Financial Mathematics 2   2011/2012
10th May 2012

1. Show that, in the Black Scholes model, if f is continuous bounded function

$$\lim_{N\to+\infty} \mathbb{E}_q\Big[f(S_N)\Big] = \mathbb{E}_Q\Big[f(S_T)\Big]$$

where $(S_n)_{n\geq 0}$ is the CRR Markov Chain.

2. Let $(X_i, i \geq 1)$ be a sequence of i.i.d. random variables $\mathbb{P}(X_i = \pm 1) = \lambda/2$ and $\mathbb{P}(X_i = 0) = 1 - \lambda$, with $0 < \lambda \leq 1$. Consider the Markov chain $S_n = X_1 + \cdots + X_n$.

   (a) Precise the conditions under which $(S_n)_{n\geq 0}$ converges in law to a standard brownian motion.

   (b) Write a Matlab or Scilab code to compute Europen Put price in the Black Scholes model.

   (c) Write a Matlab or Scilab code to compute American Put price in the Black Scholes model.

3. Let $(S_t)_{t\geq 0}$ be a geometric brownian motion. Write the expression of dS_t^2 using two different approaches.

4. Parisian options are barrier options which can be knocked in or out depending on time that the underlying asset has spent over a barrier. In the Parisian knock-out case the barrier option vanishes if the price of the underlying asset remain for a period longer than W over the barrier.
Write a Matlab or Scilab code to price Parisian option down-and-out in the Black Scholes model.
The options parameters are $K, S_0, T, down, W$.

5. Suppose that the function f is continuous and bounded. Set

$$u(t, x_1, x_2) = \mathbb{E}\Big[f(x_1 + B_t^1, x_2 + B_t^2)\Big]$$

where B_t^1 and B_t^2 are independent standard Brownian motion.
Then u is the unique smooth solution of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} & (t, x) \in]0, T] \times \mathbb{R}^2 \\ u(0, x_1, x_2) = f(x), x \in \mathbb{R}^2 \end{cases}$$

(a) Show how to compute the price of an exchange option with payoff $(S_T^1 - S_T^2)_+$ in the two-dimensional Black-Scholes using the function u.

(b) Write the explicit finite difference scheme (for sake of simplicity consider $\Delta x = \Delta x_1 = \Delta x_2$).

(c) Write the stability condition for the explicit scheme.

6. Consider a standard brownian motion with $B_0 = 0$ and $B_{t_{j+1}} = b$ wtih $t_j < t_{j+1}$. Show that

$$B_{t_j} | B_{t_{j+1}} = b \sim \mathcal{N}\left(\frac{t_j}{t_{j+1}} b, \frac{t_j}{t_{j+1}} (t_{j+1} - t_j)\right).$$