln x \quad du = \frac{1}{x}dx$

$$\int \frac{\ln x}{4x}\, dx = \frac{1}{4}\int u\, du = \frac{u^2}{8} + C = \frac{(\ln x)^2}{8} + C$$

$$\int \underset{dv}{x}\, \underset{u}{\ln x}\, dx = \ln x \frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{2\cancel{x}}\, dx = \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} + C$$

$\frac{x}{2}$

$u = \ln x \quad du = \frac{1}{x}$

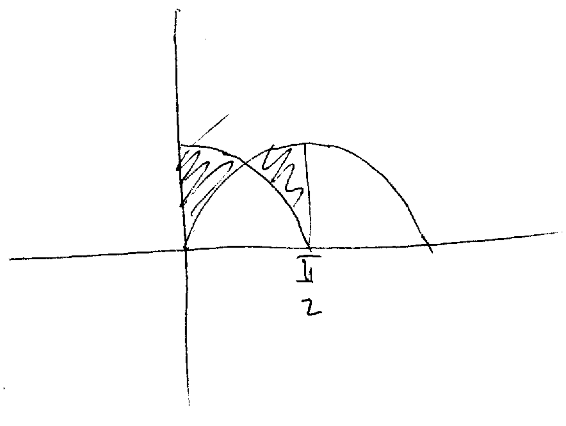
$dv = x \quad v = \frac{x^2}{2}$

$\ln\frac{1}{2} = -\ln 2$

$$\int_{1/2}^{1} \ldots = \left[\frac{(\ln x)^2}{8} - \frac{x^2}{2}\ln x + \frac{x^2}{4} \right]_{1/2}^{1} = \frac{1}{4} - \left(\frac{(\ln \frac{1}{2})^2}{8} - \frac{1}{8}\ln\frac{1}{2} + \frac{1}{16} \right) =$$

$$= \frac{3}{16} - \frac{\ln^2 2}{8} - \frac{1}{8}\ln 2$$

(a)



$a = 0, \; b = \frac{\pi}{2}$

$f(x) = \sin x, \; g(x) = \cos x$

4. Izračunaj naslednja integrala:

(a) $\int x \sin^2 x \, dx$

(b) $\int \frac{e^{2x} + 2e^x}{e^{2x} + 1} \, dx$

$$\int x \tfrac{1}{2}(1-\cos 2x)\,dx = \tfrac{1}{2}\int x(1-\cos 2x)\,dx =$$

$u = x \qquad du = dx$

$dv = 1-\cos 2x \qquad v = x - \frac{\sin 2x}{2}$

$$= \tfrac{1}{2}\left(x^2 - \tfrac{x}{2}\sin 2x - \int \left(x - \tfrac{\sin 2x}{2}\right) dx \right) =$$

$$= \tfrac{1}{2}\left(x^2 - \tfrac{x}{2}\sin 2x - \tfrac{x^2}{2} - \tfrac{\cos 2x}{4} \right) + C$$

$$= \tfrac{1}{2}\left(\tfrac{x^2}{2} - \tfrac{x}{2}\sin 2x - \tfrac{\cos 2x}{4} \right) + C$$

$$= \tfrac{x^2}{4} - \tfrac{x}{4}\sin 2x - \tfrac{\cos 2x}{8} + C$$

$t = e^x \quad dt = e^x\,dx \quad dt = t\,dx \quad dx = \frac{dt}{t}$

$$\int \frac{t^2+2t}{t^2+1}\cdot\frac{dt}{t} = \int \frac{t+2}{t^2+1}\,dt = A\ln(t^2+1) + B\,\mathrm{arctg}\,t \;\Big|'$$

$$\frac{t+2}{t^2+1} = \frac{A\cdot 2t}{t^2+1} + \frac{B}{1+t^2}$$

$A = \frac{1}{2}, \; B = 2$

$$\int \frac{t+2}{t^2+1}\,dt = \tfrac{1}{2}\ln(t^2+1) + 2\,\mathrm{arctg}\,t + C =$$

$$= \tfrac{1}{2}\ln(e^{2x}+1) + 2\,\mathrm{arctg}\,e^x + C$$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

5. Naslednjo limito izračunaj tako, da jo prepoznaš kot Riemannov integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\sqrt{\frac{k}{n}}}{\sqrt{kn}} = \lim \frac{1}{n}\sum_{k=1}^{\infty}\frac{\sin\sqrt{\frac{k}{n}}}{\sqrt{\frac{k}{n}}} =$$

$$= \int_0^1 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

$$\int_0^1 \frac{\sin\sqrt{x}}{\sqrt{x}}\,\overset{2du}{dx} = 2\int_0^1 \sin u\,du = -2\cos u\Big|_0^1 = -2\cos 1 + 2\underbrace{\cos 0}_{1}$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

Točkovanje: 1. 20%, 2. 10% +15% , 3. 10% +15%, 4. 10% +10% , 5. 10%