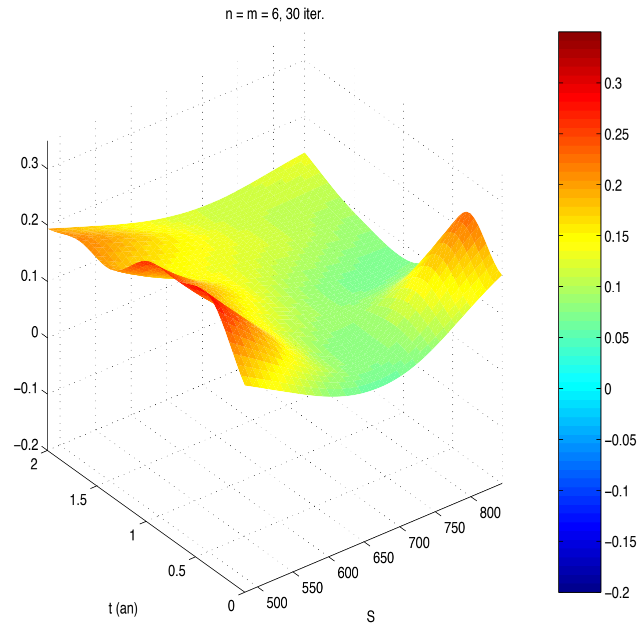
Financial Mathematics 2
Numerical methods in Finance

University of Ljubljana



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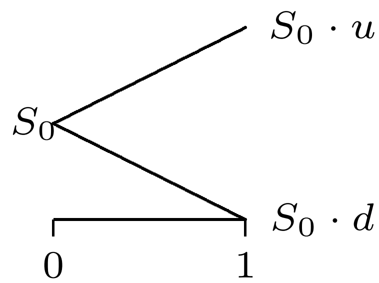
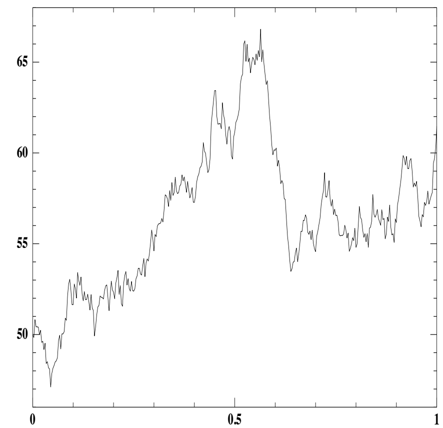
Pricing and hedging methods for derivatives

- *Cox-Ross-Rubinstein discrete model*.



- *Black-Scholes continuous model*



- Numerical methods Tree methods, Monte Carlo methods, Finite Difference methods.

Plan

1. Cox-Ross-Rubinstein model. Pricing and Delta hedging in discrete models. Markov chains. Dynamic programming equations. European and American options in CRR model.
2. Monte Carlo Methods. Simulation methods of classical law. Inverse transform method. Central Limit Theorem. Computation of expectation. Variance reduction techniques (Control Variate, Importance sampling).
3. Geometric brownian motion. Ito's Lemma. Black-Scholes model. Monte Carlo Methods for European options.
4. Greeks. Estimating sensitivities. Dynamic hedging in the Black-Scholes continuous model. Numerical algorithms for portfolio insurance.
5. Tree methods for European and American options. Convergence orders of binomial methods.
6. Monte Carlo methods for Exotic options (Barrier options, Asian options, Lookback options, Rainbow options).
7. Tree methods for exotic options. The Ritchken method. The forward shooting grid methods. The singular points method.
8. Monte Carlo Methods for American options. The Longstaff-Schwartz method.
9. Finite difference methods for the heat equation and the Black-Scholes partial differential equation. Explicit Scheme. Implicit scheme. Cranck-Nicolson scheme. Consistency and stability of the schemes.
10. Matlab sessions with the implementation of the proposed numerical algorithms.

Teaching Dates

- 5-6 April
- 12-13 April
- 19-20 April
- 24-25 April
- 3-4 May

Teaching Material

- Slides of the course.
- J.Hull Options, Futures, and Other Derivatives. Prentice Hall
- N.H. Bingham R. Kiesel. Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives. Springer Finance
- P.Glasserman. Monte Carlo methods in Financial Engineering. Springer

Examination

- The final assessment will require the solution of exercises on topics examined during lessons.
- 10 May Written Examination.
- 11 May Discussion of the Written Examination.

Financial options

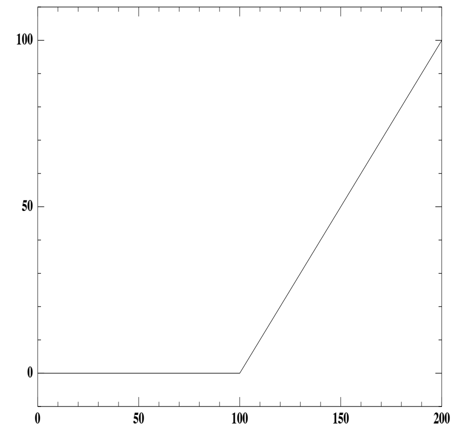
European Call options A Call option is a financial instrument giving the right (but not to the obbligation) to the owner to buy the underlying asset at a given price (called strike) at prefixed date (called maturity).
The writer will have an obbligation to sell at these conditions.
Because of the asymmetry of the contract :

- the owner of the option has to pay to the writer the prime of the option.
- the writer will provide to the owner $\max(0, S_T - K)$ at maturity.
  The quantity $\max(0, S_T - K)$ is called the payoff of the option.

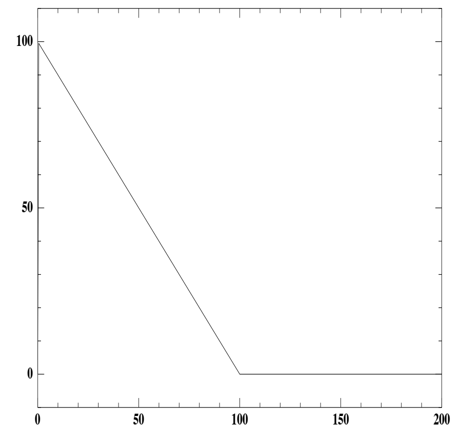
European Put options A Put option is a financial instrument giving the right (but not to the obbligation) to the owner to sell the underlying asset at a given price (called strike) at prefixed date (called maturity).

The payoff of the option is now $\max(0, K - S_T)$.

Payoff Call



Payoff Put



Pricing of financial options

What is the fair price of these financial derivatives products?
The problem of the evaluation of this contingent claim is the problem of the evaluation of a random variable $G > 0$ received at maturity.
The main message of Black-Scholes-Merton(1973) is that the fair price of a financial derivative is the price obtained using a hegding procedure under absence of arbitrage opportunities (AOA).
We will study numerical methods for two models :

- Discrete model of *Cox-Ross-Rubinstein*, based on Markov chains.
- Continous model of *Black-Scholes*, based on continous stochastic process.

Market hypothesis

- The short-term interest rate is known and is constant through time.
- The stock pays no dividends or other distributions.
- There are no penalties to short-selling.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-time interest rate.
- Absence of arbitrage opportunities. The absence of risk-free plans for making profits without any investements

Put-Call Theorem Parity

We have the following put-call parity between the prices of the underlying asset S_t and European call and put options on stocks that pay no dividends:

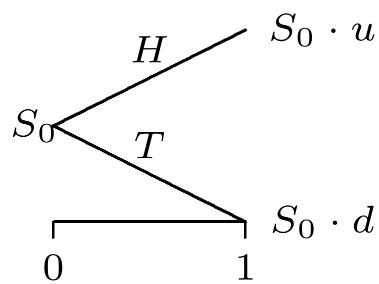
$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

.

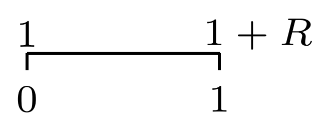
Cox-Ross-Rubinstein model

- Risky asset *H1*: $0 < d < 1 < u$.
  Let us immagine that we are tossing a coin.
  When we get “Head”, the stock price moves up.
  When we get a “Tail”, the price moves down. Consider $\Omega = \{H, T\}$, $\omega \in \Omega$.

$$S1(\omega) = \begin{cases} S_1(H) = S_0 u \\ S_1(T) = S_0 d \end{cases}$$



- Risk-free asset *H2*: $d < 1 + R < u$, R annual interest rate.



Let us consider an European call option with strike K e maturity 1.

$$V_1(\omega) = \begin{cases} \max\{0, S_0 u - K\} = (S_0 u - K)_+ & \text{if} \quad \omega = H \\ \max\{0, S_0 d - K\} = (S_0 d - K)_+ & \text{if} \quad \omega = T \end{cases}$$

Example $S_0 = 50,\ u = 1.1,\ d = 0.9,\ K = 50$

$$V_1(\omega) = \begin{cases} (55 - 50)_+ = 5 & \text{if} \quad \omega = H \\ (45 - 50)_+ = 0 & \text{if} \quad \omega = T \end{cases}$$

Replicating portfolio

The seller of the option at time 1 have to pay

$$V_1(\omega) = \begin{cases} (S_0 u - K)_+ & \text{if} \quad \omega = H \\ (S_0 d - K)_+ & \text{if} \quad \omega = T \end{cases}$$

How to compute V_0, the arbitrage price of this options at time zero?

Idea : Dynamic hedging using a portfolio $(\alpha, \beta) \in \mathbb{R}^2$ where

- α the quantity invested in the risky asset at time zero.
- β the quantity invested in the money market at time zero.

The value of the portfolio at time 0 is given by:

$$\widehat{V_0} = \alpha S_0 + \beta \Rightarrow \beta = \widehat{V_0} - \alpha S_0$$

For hedging purposes we need

$$\widehat{V_1}(\omega) = V_1(\omega)$$

No-Arbitrage conditions requires that

$$\widehat{V_0} = V_0$$

The value of the portfolio at time 1 is given by:

$$\begin{cases} (*) \quad \alpha S_1(H) + \beta(1+R) = V_1(H) \\ \alpha S_1(T) + \beta(1+R) = V_1(T) \end{cases}$$

Solving the system in the unknown variables V_0, α, β :

$$\widehat{\alpha} = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$
$$\widehat{\beta} = V_0 - \widehat{\alpha} S_0$$

Now we can compute V_0.
From (*) we have:

$$\widehat{\alpha} S_1(H) + (V_0 - \widehat{\alpha} S_0)(1+R) = \widehat{\alpha} S_1(H) + V_0(1+R) - \widehat{\alpha} S_0(1+R) = V_1(H)$$

$$V_0 = \frac{1}{(1+R)}\Big[V_1(H) - \widehat{\alpha} S_1(H) + \widehat{\alpha} S_0(1+R)\Big] = \frac{1}{(1+R)}\Big[V_1(H) + \widehat{\alpha}(S_0(1+R) - S_0 u)\Big]$$

Then

$$V_0 = \frac{1}{(1+R)}\Big[V_1(H) + \widehat{\alpha}(S_0(1+R) - S_0 u)\Big]$$

$$= \frac{1}{(1+R)}\Big[V_1(H) + \frac{V_1(H) - V_1(T)}{S_0 u - S_0 d}(S_0(1+R) - S_0 u)\Big]$$

$$= \frac{1}{(1+R)}\Big[\frac{(u-d)V_1(H) + (1+R)V_1(H) - uV_1(H) - (1+R)V_1(T) + uV_1(T)}{(u-d)}\Big]$$

$$= \frac{1}{(1+R)}\Big[\frac{((1+R)-d)}{(u-d)}V_1(H) + \frac{(u-(1+R))}{(u-d)}V_1(T)\Big]$$

Consider

$$q = \frac{(1+R)-d}{u-d}$$

and

$$\widehat{q} = \frac{u-(1+R)}{u-d}$$

Risk-neutral pricing formula

$$V_0 = \frac{1}{(1+R)}\Big[qV_1(H) + \widehat{q}V_1(T)\Big] = \mathbb{E}_q\Big[\frac{1}{(1+R)}V_1\Big] \tag{1}$$

Oss

Recall the hypothesis ($H2 : d < 1 + R < u$).

Therefore

$$q = \frac{(1+R)-d}{u-d} > 0 \qquad \widehat{q} = \frac{u-(1+R)}{u-d} > 0$$

$$q + \widehat{q} = 1$$

q is called the risk-neutral probability.

Oss

We did not define a probability measure.

Oss

The pricing formula (7) holds for each derivatives.

Risk-neutral valuation formula

$$V_0 = \mathbb{E}_q \left[\frac{1}{(1+R)} V_1 \right] \tag{2}$$

$$\mathbb{E}_q \left[\frac{V_1}{V_0} \right] = (1+R) \tag{3}$$

Oss

The price of a contingent claim is the expected value of the discounted payoff with respect to an equivalent martingale measure.

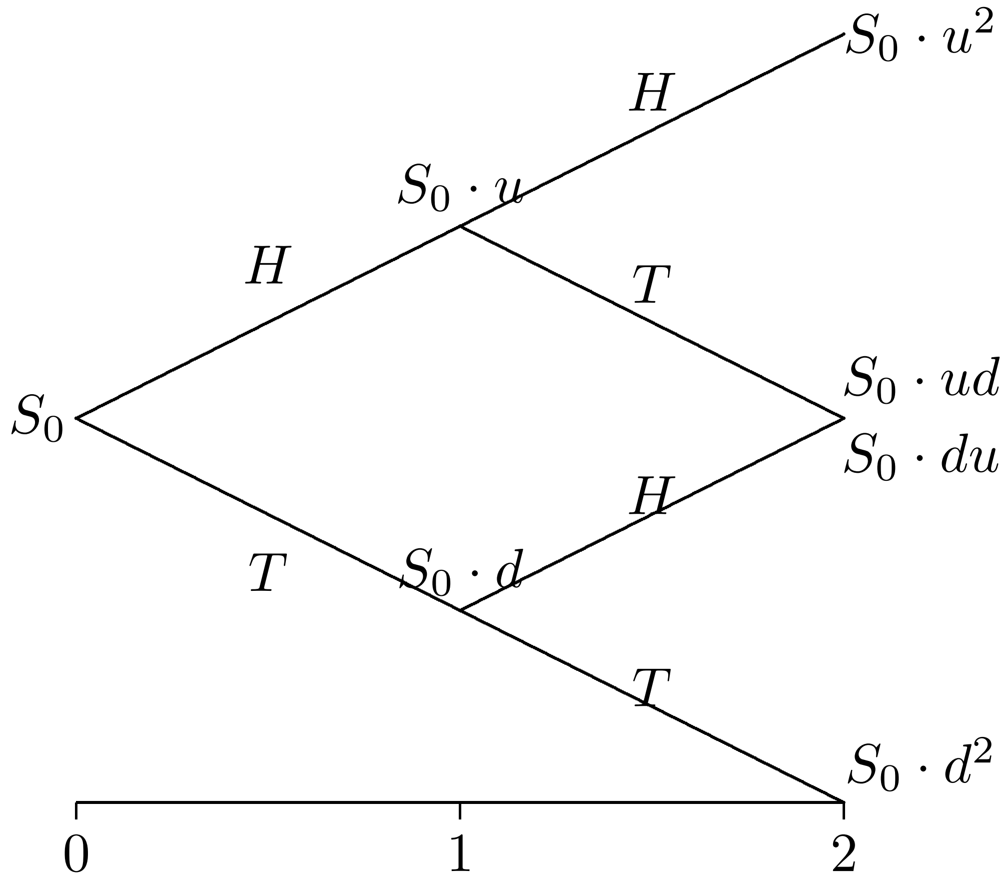
The expected return of each contingent claim is equal to the return of the risk-free asset.

Two periods CRR model

- Risky asset *H1*: $0 < d < 1 < u$.
  Consider $\Omega = \{HH, HT, TH, TT\}$, $\omega \in \Omega$ $\omega = (\omega_1, \omega_2)$.
  The asset price at time 2 is given by

$$\begin{cases} S_2(HH) = S_0 u^2 \\ S_2(HT) = S_2(TH) = S_0 ud \\ S_2(TT) = S_0 d^2 \end{cases}$$

- Risk-free asset *H2*: $d < 1 + R < u$.


$S_0 \cdot u^2$
H
$S_0 \cdot u$
H
T
S_0
$S_0 \cdot ud$
$S_0 \cdot du$
H
T
$S_0 \cdot d$
T
$S_0 \cdot d^2$
0
1
2

Let us consider an European Call option with strike K e maturity 2.
The value of the option at time 2 is given by:

$$V_2(\omega) = \begin{cases} (S_0u^2 - K)_+ & \text{if} \quad \omega = HH \\ (S_0ud - K)_+ & \text{if} \quad \omega = HT \quad or \quad \omega = TH \\ (S_0d^2 - K)_+ & \text{if} \quad \omega = TT \end{cases}$$

Example

$S_0 = 45.454545,\ u = 1.1,\ d = 0.9,\ K = 40$

$$V_2(\omega) = \begin{cases} (55 - 40)_+ = 15 & \text{if} \quad \omega = HH \\ (45 - 40)_+ = 5 & \text{if} \quad \omega = HT \quad or \quad \omega = TH \\ (36.81 - 40)_+ = 0 & \text{if} \quad \omega = TT \end{cases}$$

Dynamic hedging

The value of the portfolio at time 0 is given by:

$$\widehat{V_0} = \alpha_0 S_0 + \beta_0 \Rightarrow \beta_0 = \widehat{V_0} - \alpha S_0$$

For hedging purposes we need

$$\widehat{V_2}(\omega) = V_2(\omega)$$

No-Arbitrage conditions requires that

$$\widehat{V_0} = V_0 \qquad \widehat{V_1} = V_1$$

The value of the portfolio at time 1 is given by:

(4)

$$\begin{cases} (4.1) \quad \widehat{V_1}(H) = \alpha_0 S_1(H) + (V_0 - \alpha S_0)(1+R) = \alpha_0 S_0 u + (V_0 - \alpha_0 S_0)(1+R) =_{AOA} V_1(H) \\ \text{if} \quad \omega_1 = H \\ (4.2) \quad \widehat{V_1}(T) = \alpha_0 S_1(T) + (V_0 - \alpha S_0)(1+R) = \alpha_0 S_0 d + (V_0 - \alpha_0 S_0)(1+R) =_{AOA} V_1(T) \\ \text{if} \quad \omega_1 = T \end{cases}$$

Then $\widehat{V_1}$ depends on ω_1 the outcome of first coin toss.

Now

$$\widehat{V_1} = \alpha_1 S_1 + \beta_1 \Rightarrow \beta_1 = \widehat{V_1} - \alpha S_1$$

where α_1, β_1, S_1 depends on ω_1.

Rebalancing the portfolio

The value of the portfolio $(\widehat{V_2})$ at time 2 is given by:

$$(5) \quad \begin{cases} (5.3) \quad \widehat{V_2}(HH) = \alpha_1(H)S_2(HH) + (V_1(H) - \alpha_1(H)S_1(H))(1+R) = V_2(HH) \\ \text{if} \quad \omega_1 = H \quad and \quad \omega_2 = H \\ (5.4) \quad \widehat{V_2}(HT) = \alpha_1(H)S_2(HT) + (V_1(H) - \alpha_1(H)S_1(H))(1+R) = V_2(HT) \\ \text{if} \quad \omega_1 = H \quad and \quad \omega_2 = T \\ (5.5) \quad \widehat{V_2}(TH) = \alpha_1(T)S_2(TH) + (V_1(T) - \alpha_1(T)S_1(T))(1+R) = V_2(TH) \\ \text{if} \quad \omega_1 = T \quad and \quad \omega_2 = H \\ (5.6) \quad \widehat{V_2}(TT) = \alpha_1(T)S_2(TT) + (V_1(T) - \alpha_1(T)S_1(T))(1+R) = V_2(TT) \\ \text{if} \quad \omega_1 = T \quad and \quad \omega_2 = T \end{cases}$$

From (5.5)-(5.6) it follows

$$\alpha_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}$$

substituiting this into (5.5)

$$V_1(T) = \frac{1}{(1+R)}\Big[qV_2(TH) + \widehat{q}V_2(TT)\Big]$$

From (5.3)-(5.4) it follows

$$\alpha_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

substituiting this into (5.3)

$$V_1(H) = \frac{1}{(1+R)}\Big[qV_2(HH) + \widehat{q}V_2(HT)\Big]$$

From (4.1) and (4.2) it follows

$$\alpha_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

$$V_0 = \frac{1}{(1+R)}\Big[qV_1(H) + V_1(T)\Big]$$

Risk-neutral pricing formula

$$(6) \qquad V_0 = \frac{1}{(1+R)^2}\left[q^2 V_2(HH) + q\widehat{q}V_2(TH) + q\widehat{q}V_2(HT) + \widehat{q}^2 V_2(TT)\right] = \mathbb{E}_q\left[\frac{1}{(1+R)^2}V_2\right]$$

Oss

The price of a contingent claim is the expected value of the discounted payoff with respect to an equivalent martingale measure.

n periods CRR model

- Risky asset H1: $0 < d < 1 < u$.
  $\omega \in \Omega$ with 2^n, $\omega = (\omega_1, , \omega_n)$.
  Oss If $n = 66$, $2^{66} = 7 \times 10^{19}$, but because of recombining property there are $n + 1$ final nodes.

$$S_k(\omega_1, \ldots, \omega_{k-1}, \omega_k) = \begin{cases} S_{k-1}(\omega_1, \ldots, \omega_{k-1})u & \text{if} \quad \omega_k = H \\ S_{k-1}(\omega_1, \ldots, \omega_{k-1})d & \text{if} \quad \omega_k = T \end{cases}$$

- Risk-free asset H2: $d < 1 + R < u$.

$$S^0_{k+1} = (1 + R)S^0_k$$

$$V_n(\omega) = (S_n(\omega) - K)_+$$

V_k is the value of the option at time k. It holds:

$$V_k(\omega_1, \ldots, \omega_k) = \frac{1}{(1+R)} \Big[q V_{k+1}(\omega_1, \ldots, \omega_k, H) + \widehat{q} V_{k+1}(\omega_1, \ldots, \omega_k, T) \Big]$$

$$\alpha_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}$$

$$\beta_k(\omega_1, \ldots, \omega_k) = V_k(\omega_1, \ldots, \omega_k) - \alpha_k(\omega_1, \ldots, \omega_k) S_k(\omega_1, \ldots, \omega_k)$$

Risk-neutral pricing formula

(7)

$$V_0 = \frac{1}{(1+R)^n} \mathbb{E}_q\Big[V_n\Big] = \frac{1}{(1+R)^n} \mathbb{E}_q\Big[(S_n - K)_+\Big] = \frac{1}{(1+R)^n}\Big[\sum_{h=0}^{n} \binom{n}{h} q^h \widehat{q}^{n-h} (S_0 u^h d^{n-h} - K$$

$$V_h = \mathbb{E}_q\Big[\frac{1}{(1+R)^{n-h}} V_n | \mathcal{F}_h\Big]$$

$$\frac{1}{(1+R)^h} V_h = \mathbb{E}_q\Big[\frac{1}{(1+R)^n} V_n | \mathcal{F}_h\Big]$$

$$\mathbb{E}_q\Big[\frac{V_n}{V_h} | \mathcal{F}_h\Big] = (1+R)^{n-h}$$

Replicating portfolio algortihm

Start $t_0 = 0,\ S_0 = x.$

$V_0 = C(0, x)$

Compute $\alpha_0 = \Big(C(1, x * u) - C(1, x * d)\Big) / \Big(x * u - x * d\Big);$

$\beta_0 = V_0 - S_0 \alpha_0;$

for $k = 1, \ldots, N - 1$

BEGIN;

simulation of S_k;

$V_k = \alpha_{k-1} S_k + \beta_{k-1}(1 + R);$

rebalancing the portfolio;

$\alpha_k = \Big(C(k+1, u * S_k) - C(k+1, S_k * d)\Big) / \Big(S_k * u - S_k * d\Big);$

$\beta_k = V_k - S_k \alpha_k;$

END;

simulation of S_N;

$V_N = \alpha_{N-1} S_N + \beta_{N-1}(1 + R);$

A portfolio V is self-financing if there is no consumption or investment at any time $t > 0$.

Trading strategies

$$V_k = \alpha_k S_k + \beta_k, \quad k = 0, .., n$$

V is self-financing iff

$$V_k = \alpha_k S_k + \beta_k = \alpha_{k-1} S_k + \beta_{k-1}(1 + R)$$

The variation in its value is only due to the variations in the value of the underlying assets.

An arbitrage is a self-financing portfolio such that

$$\begin{cases} V_0 = 0 \\ V_k \geq 0 \\ P(V_N > 0) > 0 \end{cases}$$

Remark The CRR model with $d < (1 + R) < u$ is a complet market: every contigent claim with payoff $G = f(S_N)$ can be replicated perfectly with a self-financing portfolio composed of risky asset and risk-free asset.

A σ-*algebra* is a collection $\mathcal{F}$ of subsets of Ω if:

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(iii) if for any sequence $A_n \in \mathcal{F}$ we have

$$\cup_{n=1}^{\infty} A_n \in \mathcal{F}$$

.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real random variable is a function $X : \Omega \longrightarrow \mathbb{R}$ such that

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

for all Borel sets $B \in \mathcal{B}$. We say that X is $\mathcal{F}$-measurable

Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ algebra of $\mathcal{F}$.

Proposition

Let X a real random variable integrable ($\mathbb{E}(|X|) < +\infty$). Then there exists a random variable Y $\mathcal{G}$-measurable integrable such that for each $G \in \mathcal{G}$

$$\mathbb{E}(X\ \mathbb{1}_G) = \mathbb{E}(Y\ \mathbb{1}_G).$$

The random variable Y is called the conditional expectation and is denoted by

$$\mathbb{E}\left(X|\mathcal{G}\right)$$

Property

1. If X is $\mathcal{G}$-measurable, $\mathbb{E}(X|\mathcal{G}) = X$, a.s.
2. $\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{G}\right)\right) = \mathbb{E}\left(X\right)$.
3. Linearity :
$$\mathbb{E}\left(aX + bY|\mathcal{G}\right) = a\mathbb{E}\left(X|\mathcal{G}\right) + b\mathbb{E}\left(Y|\mathcal{G}\right) \text{ a.s.}$$
4. 'Taking out what is known' :
If Z is $\mathcal{G}$-measurable, $\mathbb{E}\left(ZX|\mathcal{G}\right) = Z\mathbb{E}\left(X|\mathcal{G}\right)$ a.s.
5. Tower property : if $\mathcal{A}$ is a sub-σ algebra of $\mathcal{G}$, then:
$$\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{G}\right)|\mathcal{A}\right) = \mathbb{E}\left(X|\mathcal{A}\right) \text{ a.s.}$$
and
$$\mathbb{E}\left(\mathbb{E}\left(X|\mathcal{A}\right)|\mathcal{G}\right) = \mathbb{E}\left(X|\mathcal{A}\right) \text{ a.s.}$$
6. Best approximation : if Z is $\mathcal{G}$-measurable and square integrable,
$$\mathbb{E}[(X - \mathbb{E}\left(X|\mathcal{G}\right))^2] \leq \mathbb{E}[(X - Z)^2] \text{ a.s.}$$
$\mathbb{E}\left(X|\mathcal{G}\right)$ is the best approximation in least square sense of X using a $\mathcal{G}$- measurable random variable.

Martingale

We recall that a filtration is a sequence of sub σ-algebra of $\mathcal{A}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$ for $m \leq n$.

Wwe consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that is to say a space Ω equipped with a σ-algebra $\mathcal{A}$, a probability $\mathbb{P}$ and a filtration $\mathcal{F} := (\mathcal{F}_n, n \in \mathbb{N})$.

Definition A sequence $(M_n, n \geq 0)$ of $\mathbb{R}$-valued random variables is a $\mathcal{F}$-martingale if

(i) M_n is $\mathcal{F}_n$-measurable for all n,

(ii) $\mathbb{E}(|M_n|) < +\infty$ for all n,

(iii) $\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = M_n$ for all n.

Remark If $(M_n, n \geq 0)$ is a martingale, then

$$\mathbb{E}\left[M_p|\mathcal{F}_n\right] = M_n, \ \forall p \geq n. \tag{8}$$

Markov chain

Let $(S_n, n \geq 0)$ be a sequence of random variables taking values in a finite or countable set $\mathcal{E}$. S_n is a Markov chain if:

$$\mathbb{P}\left(S_{n+1} = y | S_0 = x_0, \ldots, S_n = x_n\right) = \mathbb{P}\left(S_{n+1} = y | S_n = x_n\right).$$

The intuitive meaning of the Markov property is that the future behaviour of the process $(S_n)_{n\geq 0}$ after n depends only on the value S_n and is not influenced by the history of the process before n.

The Markov chain is said *time homogenous* if $\mathbb{P}\left(S_{n+1} = y | S_n = x\right)$ does not depend on n. One then sets:

$$P(x,y) = \mathbb{P}\left(S_{n+1} = y | S_n = x\right).$$

The matrix $(P(x,y))_{x\in\mathcal{E},\dagger\in\mathcal{E}}$ is called the transition matrix of the Markov chain

Remark

$\forall$x,y $\in \mathcal{E}$ $\quad P(x,y) \geq 0$ and, $\forall$x $\sum_{y\in\mathcal{E}} P(x,y) = 1$.

Random walks

Example

Binomial random walk Let $(X_i, i \geq 1)$ a sequence of i.i.d. (indipendent, identically distribuited) random variables with $\mathbb{P}(X_i = \pm 1) = 1/2$. Then $S_n = X_1 + \cdots + X_n$ is a homogenous Markov chain with transition matrix $P(x, x+1) = P(x, x-1) = 1/2$, $P(x, y) = 0$ otherwise.

Example

Trinomial random walk. Let $(X_i, i \geq 1)$ a sequence of i.i.d. random variables $\mathbb{P}(X_i = \pm 1) = \lambda/2$ and $\mathbb{P}(X_i = 0) = 1 - \lambda$, with $0 < \lambda \leq 1$. The transition matrix is given by
$P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$, $P(x, y) = 0$ otherwise.

Example

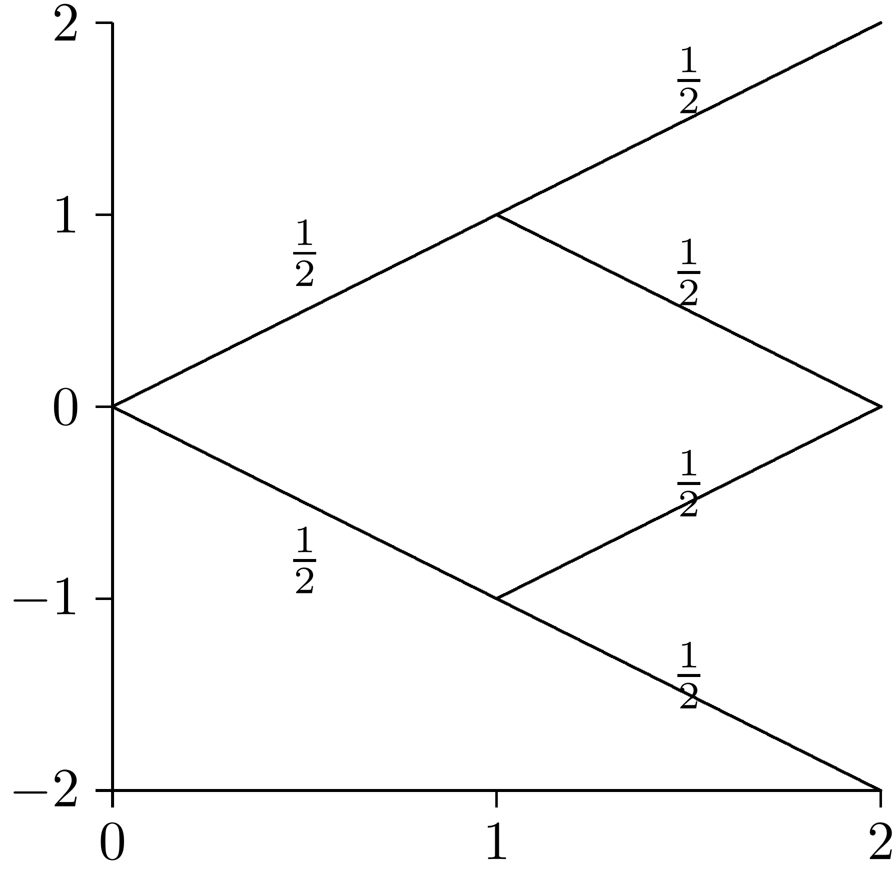
Random walk of Cox Ross-Rubinstein. Let $(U_n, n \geq 0)$ a sequence of i.i.d. random variables with $\mathbb{P}(U_n = u) = p$, $\mathbb{P}(U_n = d) = 1 - p$ and $0 < p < 1$, u and d real numbers.Let $S_0 = x$ and :

$$S_{n+1} = S_n U_{n+1}.$$

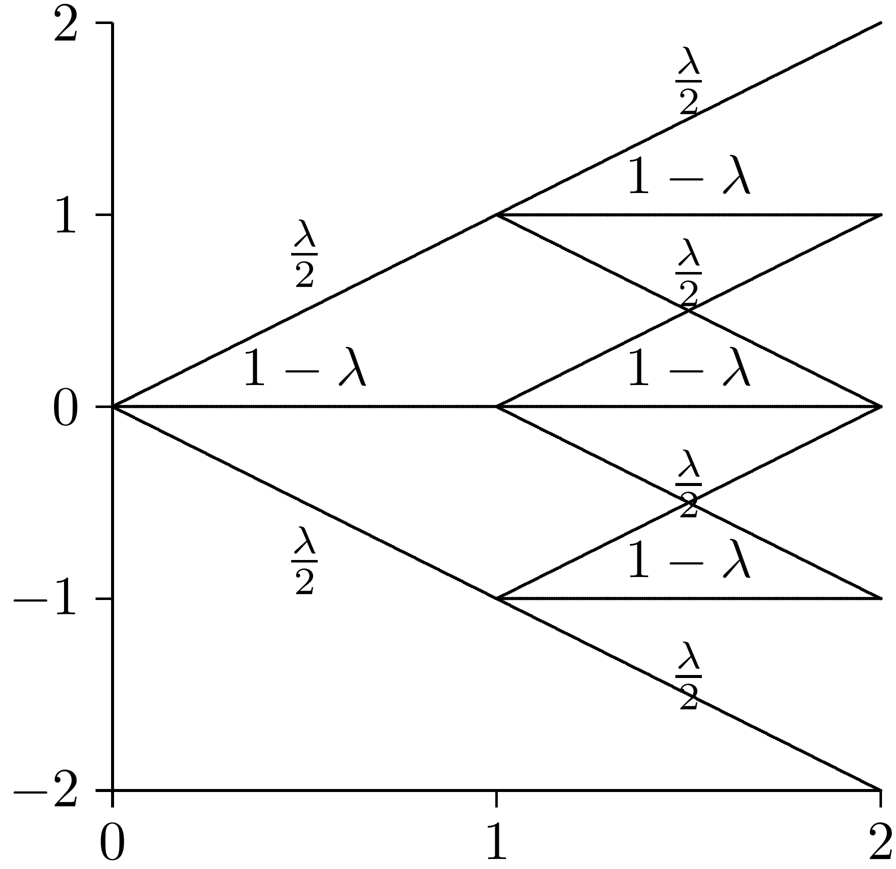
Let $S_n = x \prod_{i=1}^{n} U_i$. S_n is a homogenous Markov chain with transition matrix:

$$\begin{array}{lclcl} P(x, xu) & = & \mathbb{P}\left(S_{n+1} = xu | S_n = x\right) & = & p \\ P(x, xd) & = & \mathbb{P}\left(S_{n+1} = xd | S_n = x\right) & = & 1 - p \\ P(x, y) & = & 0 & & \text{otherwise} \end{array}$$

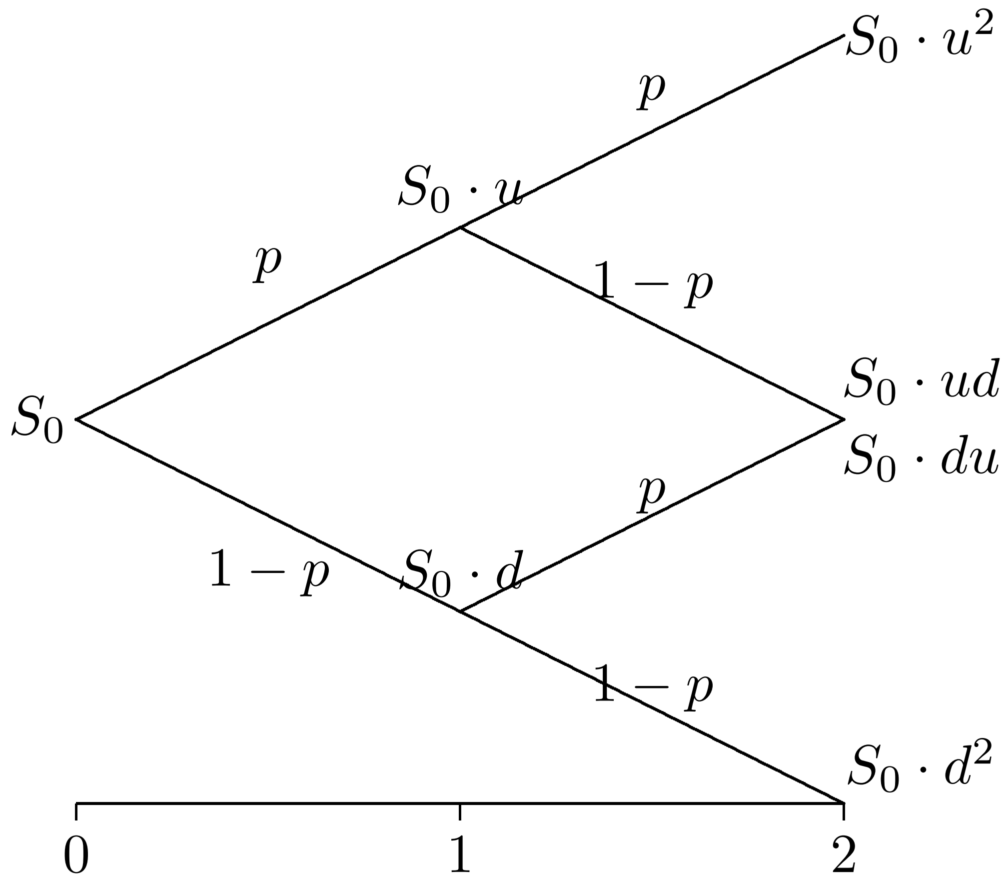
Binomial random walk



Trinomial random walk



Random walk of Cox Ross-Rubinstein



More general definition of Markov chain

Definition

Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space. Let $(\mathcal{F}_n, n \geq 1)$ *a filtration.*

A process $(S_n, n \geq 1)$ taking values in a finite or countable set $\mathcal{E}$ is a Markov chain with the family of transition matrices (P_n) if :

- For all n, S_n is $\mathcal{F}_n$-misurable.
- For any bounded function

$$I\!E\left(\phi(S_{n+1})|\mathcal{F}_n\right) = I\!E\left(\phi(S_{n+1})|S_n\right)$$

European option pricing

In the discrete models the European price could be written :

$$V_0 = \mathbb{E}\left(\frac{1}{(1+R)^N}\phi(S_N)\right),$$

where $(S_n, n \geq 0)$ is a Markov chain and r the intereste rate.

$$V_n = \mathbb{E}\left(\frac{1}{(1+R)^{N-n}}\phi(S_N)|\mathcal{F}_n\right).$$

Dynamic programming algorithm

Proposition

Let $\phi(x)$ a bounded function. Let $(S_n, n \geq 0)$ a Markov chain with transition matrix P.

Problem: compute

$$\mathbb{E}\left(\phi(S_N)\right).$$

Let u be the unique solution of:

$$\begin{cases} u(N,x) = \phi(x), \\ u(n,x) = \sum_{y \in \mathcal{E}} P(x,y) u(n+1,y). \end{cases} \tag{9}$$

Then :

$$\mathbb{E}\left(\phi(S_N) | \mathcal{F}_n\right) = u(n, S_n),$$

In particular:

$$u(0,x) = \mathbb{E}\left(\phi(S_N) | \, S_0 = x\right).$$

Proof The process (M_n) defined by $M_n := u(n, S_n)$ is a $\mathcal{F}$-martingale. In fact because u is the solution to (9) and by the Markon property

$$u(n, S_n) = \mathbb{E}\left[u(n+1, S_{n+1})|S_n\right] = E\left[u(n+1, S_{n+1})|\mathcal{F}_n\right]$$

Any martingale satisfies

$$\mathbb{E}\left[M_N|\mathcal{F}_n\right] = M_n, \ \forall n \leq N.$$

Thus,

$$\mathbb{E}\left[u(N, S_N)|\mathcal{F}_n\right] = u(n, S_n).$$

As $u(N, x) = \phi(x)$,

$$u(n, S_n) = \mathbb{E}\left[\phi(S_N)|\mathcal{F}_n\right].$$

For $n = 0$ one gets

$$u(0, S_0) = \mathbb{E}\left[\phi(S_N)|\mathcal{F}_0\right].$$

As a result, if $S_0 = x$ then

$$u(0, x) = \mathbb{E}\left[\phi(S_N)\right].$$

Example

Binomial random walk S_n is a Markov chain with transition matrix
$P(x, x+1) = P(x, x-1) = 1/2$. We have $u(0, x)) = \mathbb{E}(\phi(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = \phi(x), \\ u(n, x) = \frac{1}{2}u(n+1, x+1) + \frac{1}{2}u(n+1, x-1). \end{cases}$$

Example

Trinomial random walk S_n is a Markov chain with transition matrix
$P(x, x+1) = P(x, x-1) = \lambda/2$, $P(x, x) = 1 - \lambda$. We have $u(0, x)) = \mathbb{E}(\phi(S_N)|S_0 = x)$, where u satisfies :

$$\begin{cases} u(N, x) = \phi(x), \\ u(n, x) = \frac{\lambda}{2}u(n+1, x+1) + (1-\lambda)u(n+1, x) + \frac{\lambda}{2}u(n+1, x-1). \end{cases}$$

Corollary

Let $\phi(x)$ a bounded function from $\mathcal{E}$ to $\mathbb{R}$ and R a bounded function from $\mathcal{E}$ to $\mathbb{R}_+$. Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix P.

Problem: compute

$$\mathbb{E}\left(\prod_{i=0}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N)\right).$$

Let u be the unique solution of:

$$\begin{cases} v(N,x) = \phi(x), \\ v(n,x) = \sum_{y \in \mathcal{E}} \frac{P(x,y)}{1+R(x)} u(n+1,y). \end{cases} \tag{10}$$

Then :

$$\mathbb{E}\left(\prod_{i=n}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N) | \mathcal{F}_n\right) = v(n, S_n),$$

In particular:

$$v(0,x) = \mathbb{E}\left(\prod_{i=0}^{N-1} \frac{1}{1+R(S_i)} \phi(S_N) | S_0 = x\right).$$

Example

Cox-Ross-Rubinstein random walk

$$V_0 = \mathbb{E}\left(\frac{1}{(1+R)^N}\phi(S_N)\right),$$

we have $V_0 = v(0, S_0)$, where v satisfies :

$$\begin{cases} v(N,x) = \phi(x), \\ v(n,x) = \dfrac{p}{1+R}v(n+1,xu) + \dfrac{1-p}{1+R}v(n+1,xd). \end{cases}$$

$p = q$ Risk neutral measure

Tree algorithm
European Put options in discrete model

```
/*Risk neutral probability*/
pu=((1+R)-d)/(u-d);
pd=1-pu;

/* Conditions at maturity*/
for (j=0;j<=N;j++)
    P[j]=MAX(0.,K-S_0*pow(u,N-j)*pow(d,j));

/* Backward induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=pow(1.+R,-1.)*(pu*P[j]+pd*P[j+1]);

/*  E(\phi(S_N)|S_0=x) is given in P[0] */
```

Optimal stopping problem

One of the simplest optimal control problems is the optimal stopping problem, where at any time the only two possible control actions are:

- to stop the process (i.e. exercise the option);
- to let it continue (i.e. keeping option alive).

This problem will illustrate the basic ideas of dynamic programming for Markov chains and introduce the fundamental *principle of optimality* in a simple way.

Stopping times

Definition A random time τ is a random variable with values in $\mathbb{N} \vee \{+\infty\}$.

A random time τ is a stopping time w.r.t. a filtration $\mathcal{F} := (\mathcal{F}_n, n \in \mathbb{N})$ if $\{\tau \leq n\} \in \mathcal{F}_n$ for all n.

Proposition Let $(M_n, n \geq 0)$ be a $\mathcal{F}$-martingale and τ a stopping time w.r.t to $\mathcal{F}$. Then the stopped process

$$M_{n \wedge \tau}$$

is a martingale.

Theorem **Optional Stopping Theorem**
Let N be a strictly positive integer. Let $(M_n, n \geq 0)$ be a $\mathcal{F}$-martingale.

For any bounded stopping time τ such that $n \leq \tau \leq N, a.s.$, there holds

$$\mathbb{E}\left[M_\tau | \mathcal{F}_n\right] = M_n .$$

American option pricing

The American options can be exercised at any time between 0 and N.

The price at time 0 of an American option guaranteeing the cash–flow $\phi(S_p)$ if it is exercised at time $0 \leq p \leq N$ is given by

$$V_0^\sharp = \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[\frac{1}{(1+R)^\tau}\phi(S_\tau)\right]. \tag{11}$$

where $\tau \in \mathcal{T}_{0,N}$ is the set of $\mathcal{F}-$ stopping times taking values in $\{0, \cdots, N\}$.

Dynamic programming algorithm

Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix $P(x, y)$. Let u be the unique solution to

$$\begin{cases} u(N, x) & = & \phi(x), \\ u(n, x) & = & \max\left(\sum_{y \in \mathcal{E}} P(x, y) u(n+1, y), \phi(x) \right). \end{cases} \tag{12}$$

Then, for all $0 \leq n \leq N$,

$$\sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau) | \mathcal{F}_n\right] = u(n, S_n).$$

In particular, if $S_0 = x$ is deterministic, then

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[\phi(S_\tau)\right] = u(0, x).$$

Proof

- Set
$$Y_n := u(n, S_n) - \mathbb{E}\left[u(n, S_n)|\mathcal{F}_{n-1}\right]$$
and
$$M_n := Y_1 + \cdots + Y_n .$$
Then (M_n) is a $\mathcal{F}$–martingale. In fact
$$\begin{aligned} \mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) &= \mathbb{E}(Y_{n+1}|\mathcal{F}_n) \\ &= \mathbb{E}\left[u(n+1, S_{n+1}) - \mathbb{E}\left[u(n+1, S_{n+1})|\mathcal{F}_n\right]|\mathcal{F}_n\right] \\ &= 0. \end{aligned}$$

- Owing to the definition of u, it holds that
$$u(n, S_n) \geq \mathbb{E}\left[u(n+1, S_{n+1})|\mathcal{F}_n\right].$$
So,
$$u(n+1, S_{n+1}) - u(n, S_n) \leq u(n+1, S_{n+1}) - \mathbb{E}\left[u(n+1, S_{n+1})|\mathcal{F}_n\right] = Y_{n+1} .$$

- As $Y_{n+1} \geq u(n+1, S_{n+1}) - u(n, S_n)$, a straightforward computation leads to

$$M_p - M_n \geq u(p, S_p) - u(n, S_n),$$

for all n and all $p \geq n$.

- Besides, if τ is a stopping time such that $n \leq \tau \leq N$,

$$M_\tau - M_n \geq u(\tau, S_\tau) - u(n, S_n).$$

The Optional Stopping Theorem imply that

$$0 = \mathbb{E}\left[M_\tau - M_n | \mathcal{F}_n\right] \geq \mathbb{E}\left[u(\tau, S_\tau) | \mathcal{F}_n\right] - u(n, S_n)$$

Thus, we have just checked that

$$u(n, S_n) \geq \mathbb{E}\left[u(\tau, S_\tau) | \mathcal{F}_n\right].$$

For all stopping time taking values in $[n, N]$, there holds

$$u(n, S_n) \geq \mathbb{E}\left[\phi(S_\tau) | \mathcal{F}_n\right],$$

because from the definition u, $u(n, x) \geq \phi(x)$.

- Consequently,

$$u(n, S_n) \geq \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right]$$

- It remains to find a stopping time τ_n^* taking values in $[n, N]$ and such that

$$u(n, S_n) = \mathbb{E}\left[\phi(S_{\tau_n^*})|\mathcal{F}_n\right].$$

  To this end, set

$$\tau_n^* := \inf\left\{p > n, u(p, S_p) = \phi(S_p)\right\}.$$

  One can easily check that τ_n^* is a stopping time.
  Besides, $\tau_n^* \leq N$ since $u(N, S_N) = \phi(S_N)$.

- On the set $\{\omega; p < \tau_n^*(\omega)\}$ we have

$$u(p, S_p) = \mathbb{E}\left[u(p+1, S_{p+1})|\mathcal{F}_p\right],$$

  so that

$$Y_{p+1} = u(p+1, S_{p+1}) - u(p, S_p).$$

- Consequently,

$$\begin{aligned} Y_{n+1} &= u(n+1, S_{n+1}) - u(n, S_n) \\ Y_{n+2} &= u(n+2, S_{n+2}) - u(n+1, S_{n+1}) \\ \vdots \quad &\vdots \quad \vdots \\ Y_{\tau_n^*} &= u(\tau_n^*, S_{\tau_n^*}) - u(\tau_n^* - 1, S_{\tau_n^* - 1}). \end{aligned}$$

- Therefore,

$$M_{\tau_n^*} - M_n = u(\tau_n^*, S_{\tau_n^*}) - u(n, S_n).$$

Using the Optional Sampling Theorem, one gets

$$0 = \mathbb{E}\left[M_{\tau_n^*} - M_n | \mathcal{F}_n\right] = \mathbb{E}\left[u(\tau_n^*, S_{\tau_n^*}) | \mathcal{F}_n\right] - u(n, S_n)$$

- So we have

$$u(n, S_n) = \mathbb{E}\left[\phi(S_{\tau_n^*})|\mathcal{F}_n\right].$$

because by definition of τ_n^*

$$u(\tau_n^*, S_{\tau_n^*}) = \phi(S_{\tau_n^*}).$$

Remember that

$$u(n, S_n) \geq \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right].$$

This implies that

$$u(n, S_n) = \sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[\phi(S_\tau)|\mathcal{F}_n\right].$$

Optimal stopping time

The stopping time

$$\tau_0^* := \inf \{p > 0,\ u(p, S_p) = \phi(S_p)\}$$

satisfies

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[\phi(S_\tau)\right] = \mathbb{E}\left[\phi(S_{\tau_0^*})\right].$$

The stopping time τ_0^* is an optimal stopping time.

Dynamic programming algorithm

Let $(S_n, n \geq 0)$ be a Markov chain with transition matrix $P(x,y)$. Let u be the unique solution to

$$(13) \qquad \begin{cases} u(N,x) & = & \phi(x), \\ u(n,x) & = & \max\left(\sum_{y\in\mathcal{E}} \frac{1}{1+R} P(x,y) u(n+1,y), \phi(x) \right). \end{cases}$$

Then, for all $0 \leq n \leq N$,

$$\sup_{\tau \in \mathcal{T}_{n,N}} \mathbb{E}\left[(1+R)^{-(\tau-n)} \phi(S_\tau) | \mathcal{F}_n \right] = u(n, S_n).$$

In particular, if $S_0 = x$ is deterministic, then

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}\left[(1+R)^{-\tau} \phi(S_\tau) \right] = u(0, x).$$

Binomial algorithm
American Put option in discrete model

```
/*Risk neutral probability*/
pu=((1+R)-d)/(u-d);
pd=1-pu;

/*Intrinsic values*/
for (j=0;j<=2*N;j++)
    InV[j]=max(0.,K-x*pow(u,N-j));
/*Terminal condition*/
for (j=0;j<=N;j++)
    P[j]=InV[2*j];

/*Dynamic programming*/
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=MAX(pow(1.+R,-1.)*(pu*P[j]+pd*P[j+1]),InV[i+2*j]);

  /* Price in P[0] */
```