Greeks, Dynamic Hedging

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Estimating Sensitivities

We will see that in a idealized setting of continuous trading in a complete market, the payoff of a contingent claim can be hedged through trading in underlying assets.

Implementation of the strategy requires knowledge of the pricing sensitivities.
The sensitivieties are very usefull in risk management.

Black-Scholes formula

$$C = xN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

We will consider the delta Δ, gamma Γ, rho ρ, vega $Vega$ and theta Θ.

Delta

$$\Delta = \frac{\partial C}{\partial x} = N(d_1) > 0$$

The price of a Call option is a increasing function w.r.t. x the initial price.

Gamma

$$\Gamma = \frac{\partial^2 C}{\partial x^2} = \frac{N'(d_1)}{x\sigma\sqrt{T-t}} > 0$$

with

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The price of a Call option is a convex function w.r.t. x the initial price.

Rho

$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) > 0$$

The price of a Call option is a increasing function w.r.t. r.

Vega

$$Vega = \frac{\partial C}{\partial \sigma} = x\sqrt{T-t}N'(d_1) > 0$$

The price of a Call option is a increasing function w.r.t. σ.

Theta

$$\Theta = \frac{\partial C}{\partial \tau} = -\frac{xN'(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2) < 0$$

The price of a Call option is a decreasing function w.r.t. τ.

Greeks : Monte Carlo Method

There are two ways to tackle this problem:

- finite difference approximation.
- the pathwise method.

Finite difference approximation : Delta

Consider a function $u(x) : \mathbb{R} \to \mathbb{R}$, $u \in C^4(\mathbb{R})$.
By Taylor expansion

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2 u''(x+\nu h), \quad 0 \leq \nu \leq 1$$

So we have

$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h)$$

Moreover and

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2 u''(x) + \frac{1}{6}h^3 u^{(3)}(x+\nu^+ h)$$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2 u''(x) - \frac{1}{6}h^3 u^{(3)}(x+\nu^- h)$$

with $-1 \leq \nu_x^- \leq 0$, $0 \leq \nu_x^+ \leq 1$. Therefore

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

Finite difference approximation: Gamma

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u^{(3)}(x) + \frac{1}{24}h^4u^{(4)}(x+\nu^+h)$$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u^{(3)}(x) + \frac{1}{24}h^4u^{(4)}(x+\nu^-h)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

Delta and Gamma approximations

We approximate

Delta

$$\Delta = \frac{\partial \mathbb{E}\Big[\psi(S_T^x)\Big]}{\partial x} \approx \frac{\mathbb{E}\Big[\psi(S_T^{x+h})\Big] - \mathbb{E}\Big[\psi(S_T^x)\Big]}{h} = \frac{\mathbb{E}\Big[\psi(S_T^{x+h}) - \psi(S_T^x)\Big]}{h}$$

or otherwise

$$\Delta = \frac{\partial \mathbb{E}\Big[\psi(S_T^x)\Big]}{\partial x} \approx \frac{\mathbb{E}\Big[\psi(S_T^{x+h}) - \psi(S_T^{x-h})\Big]}{2h}$$

Gamma

$$\Gamma = \frac{\partial^2 \mathbb{E}\Big[\psi(S_T^x)\Big]}{\partial x^2} \approx \frac{\mathbb{E}\Big[\psi(S_T^{x+h}) - 2\psi(S_T^x) + \psi(S_T^{x-h})\Big]}{h^2}$$

```
function [Y] = Price(N,K,S0,INC,T,r,sigma)

  mean_price=0;
  mean_price_inc=0;
  var_price=0;

  drift=(r-0.5*sigma^2)*T;
  S0_inc=S0*(1+INC) //S0 incremented

   for i = 1:N
  //g=rand(1,'normal');
  g=gaussian()
  brownian=g*sqrt(T);
  spot=S0*exp(drift+sigma*brownian);
  price_sample=payoff_call(spot,K);

  spot_inc=S0_inc*exp(drift+sigma*brownian);
  price_sample_inc=payoff_call(spot_inc,K);
```

```
  mean_price=mean_price+price_sample;
  var_price=var_price+price_sample^2;

  mean_price_inc=mean_price_inc+price_sample_inc;
  end

  price=exp(-r*T)*mean_price/N;
  error_price=sqrt(exp(-2*r*T)*var_price/N-price^2)/sqrt(N-1);
  inf_price=price-1.96*error_price;
  sup_price=price+1.96*error_price;

  price_inc=exp(-r*T)*mean_price_inc/N;
  delta=(price_inc-price)/(S0*INC)
  //disp('inf',inf_price);
  //disp('sup',sup_price);
  Y(1) =inf_price;
  Y(2)= price;
  Y(3)= sup_price;
  Y(4)= delta
endfunction;
```

Remark

- E$\Big[$f(X)-g(X)$\Big]$. It is better to perform:
- indipendent simulations $I_N^1 = \frac{1}{N}\sum_{i=1}^N f(X_i) - \frac{1}{N}\sum_{i=1}^N g(X_i')$ with X_i and X_i' i.i.d. $i = 1, .., N$?
- or common simulations $I_N^2 = \frac{1}{N}\sum_{i=1}^N \Big[f(X_i) - g(X_i)\Big]$?
- $Var\Big[I_N^1\Big] = \frac{1}{N}\Big[Var(f(X)) + Var(g(X))\Big]$
- $Var\Big[I_N^2\Big] = \frac{1}{N}\Big[Var(f(X)) + Var(g(X)) - 2\ cov(f(X), g(X))\Big]$
- If $f(X)$ and $g(X)$ have positive correlation it is better to perform common simulations.

Pathwise method

Interchange of differentiation and expectation.

The pathways approach supposes that $x \mapsto S_t^x(\omega)$ is differentiable for almost every ω (and this is the case) and the payoff function ϕ is differentiable also.

Then

$$\partial_x \mathbb{E}\Big[\phi(S_t^x)\Big] = \mathbb{E}\Big[\phi'(S_t^x)\partial_x S_t^x\Big].$$

Black-Scholes equation

F.Black e M.Scholes The pricing of Options and Corporate Liabilites
Journal of Political Economy 73

$$\Theta + \frac{1}{2}\sigma^2 x^2 \Gamma + rx\Delta - rC = 0.$$

Adding the boundary condition at maturity they obtains the Black-Scholes equation:

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} + -rC = 0 \ \text{ in } [0,T[\times[0,+\infty) \\ C(T,x) = \psi(x), \quad x \in [0,+\infty) \end{cases}$$

The Black-Scholes equation is a partial differential equation.

$C(t,x)$, the price of the option at time t with initial underlying asset x, is solution of this PDE.

Idea of the proof: using portfolio with short position in the risk asset and long positions in the Call options that replicates the risk-free asset on $[0,T]$.

Risk-free replicating portfolio

At time

- we buy m_t Call options with maturity T
- we sell $m_t n_t$ stocks.

The value of the portfolio at time t is given by

$$V_t^0 = -m_t C_t + m_t n_t S_t$$

The portfolio is self-financing, so that:

$$dV_t^0 = -m_t dC_t + m_t n_t dS_t.$$

By Ito's Lemma

$$dC(t, S_t) = \mu(t, S_t)dt + \frac{\partial C}{\partial S_t} dS_t,$$

with

$$\mu(t, S_t) = \frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2}.$$

$$dV_t^0 = -m_t dC_t + m_t n_t dS_t = -m_t(\mu(t,S_t)dt + \frac{\partial C}{\partial S_t}dS_t) + m_t n_t dS_t =$$

$$-m_t\mu(t,S_t)dt + (m_t n_t - m_t\frac{\partial C}{\partial S_t})dS_t.$$

In order to obtain a risk-free portfolio we need

$$n_t = \frac{\partial C}{\partial S_t}.$$

The arbitrage free hyphotesis says us that ($V_t^0 = S_t^0$)

$$dV_t^0 = rV_t^0 dt = -m_t\mu(t,S_t)dt.$$

$$dV_t^0 = rV_t^0 dt = r(-m_t C_t + m_t\frac{\partial C}{\partial S_t}S_t)dt = rm_t(\frac{\partial C}{\partial S_t}S_t - C_t)dt = -m_t\mu(t,S_t)dt.$$

Then

$$r(\frac{\partial C}{\partial S_t}S_t - C_t) = -\mu(t,S_t),$$

that provides the Black Scholes equation

$$r(\frac{\partial C}{\partial S_t}S_t - C_t) = -(\frac{\partial C}{\partial t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2}).$$

The Black Scholes equation

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} + r S_t \frac{\partial C}{\partial S_t} - r C_t = 0$$

We have to add the following terminal condition $C(T, S_T) = \psi(S_T)$.
Moreover

$$m_t = \frac{V_t^0}{\frac{\partial C}{\partial S_t} S_t - C_t}$$

Dynamic Delta

We want to replicate the option on $[0, T]$ using risk asset S_t and risk-free asset S_t^0.
We construct a portfolio

$$V_t = \alpha(t, S_t)S_t + \gamma(t, S_t)S_t^0$$

that equals C_t.
In order to achieve a perfect replication we need

-
$$\alpha(t, S_t) = n_t = \frac{dC(t, S_t)}{dS_t} = N(d_1),$$

  unit of risk asset S_t

-
$$\gamma(t, S_t) = -\frac{1}{m_t} = \left(C(t, S_t) - S_t \frac{dC(t, S_t)}{dS_t}\right)\frac{1}{S_t^0},$$

  unit of risk-free asset S_t^0

At maturity, we will have

$$V_T = (S_T - K)_+.$$

Proof

The value of the risk-free portfolio at time t is given by:

$$V_t^0 = -m_t C_t + m_t n_t S_t = S_t^0.$$

We replicate the options with this portfolio

$$V_t = \alpha_t S_t + \gamma_t S_t^0 = C_t = n_t S_t - \frac{S_t^0}{m_t}.$$

Discrete Dynamic Hedging

Osservazione The Black-Scholes model is a complet market: every contigent claim with payoff $G = f(S_T)$ can be replicated perfectly with a self-financing portfolio.
Theoretically the risk is exactly zero.
The Black-Scholes analysis requires continuous hedging, which is possible in theory but impossible in practice.

The simpliest model for discrete hedging is to rehedge at fixed intervals of time $\Delta T = \frac{T}{N}$; a strategy commonly used with ΔT ranging from one day to one week.

So we will have errors in following a pure Black-Scholes hedging strategy in discrete time.

Start $t_0 = 0,\ S_0 = x, S_0^0 = 1, \Delta T = \frac{T}{N}$

$V_0 = C(0, T, K, r, \sigma, x)$

$\alpha_0 = N(d1(S_0))$, $\gamma_0 = \left(V_0 - S_0\alpha_0\right)\frac{1}{S_0^0}$; $\beta_0 = V_0 - S_0\alpha_0$

for $k = 1, \ldots, N-1$

BEGIN;

$t_k = t_{k-1} + \Delta T$;

simulation of $g \sim N(0,1)$; $S_k = S_{k-1}e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g\sqrt{\Delta T}}$;

$S_k^0 = S_{k-1}^0 e^{r\Delta T}$;

$V_k = \alpha_{k-1}S_k + \gamma_{k-1}S_k^0$; $V_k = \alpha_{k-1}S_k + \beta_{k-1}e^{r\Delta T}$

rebalancing the portfolio;

$\alpha_k = N(d_1(S_k))$; $\gamma_k = \left(V_k - S_k\alpha_k\right)\frac{1}{S_k^0}$; $\beta_k = V_k - S_k\alpha_k$

END;

$S_N = S_{N-1}e^{(\mu - \frac{1}{2}\sigma^2)\Delta T + \sigma g\sqrt{\Delta T}}$;

$S_N^0 = S_{N-1}^0 e^{r\Delta T}$;

$V_N = \alpha_{N-1}S_N + \gamma_{N-1}S_N^0$; $V_N = \alpha_{N-1}S_N + \beta_{N-1}e^{r\Delta T}$

Portfolio insurance

We want to obtain the quantity

$$\max(K, S_T)$$

It is easy to show that

$$\max(K, S_T) = (K - S_T)_+ + S_T$$

The sum

$$V_t + S_t = \alpha(t, S_t)S_t + \gamma(t, S_t)S_t^0 + S_t$$

provides us a portfolio with final value $\max(K, S_T)$ at maturity.