The Black Scholes model

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Continuous Stochastic Processes

The origin of stochastic processes can be traced back to the field of statistical physics. A physical process is a physical phenomenon whose evolution is studied as a function of time.

In a financial framework, the idea is to give a model of stock price fluctuations in continuous time.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A continuous-time stochastic process is a family $(X_t)_{t\geq 0}$ of $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

- the index t stands for the time.
- for each time t fixed:

$$X_t : \Omega \longrightarrow \mathbb{R}$$

- for each $\omega \in \Omega$ the map $t \longrightarrow X_t(\omega)$ is called the path of the process.

Brownian motion

In finance, the most common models are constructed on the Brownian motion.

Definition

A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t\geq 0}$ with indipendent, normally distributied and stationary increments. In other words :

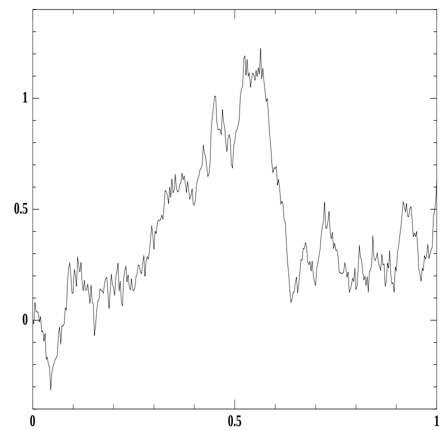
- P1 $B_0 = 0$.
- P1 the function $s \mapsto B_s(\omega)$ is a continuous function.
- P2 indipendent increments : for each $k, 0 \leq t_0 < t_1 < \ldots < t_k$, the increments $B_{t_0}, B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, .., B_{t_k} - B_{t_{k-1}}$ are indipendent.
- P3 for each $t > s \geq 0$, $B_t - B_s \sim N(0, t-s) \Rightarrow$
  $\mathbb{E}_{\mathbb{P}}(B_t - B_s) = 0$ and $\mathbb{E}_{\mathbb{P}}\left[(B_t - B_s)^2\right] = t - s$.
  In particular for $s = 0$ it follows that $\mathbb{E}_{\mathbb{P}}(B_t) = 0$ e $Var(B_t) = t$.

A particle that is undergoing a Brownian motion B_t has the following property.

- P2 Absence of memory.
- P3 The mean of $B_t - B_s$ is 0, so there is not a privilegiate direction.
  The variance of the particle movement is proportional to the observed time.

Brownian motion

The path of the Brownian motion are continuous, but not differentiable.



Financial example 2

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - e^{\sigma B_T})_+\right].$$

with B_T B.M a time T.
We consider a put option with underlying asset

$$S_t = e^{\sigma B_t}$$

Under $\mathbb{P}$, $B_T \sim N(0,T)$. So $B_T = g\sqrt{T}$ con $g \sim N(0,1)$.
We can approximate the put price with

$$P \approx e^{-rT}\frac{f(X_1) + \cdots + f(X_n)}{n}$$

$X_1, .., X_n \sim N(0,T)$.

Monte Carlo algorithm

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {
      /*Brownian motion simulation*/
      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-exp(sigma*brownian));

      mean_price= mean_price+price_sample;
      mean2_price= mean2_price+SQR(price_sample);
    }
  /* Price */
  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```

Simulation of Brownian motion path

Let $[0, T]$ be divided using N time intervals of lenght $\Delta T = \frac{T}{N}$.

$$B_T = B_{N\Delta T} = \sum_{k=1}^{N} (B_{k\Delta T} - B_{(k-1)\Delta T}) = \sum_{k=1}^{N} \Delta B_k$$

The increments $\Delta B_k \sim N(0, \Delta T)$ are indipendent and normally distribuited :

$$\Delta B_k = g\sqrt{\Delta T}$$

with $\quad g \sim N(0, 1)$.

Start $t_0 = 0,\ B_0 = 0,\ \Delta T = \frac{T}{N}$

for $k = 1, \ldots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0, 1)$;

$B_{k\Delta T} = B_{(k-1)\Delta T} + g\sqrt{\Delta T}$;

END;

Algorithm

```
main()
{
  double k,T,brownian,B_T,time;
  int N;

  k=T/N;
  brownian=0.;
  time=0.;
  for(i=1;i<=N;i++)
    {
      /*Time*/
      time+=k;

      /*Brownian path simulation*/
      brownian=brownian+gaussian()*sqrt(k);
    }

  /* B_T */
  B_T=brownian;
}
```

Brownian motion and random walk

One of the standard way used to approximate a Brownian motion is to use a random walk. Here we use the standard symmetric random walk.

Proposition

Let $(X_i, i \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(X_i = \pm 1) = 1/2$.
Set $S_n = X_1 + \cdots + X_n$.
Let $\Delta T = T/N$ be the time step. Set

$$B_N = \sqrt{\Delta T} S_N .$$

Then, the sequence B_N converges in distribution to B_T.
Consequently, if f is a bounded continuous function then

$$\mathbb{E}_{\mathbb{P}}\Big[f(B_N)\Big] \text{ converges to } \mathbb{E}_{\mathbb{P}}\Big[f(B_T)\Big].$$

In the financial example 2, $f(B_T) = (K - e^{\sigma B_T})_+$. We use $f(B_N) = (K - e^{\sigma B_N})_+$, or $g(S_N) = (K - e^{\sigma \sqrt{\Delta T} S_N})_+$

Besides, $E(g(SN))$ can be computed as follows :

$$\begin{cases} u(N\Delta T, x) = g(x), \\ u(n\Delta T, x) = \dfrac{1}{2}u((n+1)\Delta T, x+1) + \dfrac{1}{2}u((n+1)\Delta T, x-1). \end{cases}$$

Exercise

Compute

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - e^{\sigma B_T})_+\right].$$

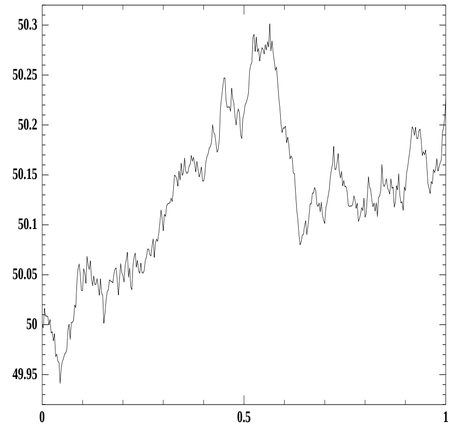
with $K = 100$, $r = 0.03$, $\sigma = 0.2$ using

- Monte Carlo algorithm
- Tree method

Brownian motion with drift

$$X_t = \mu t + \sigma B_t$$

with B_t standard brownian motion, with μ and σ costants.



A french matematician Bachelier introduces it in first years of 1900 for modelling stock prices. But there is the problem of the negative prices:

$$X_t \sim N(\mu t, \sigma^2 t).$$

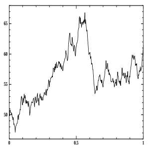
Geometric Brownian motion

Definition

A Geometric Brownian motion S_t is a continuous stochastic process such that:

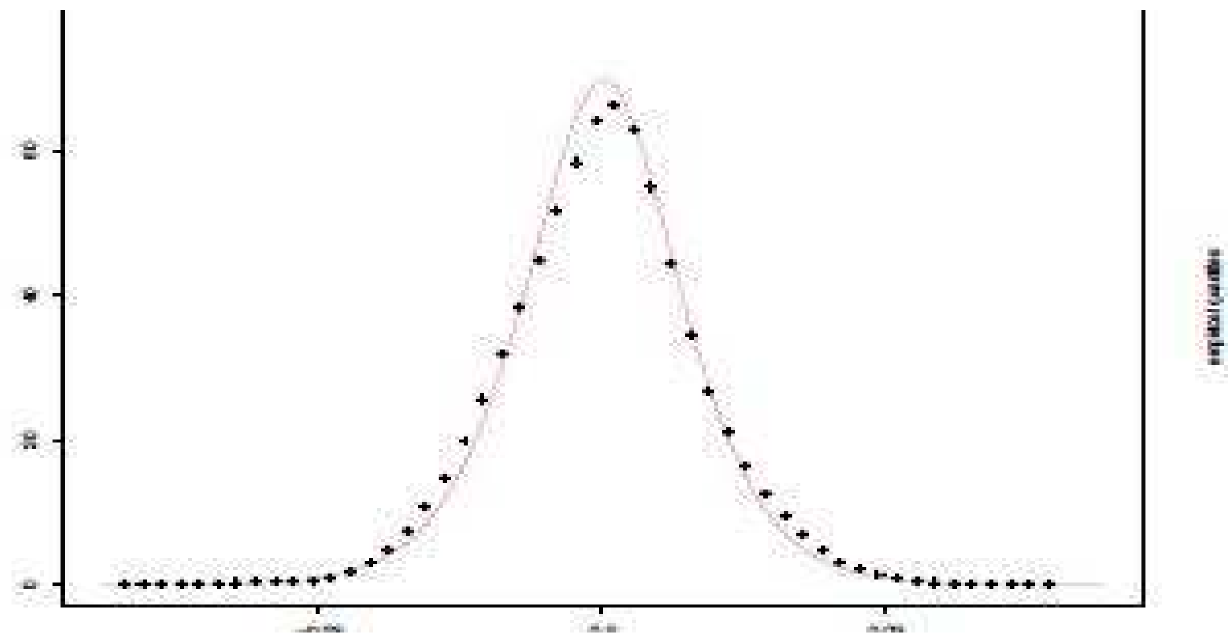
- P1 $S_0 = x$.
- P2 $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$

B_t standard Brownian motion, μ and σ costant.



The log-returns $\log \frac{S_t}{S_0}$ have normal distribution (the returns are normal).

$\frac{S_t}{S_0}$ is log-normal of parameters $(\mu - \frac{1}{2}\sigma^2)t$ and $\sigma^2 t$.



Property of the GBM

P1 Consider s<t. Then

$$\log(\frac{S_t}{S_s}) \sim N((\mu - \frac{1}{2}\sigma^2)(t-s), \sigma^2(t-s))$$

Expectation

$$\mathbb{E}(\frac{S_t}{S_s}) = e^{\mu(t-s)}$$

Variance

$$Var(\frac{S_t}{S_s}) = e^{2\mu(t-s)}(e^{\sigma^2(t-s)} - 1)$$

P2 for each $0 \leq t_0 < t_1 < \ldots < t_n$, the relative increments $S_{t_k}/S_{t_{k-1}}$ are indipendent and have common law.

Financial example 3

$$P = e^{-rT}\mathbb{E}_{\mathbb{P}}\left[(K - S_T)_+\right],$$

with S_T value of the GBM at time T T.
We consider a put option with underlying asset

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

The payoff can be written

$$h(B_T) = (K - S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T})_+$$

Then

$$P \approx e^{-rT}\frac{h(X_1) + \cdots + h(X_n)}{n}$$

$X_1, .., X_n \sim N(0, T)$.

Monte Carlo algorithm

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {

      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-x*exp((mu-0.5*sigma*sigma)*T+sigma*brownian));

      mean_price=mean_price+price_sample;
      mean2_price=mean2_price+SQR(price_sample);
    }

  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```

Simulation of Geometric Brownian motion path

Let $[0,T]$ be divided using N time intervals of lenght $\Delta T = \frac{T}{N}$.

$$S_T = S_{N\Delta T} = S_0 \prod_{k=1}^{N} \frac{S_{k\Delta T}}{S_{(k-1)\Delta T}} = S_0 \prod_{k=1}^{N} e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma \Delta B_k},$$

with

$$(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma \Delta B_k = (\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g \sqrt{\Delta T} \quad con \quad g \sim N(0,1)$$

Simulation of the GBM path $(S_t)_{0 \le t \le T}$:

Start $t_0 = 0$, $S_0 = x$, $\Delta T = \frac{T}{N}$

for $k = 1, \ldots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0,1)$;

$S_{k\Delta T} = S_{(k-1)\Delta T} e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g \sqrt{\Delta T}}$;

END.

Algorithm

```
main()
{
  double k,T,w_derive,s,S_N,mu=0.1,sigma=0.2,time;
  int N;

  k=T/N;
  s=50.;
  time=0.;
  for(i=1;i<=N;i++)
    {
      /*Timew*/
      time=time+k;
      /*Geometric Brownian simulation*/
      s=s*exp((mu-0.5*sigma*sigma)*k+sigma*gaussian()*sqrt(k));
    }
  /* S_T */
   S_T=s;
}
```

Differential property of the Brownian motion

$$\mathbb{E}\Big[(B_t - B_s)^2\Big] = t - s$$

Let us consider the random variable.

$$X = \Big(B_{t+\Delta t} - B_t\Big)^2$$

Then

$$\mathbb{E}\Big[X\Big] = (t + \Delta t) - t = \Delta t$$

and

$$V\Big[X\Big] = 2(\Delta t)^2$$

When Δt is close to zero the r.v. X is "not to much random" and is very close to his mean Δt:

$$X = \Big(B_{t+\Delta t} - B_t\Big)^2 \approx \Delta t$$

We write

$$\Big(dB_t\Big)^2 = dB_t dB_t = dt$$

and

$$dB_t = \sqrt{dt}$$

The quadratic variation of the Brownian motion in $[0,T]$ is equal to his variance.

Let $(B_t, t \geq 0)$ be a standard Brownian motion. For each $T > 0$ and partition $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ so that $\pi = \sup_{i \leq n}(t_i^n - t_{i-1}^n)$ goes to zero when $n \to \infty$:

$$\sum_{i=1}^{n} \Big(B(t_i^n) - B(t_{i-1}^n) \Big)^2 \to T,$$

in the sense if the quadratic mean, for $n \to \infty$

Proof

$$\mathbb{E}\left[\sum_{i=1}^{n} \Big(B(t_i^n) - B(t_{i-1}^n) \Big)^2 \right] = T$$

The random variables $\Big(B(t_i^n) - B(t_{i-1}^n) \Big)^2$, $i = 1, 2, \ldots, n$ are indipendent.

$$\begin{aligned} \mathrm{Var}\left[\sum_{i=1}^{n} \Big(B(t_i^n) - B(t_{i-1}^n) \Big)^2 \right] &= \sum_{i=1}^{n} \mathrm{Var}\left[\Big(B((t_i^n) - B(t_{i-1}^n) \Big)^2 \right] \\ &= 2 \sum_{i=1}^{n} \left(t_i^n - t_{i-1}^n \right)^2 \leq 2\pi T. \end{aligned}$$

This variance goes to 0 when $n \to \infty$.

Stochastic integral

Consider the stochastic integral

$$\int_0^T f(t, B_t)dB_t.$$

We can define $X_t = \int_0^T f(t, B_t)dB_t$ as the limit of discrete sums of the type

$$X_n = \sum_{j=0}^{n-1} f(t_j^n, B_{t_j^n})(B_{t_{j+1}^n} - B_{t_j^n}),$$

as n goes to infinity.

When can think X_n as a "Riemann sum" in which the representative point inside each subinterval is the left-most point.

This definition of the stochastic integral is called the Ito integral.

Of course, conditions on f are necessary to ensure that X_n converge in a reasonable sense and that the limit does not depende on the sequence on meshes t_i^n.

Example

$$\int_0^T dB_s = B_T$$

Example

$$\int_0^T B_s dB_s = -\frac{1}{2}T + \frac{1}{2}B_T^2$$

$$
\begin{aligned}
\int_0^T B_s \, dB_s &= \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{t_j} \cdot \left(B_{t_{j+1}} - B_{t_j}\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(B_{t_j} B_{t_{j+1}} - B_{t_j}^2\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(-\frac{1}{2}\left(B_{t_{j+1}} - B_{t_j}\right)^2 - \frac{1}{2} B_{t_j}^2 + \frac{1}{2} B_{t_{j+1}}^2\right) \\
&= \lim_{n\to\infty} \frac{1}{2}\left[-\sum_{i=0}^{n-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 + \sum_{i=0}^{n-1}\left(B_{t_{j+1}}^2 - B_{t_j}^2\right)\right] \\
&= -\frac{1}{2} T + \frac{1}{2} B_T^2.
\end{aligned}
$$

Ito integral property

$$\int_0^T f(t, B_t)dB_t.$$

- Linearity
- Expectation

$$\mathbb{E}\Big[\int_0^T f(t, B_t)dB_t\Big] = 0.$$

- Quadratic mean

$$\mathbb{E}\Big[\Big(\int_0^T f(t, B_t)dB_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T f^2(t, B_t)dt\Big].$$

Stochastic differential equations

Definition A process $(X_t)_{t\geq 0}$ which satisfies

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{1}$$

is called a solution of the stochastic differential equation with coefficient μ and σ, intial condition x and Brownian motion $(B_t)_{t\geq 0}$.

$(X_t)_{t\geq 0}$ is called the diffusion process corresponding to the coefficients μ and σ. We can write the differential simbolic notation

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x. \end{cases}$$

Example

The standard Brownian motion, the Brownian motion with drift and the geometric Brownian motion are solution of particular s.d.e.

Example : Brownian motion with drift

The Brownian motion with drift is solution of the following s.d.e.

$$\begin{cases} dX_t = \mu dt + \sigma dB_t \\ X_0 = x. \end{cases}$$

It is the diffusion process corresponding to the coefficients $\mu(t, X_t) = \mu$ and $\sigma(t, X_t) = 1$.

Stochastic Integral

$$X_t = x + \int_0^t \mu ds + \int_0^t \sigma dB_s$$

The solution is

$$X_t = x + \mu t + \sigma B_t$$

Example : Geometric Brownian motion

The Geometric Brownian motion with drift is solution of the following s.d.e.

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_0 = x. \end{cases}$$

Stochastic Integral

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u \tag{2}$$

The solution is

$$S_t = x e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

The results is obtained using the Ito's Lemma.

Ito's Lemma

Lemma

Let $(X_t)_{t\geq 0}$ *the solution of*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$
$$X_0 = x_0$$

and let $f(t, X_t)$ *be a real-valued function of class* $C^{1,2}$*.*
Then

$$df(t, X_t) = \left(\frac{\partial f(t,X_t)}{\partial t} + \mu(t, X_t)\frac{\partial f(t,X_t)}{\partial X_t} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f(t,X_t)}{\partial X_t^2}\right)dt + \sigma(t, X_t)\frac{\partial f(t,X_t)}{\partial X_t}dB_t$$

We can write

$$df(t, X_t) = \alpha(t, X_t)dt + \frac{\partial f}{\partial X_t}dX_t$$

with

$$\alpha(t, X_t) = \frac{\partial f}{\partial t} + \frac{\sigma^2(t, X_t)}{2}\frac{\partial^2 f}{\partial X_t^2}$$

Example

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$
$$S_0 = x$$

Using Ito's lemma

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Let us consider $X_t = B_t$ and

$$S_t = f(t, B_t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Ito's lemma implies that

$$dS_t = df(t, B_t) = \Big((\mu - \frac{1}{2}\sigma^2)S_t + \frac{1}{2}\sigma^2 S_t\Big)dt + \sigma S_t dB_t = \mu S_t dt + \sigma S_t dB_t$$

On the contrary, let is consider $X_t = S_t$ and

$$f(t, S_t) = \ln(S_t)$$

Ito's lemma implies that

$$dln(S_t) = df(t, S_t) \quad = \quad (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$$

or in the integral form

$$\begin{aligned}
\int_0^T 1 \; dln(S_t) &= \int_0^T (\mu - \frac{1}{2}\sigma^2) \; dt + \int_0^T \sigma \; dB_t \\
[ln(S_t)]_0^T &= (\mu - \frac{1}{2}\sigma^2) \; [t]_0^T + \sigma \; [B_t]_0^T \\
ln(\frac{S_T}{S_0}) &= (\mu - \frac{1}{2}\sigma^2)T + \sigma(B_T - B_0) \\
\frac{S_T}{S_0} &= \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T\right]
\end{aligned}$$

Then

$$S_T = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T .\right] \tag{3}$$

Example

Consider $f(t,x) = x^2$ and $X_t = B_t$. Then

$$f(t, B_t) = B_t^2$$

Ito's lemma implies that

$$dB_t^2 = df(t, B_t) = \left(\frac{1}{2}2\right)dt + 2B_t dB_t = dt + 2B_t dB_t$$

In the integral form

$$B_t^2 = B_0^2 + \int_0^t \frac{1}{2}2du + \int_0^t 2B_u dB_u$$

so that

$$\int_0^t B_u dB_u = \frac{1}{2}(B_t^2 - t).$$

Theorem (Existence and Uniqueness)

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{4}$$

If μ and σ are continuous functions, and if there exists a constant $K < +\infty$, such that :

1. $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$
2. $|\mu(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$

then, for any $T \geq 0$, (4) admist a unique solution in the interval $[0,T]$.
Moreover, this solution $(X_s)_{0\leq s\leq T}$ satisfies :

$$\mathbb{E}\left(\sup_{0\leq s\leq T} |X_s|^2\right) < +\infty$$

Simulation diffusions paths

Euler Discretization Scheme

$$\Delta X_t = X_{t+\Delta T} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)\Delta B_t$$
$$X_0 = x_0$$

Start $t_0 = 0,\ x_0,\ \Delta T = \frac{T}{N}$

for $k = 1, \ldots, N$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0, 1)$;

$x_{k\Delta T} = x_{(k-1)\Delta T} + \mu(x_{(k-1)\Delta T}, t_{k-1})\Delta T + \sigma(x_{(k-1)\Delta T}, t_{k-1})g\sqrt{\Delta T}$

END;

Brownian motion

Definition

A Brownian motion is a real-valued, continuous stochastic process $(X_t)_{t\geq 0}$ with indipendent, normally distribuited and stationary increments. In other words :

- $B_0 = 0$.
- continuity.
- *indipendent increments* : if $s \leq t$, $B_t - B_s$ is indipendent of $\mathcal{F}_s = \sigma(B_u, u \leq s)$.
- *stationary increments* : if $s \leq t$, $B_t - B_s$ and B_{t-s} have the same law.

Continuous-time martingale

Let us consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $\mathcal{F} := (\mathcal{F}_t, t \geq 0)$ on this space.

Definition An adapted family $(M_t, t \geq 0)$ of integrable random variables is a $(\mathcal{F}_t)$-martingale if for each $s \leq t$,

$$\mathbb{E}\,(M_t | \mathcal{F}_s) = M_s\,.$$

It follows from the definition that if $(M_t)_{t \geq 0}$ is a martingale, then $\mathbb{E}(M_t) = \mathbb{E}(M_0)$, for each t.

Example

B_t is an $\mathcal{F}_t$-martingale.

Markov property

The intuitive meaning of the Markov property is that the future behaviour of the process $(X_t)_{t\geq 0}$ after t depends only on the value X_t and is not influenced by the history of the process before t.

Mathematically speaking, $(X_t)_{t\geq 0}$ satisfies the Markov property if, for any function f bounded and measurable and for any s and t, such that $s \leq t$, we have :

$$\mathbb{E}\left(f\left(X_t\right)|\mathcal{F}_s\right) = \mathbb{E}\left(f\left(X_t\right)|X_s\right).$$

This property is satisfied for a solution of the equation (4).

This is a crucial property of the Markovian model and it will have great conseguences in the pricing of options.

Black-Scholes model

- Risk-free asset

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ S_0^0 = 1. \end{cases}$$

- Risk asset

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_0 = x. \end{cases}$$

  with $(B_t)_{t\geq 0}$ standard brownian motion under the historical probability $\mathbb{P}$.

- The short-term interest rate is known and is costant through time.
- The stock pays no dividends or other distributions.
- There are no penalties to short-selling.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-time interest rate.
- Absence of arbitrage opportunities.

Financial interpretation of the parameters

- r istantaneous interest rate : $[0\%, 12\%]$

- μ expected return of the risky asset.

$$\mathbb{E}(\frac{S_t}{S_0}) = e^{\mu t}$$

- σ is the volatility σ.
  This is vey important parameters : $[30\%, 70\%]$ in the equity market.

- risk premium λ

$$\lambda = \frac{\mu - r}{\sigma}$$

  Then

$$\mu = r + \lambda\sigma$$

  The expected return μ of the risky asset is the sum of the return of the no-risky asset plus something proportional to σ.

We can write

$$dS_t = rS_t dt + \sigma S_t (dB_t + \lambda dt)$$

The Girsanov theorem gives

$$d\widehat{B}_t = dB_t + \lambda dt$$

with $\widehat{B}_t$ standard Brownian motion under the risk neutral probability Q.

Dinamics under the risk neutral probability measure

$$(5) \qquad \begin{cases} dS_t = rS_t dt + \sigma S_t d\widehat{B}_t \\ S_0 = x. \end{cases}$$

with $\widehat{B}_t$ standard Brownian motion under Q.
The solution(5) is

$$S_t = xe^{(r-\frac{1}{2}\sigma^2)t+\sigma\widehat{B}_t}$$

Then

$$\mathbb{E}_Q\left(\frac{S_T}{S_t}|\mathcal{F}_t\right) = e^{r(T-t)}$$

Radon-Nikodyn Theorem

Let $\mathbb{P}$ and Q be two probabilty measure on $(\Omega, \mathcal{F})$
If Q is absolutely continuous with respect to $\mathbb{P}$, $(A \in \mathcal{F}, \mathbb{P}(A) = 0 \rightarrow Q(A) = 0)$, then there existe a unique r.v. $X \geq 0$, $\mathcal{F}$-misurable such that

$$Q(A) = \int_A X \, d\mathbb{P}$$

The random variable X is commonly written as

$$\frac{dQ}{d\mathbb{P}} = X$$

X is called the Radon-Nikodyn derivative.

Change of probability measure in the Gaussian case

Let use consider $Z \sim N(\mu, 1)$ under $\mathbb{P}$.
Then there exists Q so that $Z(0,1)$ under Q where

$$dQ = e^{-\mu Z + \frac{1}{2}\mu^2} d\mathbb{P}.$$

In fact

$$\mathbb{P}(Z \leq z) = \int_{\{\omega : Z(\omega) \leq z\}} d\mathbb{P}(\omega) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2}} dx,$$

and

$$Q(Z \leq z) = \int_{\{\omega : Z(\omega) \leq z\}} dQ(\omega) = \int_{\{\omega : Z(\omega) \leq z\}} e^{-\mu Z(\omega) + \frac{1}{2}\mu^2} d\mathbb{P}(\omega) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

Moreover it holds

$$\mathbb{E}_{\mathbb{P}}\Big[f(Z)\Big] = \mathbb{E}_Q\Big[f(Z)e^{\mu Z - \frac{1}{2}\mu^2}\Big] \quad and \quad \mathbb{E}_Q\Big[f(Z)\Big] = \mathbb{E}_P\Big[f(Z)e^{-\mu Z + \frac{1}{2}\mu^2}\Big].$$

Girsanov's Theorem

Let B_t be a Brownian motion under $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\mathcal{F}_t$.
Let $(Z_t)_{0 \leq t \leq T}$ be the process defined by :

$$Z_t = \exp\Big(- \lambda B_t - \frac{1}{2}\lambda^2 t\Big).$$

Then, under the probability measure Q with density Z_T with respect to $\mathbb{P}$

$$dQ = Z_T d\mathbb{P}$$

the process $(\widehat{B}_t)_{0 \leq t \leq T}$ given by $\widehat{B}_t = B_t + \lambda t$, is a standard Brownian motion under Q.

Risk neutral pricing formula

$$\mathbb{E}_Q \left(\frac{S_T}{S_t} | \mathcal{F}_t \right) = e^{r(T-t)}$$

This holds for each asset:

$$\mathbb{E}_Q \left[\frac{V_T}{V_t} | \mathcal{F}_t \right] = e^{r(T-t)}$$

Equivalently

$$V_t = \mathbb{E}_Q \left(e^{-r(T-t)} V_T | \mathcal{F}_t \right).$$

The price of a contingent claim is the expected value of the discounted payoff.

$$e^{-rt} V_t = \mathbb{E}_Q \left(e^{-rT} V_T | \mathcal{F}_t \right).$$

Discounted prices are martingales.

Black-Scholes formula for European Call options

The price at time t of an European Call option in the Black-Scholes model

$$C_t = \mathbb{E}_Q\left(e^{-r(T-t)}C_T|\mathcal{F}_t\right) = \mathbb{E}_Q\left(e^{-r(T-t)}(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} - K)_+\right)$$

is given by

$$C_t = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

with

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and $N(x)$ the distribution function of the standard Gaussian variable

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}dx.$$

Black-Scholes formula for European Put options

The price at time t of an European put option in the Black-Scholes model

$$P_t = \mathbb{E}_Q\left(e^{-r(T-t)}P_T|\mathcal{F}_t\right) = \mathbb{E}_Q\left(e^{-r(T-t)}(K - S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)})_+\right)$$

is given by

$$P_t = Ke^{-r(T-T)}N(-d_2) - S_t N(-d_1)$$

Implementation of the formula

The price of the call option depends on six parameters.

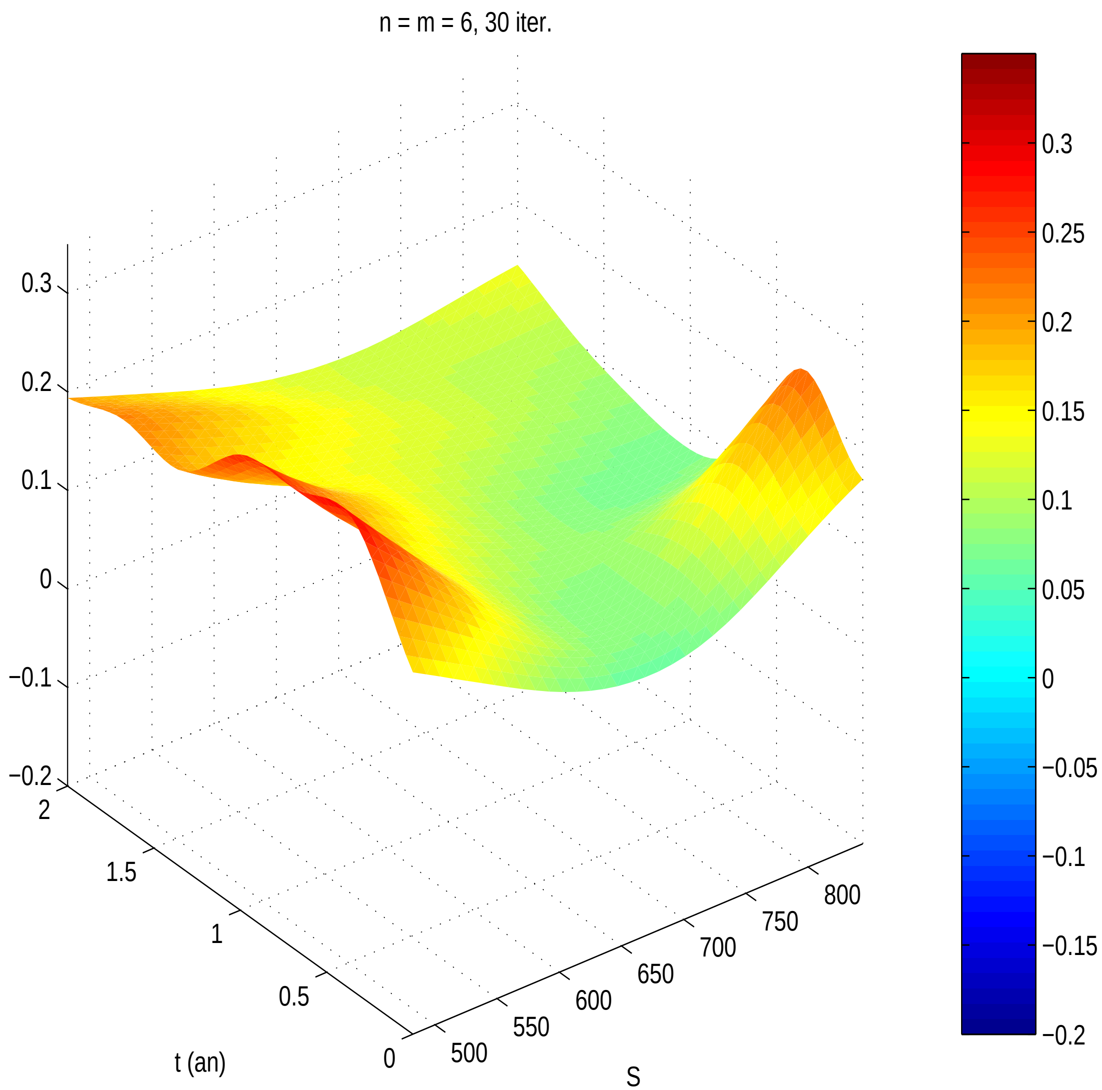
$$C = C(S_t = x, t, T, K, \sigma, r)$$

- The strike K and the maturity T are specified in the contract.
- r is constant. But in general this is not true (Vasicek or CIR model).

- The volatility cannot be observed directly. In practice, two methods are used to evaluate σ
  – The historical method: in the BS model, $\sigma^2 T$ is the variance of $\log(S_T)$ and the variables $\log(S_{\Delta T}/S_0)$, $\log(S_2/S_{\Delta T})$, $\ldots$, $\log(S_{N\Delta T}/S_{(N-1)\Delta T})$ are i.i.d random variables.
    Therefore, σ can be estimated by statistical means using past observations of the asset price.
  – the "implied volatility" method: some options are quoted on organized markets; the price of options being an increasing function of σ, we can associate an "implied" volatility to each quoted option, by inversion of the Black-Scholes formula.

$$C^{Obs}(S_0, 0, T, K) = C(S_0, 0, T, K, \Sigma(K, T), r)$$

Σ is called implied volatility. Due to the market imperfections Σ has a typical dependence on K called SMILE EFFECT.



n = m = 6, 30 iter.
0.3
0.2
0.1
0
−0.1
−0.2
2
1.5
1
0.5
0
t (an)
500
550
600
650
700
750
800
S
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2

e

Approximating the distribution function of $g \sim N(0,1)$

Set $t = \frac{1}{1+px}$, then:

$$N(x) = \begin{cases} 1 - \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) & \text{if} \quad x \geq 0 \\ \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) & \text{if} \quad x < 0 \end{cases}$$

with the following constants :

$$\begin{aligned} p &= 0.2316419; \\ b_1 &= 0.319381530; \\ b_2 &= -0.356563782; \\ b_3 &= 1.781477937; \\ b_4 &= -1.821255978; \\ b_5 &= 1.330274429; \end{aligned}$$

Approximating the distribution function of $g \sim N(0,1)$

```
double N(double x)
{ const double p= 0.2316419;
  const double b1= 0.319381530;
  const double b2= -0.356563782;
  const double b3= 1.781477937;
  const double b4= -1.821255978;
  const double b5= 1.330274429;
  const double one_over_twopi= 0.39894228;
  double t;
  if(x >= 0.0)
    {
      t = 1.0 / ( 1.0 + p * x );
      return (1.0 - one_over_twopi * exp( -x * x / 2.0 )
* t * ( t *( t * ( t * ( t * b5 + b4 ) + b3 ) + b2 ) + b1 ));
    }
  else
    {/* x < 0 */
      t = 1.0 / ( 1.0 - p * x );
      return ( one_over_twopi * exp( -x * x / 2.0 ) *
       t * ( t *( t * ( t * ( t * b5 + b4 ) + b3 ) + b2 ) + b1 ));
    }
}
```

Scilab

```
function [y]=Norm(x)
  [y,Q]=cdfnor("PQ",x,0,1);
endfunction
```

Price of a Call option

```
main()
{
    double sigmasqrt,d1,d2,delta,price;

    sigmasqrt=sigma*sqrt(t);
    d1=(log(s/k)+r*t)/sigmasqrt+sigmasqrt/2.;
    d2=d1-sigmasqrt;
    delta=N(d1);
    /*Price*/
    price=s*delta-exp(-r*t)*k*N(d2);
}
```

Price of a put option

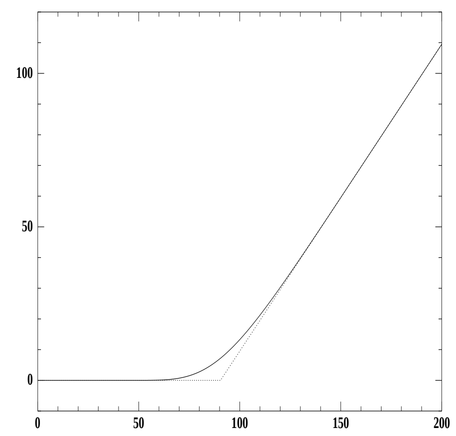
```
main()
{
    double sigmasqrt,d1,d2,delta,price;

    sigmasqrt=sigma*sqrt(t);
    d1=(log(s/k)+r*t)/sigmasqrt+sigmasqrt/2.;
    d2=d1-sigmasqrt;
    delta=N(-d1);

    /*Price*/
    price=exp(-r*t)*k*N(-d2)-delta*s;
}
```

Put-Call Theorem Parity

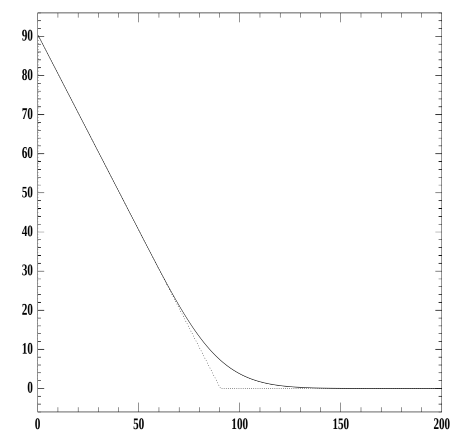
We have the following put-call parity between the prices of the underlying asset S_t and European call and put options on stocks that pay no dividends:

$$C_t = P_t + S_t - Ke^{-r(T-t)}.$$

Payoff Call



Payof Put



Proof of the Black-Scholes formula

$$C(t,x) = E_Q\left[e^{-r(T-t)}(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)} - K)_+\right]$$

Then

$$C(t,x) = E_Q\left[\left(xe^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}g} - Ke^{-r(T-t)}\right)_+\right]$$

with $g \sim N(0,1)$.
Set

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\begin{aligned} C(t,x) &= \mathbb{E}\left[\left(xe^{\sigma\sqrt{(T-t)}g-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right)\mathbf{1}_{g\geq -d_2}\right] \\ &= \int_{-d_2}^{+\infty}\left(xe^{\sigma\sqrt{(T-t)}y-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \\ &= \int_{-\infty}^{d_2}\left(xe^{-\sigma\sqrt{(T-t)}y-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy. \end{aligned}$$

$$C(t,x) = \int_{-\infty}^{d_2} \left(x e^{-\sigma\sqrt{(T-t)}y - \sigma^2 (T-t)/2} - K e^{-r(T-t)} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

The change of variable $z = y + \sigma\sqrt{(T-t)}$, gives :

$$C(t,x) = xN(d_1) - Ke^{-r(T-t)} N(d_2),$$

with :

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx.$$

Monte Carlo method in the Black-Scholes model

We want compute

$$P = e^{-rT}\mathbb{E}_Q\left[(K - S_T)_+\right].$$

in the Black-Scholes model

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T}$$

The payoff function can be written in the following way

$$h(B_T) = (K - S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T})_+$$

We can approximate the price with

$$P \approx e^{-rT}\frac{h(X_1) + \cdots + h(X_n)}{n}$$

$X_1, .., X_n \sim N(0, T)$.

Monte Carlo algorithm

European Put in the Black-Scholes model

```
main()
{
  double mean_price,mean2_price,brownian,price,price_sample,error_price,inf_price,sup_price;
  mean_price= 0.0;
  mean2_price= 0.0;
  for(i=1;i<=N;i++)
    {

      brownian=gaussian()*sqrt(T);

      price_sample=MAX(0.0,K-x*exp((r-0.5*sigma*sigma)*T+sigma*brownian));

      mean_price= mean_price+price_sample;
      mean2_price= mean2_price+SQR(price_sample);
    }
  /* Price */
  price=exp(-r*T)*(mean_price/N);
  error_price= sqrt(exp(-2.*r*T)*mean2_price/N - SQR(price))/sqrt(N-1);
  inf_price= price - 1.96*(error_price);
  sup_price= price + 1.96*(error_price);
}
```