ANALIZA 4 (fin) - 1. kolokvij

6. 12. 2011

Ime in priimek:
Vpisna št.:

1. [20] Poišči zaporedje $(a_n)_{n\geq 0}$, ki zadošča zvezi
$$a_n = 5a_{n-1} - 6a_{n-2} + 3 \cdot 2^n, \quad n = 2, 3, \ldots$$
in velja $a_0 = 1, a_1 = 2$.

2. [20] Poišči rešitev naslednje NDE
$$y''' = y^2 y',$$
ki zadošča pogojem $y(0) = 1, y'(0) = 1/\sqrt{6}, y''(0) = 1/3$.

3. [25] Naj bo $\vec{\omega} = (2, 4, 4)^T, \vec{a} = (0, -2, 1)^T$. Poišči splošno rešitev $\vec{x} = \vec{x}(t)$ vektorske NDE
$$\dot{\vec{x}} = \vec{\omega} \times \vec{x} + \vec{a}.$$
Pomoč: Izpiši dano NDE po komponentah.

4. Opazujemo populacijo šimpanzov. Vsako leto umre delež $\alpha > 0$ samcev in enako za samice. Prav tako se v vsakem letu vsak samec pari z vsako samico in delež $\beta > 0$ teh paritev se konča z rojstvom enega mladiča, ki je z verjetnostjo $\frac{1}{2}$ samec in z enako verjetnostjo samica.

   (a) [5] Če $X(t)$ pomeni število samcev v letu t ter $Y(t)$ pomeni število samic v letu t, zapiši sistem NDE za $X(t), Y(t)$, ki opisuje zgoraj opisano dinamiko populacije.

   (b) [15] Skiciraj fazni portret sistema NDE iz (a) pri $\alpha = \beta = 1$.

   (c) [5] Na faznem portretu iz (b) (tj. pri $\alpha = \beta = 1$) označi množico točk (X_0, Y_0) z naslednjo lastnostjo: če je število samcev v letu $t = 0$ enako $X_0 \geq 0$ ter število samic $Y_0 \geq 0$, populacija izumre.

   (d) [10] Reši sistem iz (a).

   Pozor! Rešitev vsebuje integral, ki ga ni moč izraziti z elementarnimi funkcijami!

1

$$a_n - 5a_{n-1} + 6a_{n-2} = 3\cdot 2^n$$

Hom.:

$$a_n = \mu^n \to \mu^2 - 5\mu + 6 = 0$$

$$(\mu-2)(\mu-3) = 0$$

$$a_n^{hom} = A\cdot 2^n + B\cdot 3^n$$

part.:

$$a_n = 2^n \cdot n \cdot C$$

$$2^n \cdot n \cdot C - 5\cdot 2^{n-1}(n-1)C + 6\cdot 2^{n-2}(n-2)C = 3\cdot 2^n \quad / : 2^{n-2}$$

$$4Cn - 10(n-1)C + 6(n-2)C = 3\cdot 4$$

$$+10C - 12C = 3\cdot 4$$

$$C = -3/2 \cdot 4 = -6$$

$$a_n = A\cdot 2^n + B\cdot 3^n - 6n\cdot 2^n$$

$a_0 = 1$: $\quad 1 = A + B \quad , \quad 1 = A + B$

$a_1 = 2$: $\quad 2 = 2A + 3B - 3\cdot 4, \quad 2A + 3B = 14$

$\cancel{B = 3, \ A = -2}$ $B =$ $\quad A = -11$

$B = 12$

$$\boxed{a_n = -11\cdot 2^n + 12\cdot 3^n - 6n\cdot 2^n}$$

$$\cancel{a_n = -2\cdot 2^n + 3\cdot 3^n - \frac{3}{2} n 2^n}$$

$$\cancel{= -2^{n+1} + 3^{n+1} - \frac{3}{2} n\cdot 2^n}$$

(2)

$$y''' = y^2 y'$$

$$(y'')' = (y^3/3)' \quad / \int$$

$$y'' = y^3/3 + A \quad \leftarrow \quad \begin{matrix} y''(0) = 1/3 \\ y(0) = 1 \end{matrix} \Rightarrow A = 0$$

$$y'' = y^3/3 \qquad y'(y) = z(y) \qquad \dot{} = \frac{d}{dy}$$

$$y'' = \frac{d}{dx} y' = \frac{dy}{dx}\frac{d}{dy} z = \dot{z} \cdot z$$

$$\dot{z} z = y^3/3$$

$$(z^2/2)^{\cdot} = y^3/3 \quad / \int dy$$

$$z^2/2 = B + \frac{y^4}{12} \qquad \cancel{z = y'}$$

$$y'^2 = 2B + y^4/6 \qquad \left\{ \begin{matrix} y'(0) = 1/\sqrt{6} \\ y(0) = 1 \end{matrix} \right. \Rightarrow B = 0$$

$$y' = \pm\sqrt{y^4/6} = \pm \frac{y^2}{\sqrt{6}}$$

$$y' = y^2/\sqrt{6}$$

$$dy/y^2 = dx/\sqrt{6}$$

$$-1/y = C + x/\sqrt{6} \qquad y(0) = 1 \rightarrow C = -1$$

$$\boxed{y = \frac{1}{1 - x/\sqrt{6}}}$$

③ $$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & -4 & 4 \\ 4 & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

l. vr., l. vekt.: $\lambda_1 = 0, \quad v_1 = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$

$\lambda_2 = 6i, \quad v_2 = \begin{pmatrix} 4i/3 \\ 1 - i/3 \\ -1 - i/3 \end{pmatrix}$

$\lambda_3 = -6i, \quad v_3 = \begin{pmatrix} -4i/3 \\ 1 + i/3 \\ -1 + i/3 \end{pmatrix}$

res. hom.: $$\vec{x} = \underbrace{\begin{pmatrix} 2 & 4i/3\, e^{6it} & -\frac{4i}{3} e^{-6it} \\ 4 & (1 - \frac{i}{3}) e^{6it} & (1 + \frac{i}{3}) e^{-6it} \\ 4 & (-1 - \frac{i}{3}) e^{6it} & (-1 + \frac{i}{3}) e^{-6it} \end{pmatrix}}_{X} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

var. konst.: $$\begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \\ \dot{C}_3 \end{pmatrix} = X^{-1} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cancel{x} & 1/9 & 1/9 \\ \cancel{x} & (1 + \frac{i}{3}) e^{-6it}/\beta & (-1 + \frac{i}{3}) e^{-6it}/\beta \\ \cancel{x} & (1 - \frac{i}{3}) e^{6it}/\beta & (-1 - \frac{i}{3}) e^{6it}/\beta \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} =$$

$\beta = 4$

$$= \begin{pmatrix} -1/9 \\ e^{-6it}/\beta \, (-3 - i/3) \\ e^{6it}/\beta \, (-3 + i/3) \end{pmatrix} \Rightarrow$$

$C_1 = -t/9 + K_1$

$C_2 = \frac{e^{-6it}}{-6i\beta} (-3 - i/3) + K_2$

$C_3 = \frac{e^{6it}}{6i\beta} (-3 + i/3) + K_3$

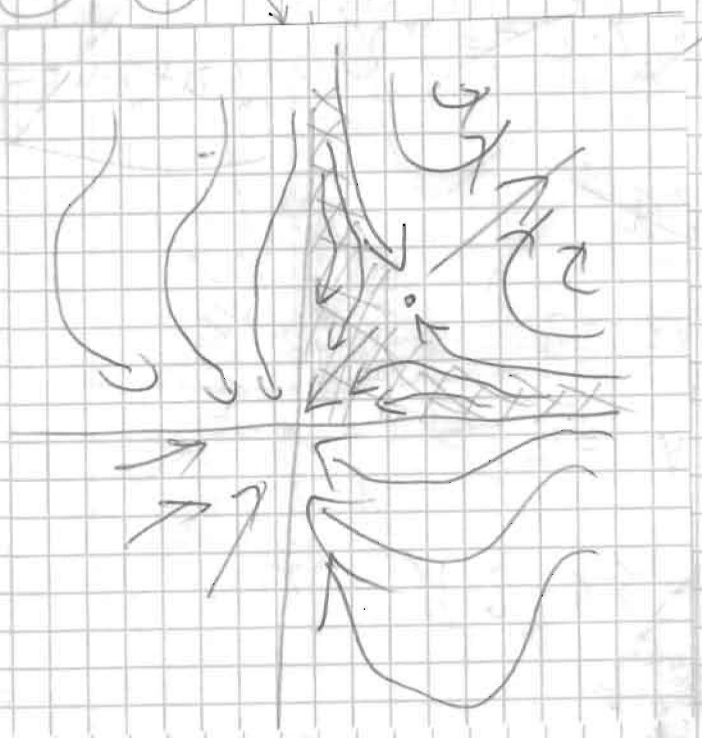
ge. res.:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = K_1 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + K_2 \begin{pmatrix} 4i/3 \\ (1 - i/3) \\ -1 - i/3 \end{pmatrix} e^{6it} + K_3 \begin{pmatrix} -4i/3 \\ 1 + i/3 \\ -1 + i/3 \end{pmatrix} e^{-6it} + \begin{pmatrix} -2t/9 + 1/3 \\ -4t/9 - 1/18 \\ -4t/9 - 1/9 \end{pmatrix}$$

(4a) $\dot{X} = \frac{1}{2}\beta XY - \alpha X$

$\dot{Y} = \frac{1}{2}\beta XY - \alpha Y$

(4b) & (4c)



$\dot{X} = \frac{1}{2}XY - X$

$\dot{Y} = \frac{1}{2}XY - Y$

stac. točke: $T_1(0,0)$

$T_2(2,2)$

(4d) $(X-Y)^{\cdot} = -\alpha(X-Y)$, $X-Y = Ce^{-\alpha t} \Rightarrow Y = X - Ce^{-\alpha t}$

$\Rightarrow \dot{X} = X\left(\frac{\beta}{2}(X - Ce^{-\alpha t}) - \alpha\right)$

$\dot{X} = X\left(\frac{\beta}{2}X - (\alpha + \frac{\beta}{2}Ce^{-\alpha t})\right)$, $z = 1/X$

$-\dot{z} = \frac{\beta}{2} - (\alpha + \frac{\beta}{2}Ce^{-\alpha t})z$

hom: $-\frac{dz}{dt} = -(\alpha + \frac{C\beta}{2}e^{-\alpha t})z$

$z = Ke^{\alpha t - \frac{C\beta}{2\alpha}e^{-\alpha t}}$, var. konst. $K \to K(t)$

$\dot{K} = -\frac{\beta}{2}e^{-\alpha t + \frac{C\beta}{2\alpha}e^{-\alpha t}} \Rightarrow z(t) = De^{\alpha t - \frac{C\beta}{2\alpha}e^{-\alpha t}} - \frac{\beta}{2}e^{\alpha t - \frac{C\beta}{2\alpha}e^{-\alpha t}} \int^t \left(e^{-\alpha t' + \frac{C\beta}{2\alpha}e^{-\alpha t'}}\right) dt'$

$\boxed{X = \frac{1}{(\ldots)}, \quad Y = \frac{1}{(\ldots)} - Ce^{-\alpha t}}$