Tree methods for continuous models

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European options in Black-Scholes continuous model

Problem: compute European Put options

$$P = e^{-rT} \mathbb{E}_Q \left[(K - S_T)_+ \right],$$

where $(S_t)_{t \geq 0}$ is a geometric brownian motion.

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t, S_0 = x,$$

$$S_T = x e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T}.$$

G.M.B. and random walk

Let $S_{0\leq t\leq T}$ a g.b.m. Then :

Proposition

Let $(X_i, i \geq 1)$ *be a sequence of i.i.d. random variables,* $\mathbb{P}(X_i = u) = q$ *and* $\mathbb{P}(X_i = d) = 1 - q$*.*
Let $(S_n)_{n\geq 0}$ *be the CRR random walk with* $S_0 = x$ *and*

$$S_{n+1} = S_n X_{n+1}.$$

Let $\Delta T = T/N$ *be the time discretization step,*

$$u = e^{\sigma\sqrt{\Delta T}},$$

$$d = e^{-\sigma\sqrt{\Delta T}},$$

$$q = \frac{e^{r\Delta T} - d}{u - d}.$$

Then S_N *converges in law to* S_T*.*

European price with CRR random walk

We can approximate

$$P = e^{-rT}\mathbb{E}_Q\left[(K - S_T)_+\right]$$

by

$$e^{-rT}\mathbb{E}_q\Big[(K - S_N)_+\Big].$$

To compute $e^{-rT}\mathbb{E}_q\Big[f(S_N)\Big]$ we have to solve:

$$\begin{cases} u(N\Delta T, x) = f(x), \\ u(n\Delta T, x) = e^{-r\Delta T}\Big[qu\Big((n+1)\Delta T, xu\Big) + (1-q)u\Big((n+1)\Delta T, xd\Big)\Big]. \end{cases}$$

For the put option, $f(S_N) = (K - S_N)_+$ we obtain the following algorithm :

CRR Algorithm for put option in BS model

```
/*Up-Down factors*/
h=T/N;
u=exp(sigma*sqrt(h));
d=1./u;

/*Risk neutral probability*/
pu=(exp(r*h)-d)/(u-d);
pd=1-pu;

/* Condition at maturity */
for (j=0;j<=N;j++)
    P[j]=MAX(0.,K-x*pow(u,N-j)*pow(d,j));

/* Backward induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=exp(-r*h)*(pu*P[j]+pd*P[j+1]);

/* E(f(S_N)|S_0=x) is given in P[0] */
```

Proof Let us consider $S_0 = 1$ and $\lambda \in \mathbb{R}$

$$\begin{aligned} & E_q\left[\exp\left(i\lambda \ln S_N\right)\right] \\ = \quad & E_q\left[\exp\left(i\lambda \ln \prod_{n=0}^{N-1} \frac{S_{n+1}}{S_n}\right)\right] \\ = \quad & E_q\left[\exp\left(i\lambda \ln \frac{S_1}{S_0}\right)\right]^N \\ = \quad & \left(q \exp\left(i\lambda\sigma\sqrt{\Delta T}\right) + (1-q)\exp\left(-i\lambda\sigma\sqrt{\Delta T}\right)\right)^N \end{aligned}$$

and since $q = \frac{e^{r\Delta T} - d}{u-d} \sim \frac{1}{2} + \frac{\left(r - \frac{\sigma^2}{2}\right)}{2\sigma}\sqrt{\Delta T}$.

$$\begin{aligned} E_q\left[\exp\left(i\lambda \ln S_N\right)\right] \quad & \sim \quad \left(1 + \left[i\lambda\left(r - \frac{\sigma^2}{2}\right) - \lambda^2\frac{\sigma^2}{2}\right]\frac{T}{N}\right)^N \\ & \to \quad \exp\left(\left[i\lambda\left(r - \frac{\sigma^2}{2}\right) - \lambda^2\frac{\sigma^2}{2}\right]T\right) \\ & = \quad E_Q\left[\exp\left(i\lambda\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T\right)\right)\right] \\ & = \quad E_Q\left[\exp\left(i\lambda \ln S_T\right)\right] \end{aligned}$$

$$S_N \to S_T$$

in law for $N \to \infty$.

Local consistency

Kushner's theorem says that the local consistency conditions, that is the matching at the first order of the first and second moments of the logarithmic increments of the approximating chain with those of the continuous-time limit grant the convergence of the expectations of smooth functionals.

DISCRETE CONTINUOS

$S_{(n+1)\Delta T} = S_{n\Delta T} X_{(n+1)\Delta T}$ $dS_t = S_t dt + \sigma S_t dB_t$

$$qu + (1-q)d = \mathbb{E}_q[\frac{S_{(n+1)\Delta T}}{S_{n\Delta T}}] \approx \mathbb{E}_Q[\frac{S_{t+\Delta t}}{S_t}] = e^{r\Delta T},$$

$$qu^2 + (1-q)d^2 = E_q[(\frac{S_{(n+1)\Delta T}}{S_{n\Delta T}})^2] \approx \mathbb{E}_Q[(\frac{S_{t+\Delta t}}{S_t})^2] = e^{2r\Delta t + \sigma^2 \Delta t}.$$

Option pricing in continuous models

Let $b:\mathbb{R}\to\mathbb{R}$ and $\sigma:\mathbb{R}\to\mathbb{R}$ two functions such that :

$$|b(x)-b(y)| \leq K|x-y| \text{ and } |\sigma(x)-\sigma(y)| \leq K|x-y|.$$

Let $(W_t, t \geq 0)$ a standard Brownian motion w.r.t $(\mathcal{F}_t, t \geq 0)$. We will denote $(S_t, t \geq 0)$ the diffusion which is the unique solution of :

$$dS_t = b(S_t)dt + \sigma(S_t)dW_t, S_0 = x_0.$$

The price of an European option is given by :

$$V_t = \mathbb{E}\left(e^{-r(T-t)} f(S_T) | \mathcal{F}_t\right),$$

The price of an American option is given by :

$$V_t = \sup_{\tau, \mathcal{F}_t s.t., t \leq \tau \leq T} \mathbb{E}\left(e^{-r(\tau-t)} f(S_\tau) | \mathcal{F}_t\right),$$

Approximation results

We now consider a sequence of Markov chain $(\hat{S}_n^{(N)}, n \geq 0)$ with transition matrix $P^{(N)}(x, y)$ such that $\hat{S}_0^{(N)} = x_0$. Let $k = T/N$ and let:

$$S_t^{(N)} = \hat{S}_{[t/k]}^{(N)}.$$

We will give sufficient conditions implying that $(S_t^{(N)})_{0 \leq t \leq T}$ converges in law to $(S_t)_{0 \leq t \leq T}$.

The basic idea is that we need to satisfy a consistency conditions for the first two moments.

Hypothesis (H)

(H0) $P^{(N)}(x,y) = 0$ except for a finite number of values. Moreover, $P^{(N)}(x,y) = 0$, if $|x-y| > A$, for $A > 0$.

(H1) If :

$$b^N(x) = \frac{1}{k} \sum_y P^{(N)}(x,y)(y-x),$$

then for all $R > 0$, $\lim_{N\to+\infty} \sup_{|x|\leq R} |b^N(x) - b(x)| = 0$.

(H2) If :

$$a^N(x) = \frac{1}{k} \sum_y P^{(N)}(x,y)(y-x)^2,$$

and if : then for all $R > 0$, $\lim_{N\to+\infty} \sup_{|x|\leq R} |a^N(x) - \sigma^2(x)| = 0$.

(H3) for all $\epsilon > 0$ and for all R :

$$\lim_{N\to+\infty} \sup_{|x|\leq R} \frac{1}{k} \sum_{y,|y-x|>\epsilon} P^{(N)}(x,y) = 0.$$

Convergence results in the European case

Theorem Under Hypothesis (H), for all $t \leq T$, $S_t^{(N)}$ converges in law to S_t when N tends to infinity.

Corollary Let f be a continous and bounded function. Let $P^{(N)}$ be a transition probability satisfying Hypothesis (H). Then :

$$\mathbb{E}\left(e^{-rT} f(S_T)\right) = \lim_{N \to +\infty} \mathbb{E}\left(e^{-rT} f(S_T^{(N)})\right).$$

Moreover :

$$\mathbb{E}\left(e^{-rT} f(S_T^{(N)})\right) = \mathbb{E}\left(e^{-rT} f(S_N^{(N)})\right) = \bar{u_e}(0, x_0),$$

$\bar{u}(n, x_0)$ can be computed solving

$$\begin{array}{rcl} \bar{u_e}(N, x) & = & f(x) \\ \bar{u_e}(n, x) & = & e^{-rk} \sum_y P^{(N)}(x, y) \bar{u_e}(n+1, y). \end{array}$$

Convergence results in the American case

Theorem Under Hypothesis (H)

$$\sup_{\tau,\mathcal{F}_t-\text{s.t.},\tau\leq T}\mathbb{E}\left(e^{-r\tau}f(S_\tau)\right) \quad = \quad \lim_{N\to+\infty}\sup_{\mathcal{T},\mathcal{G}_n^{(N)}-\text{s.t.},\mathcal{T}\leq N}\mathbb{E}\left(e^{-rk\mathcal{T}}f(S_{\mathcal{T}}^{(N)})\right),$$

with $\mathcal{G}_n^{(N)} = \sigma(S_1^{(N)},\ldots,S_n^{(N)})$.

Corollary Let f be a continous and bounded function. Let $P^{(N)}$ be a transition probability satisfying Hypothesis (H). Then :

$$\sup_{\tau,\mathcal{F}_t-\text{s.t.},\tau\leq T}\mathbb{E}\left(e^{-r\tau}f(S_\tau)\right) = \lim_{N\to+\infty}\sup_{\mathcal{T},\mathcal{G}_n^{(N)}-\text{s.t.},\mathcal{T}\leq N}\mathbb{E}\left(e^{-rk\mathcal{T}}f(S_{\mathcal{T}}^{(N)})\right) = \bar{u_a}(0,x_0).$$

$\bar{u}(n,x_0)$ can be computed solving

$$\begin{aligned}
\bar{u_a}(N,x) &= f(x)\\
\bar{u_a}(n,x) &= \sup\left(e^{-rk}\textstyle\sum_y P^{(N)}(x,y)\bar{u_a}(n+1,y), f(x)\right).
\end{aligned}$$

G.B.M. and Kamrad Ritchken tree

Kamrad and Ritchken choose to take a symmetric 3-points approximation to $\log\left(\frac{S_{n\Delta T}}{S_0}\right)$

$$(1) \qquad \log S_{(n+1)\Delta T} = \begin{cases} \log S_{n\Delta T} + \log u & \text{with} \quad p_u \\ \log S_{n\Delta T} & \text{with} \quad p_m \\ \log S_{n\Delta T} + \log d & \text{with} \quad p_d \end{cases}$$

In order to obtain the convergence, they match the 2 first moments of $\log\left(\frac{S_{n\Delta T}}{S_0}\right)$.
By replacing $\log u$ by $\lambda\sigma\sqrt{\Delta T}$ this leads to

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta T}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \\ p_d &= \frac{1}{2\lambda^2} - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta T}}{2\lambda\sigma}. \end{aligned}$$

The parameter λ appears as a free parameter of the geometry of the tree, which may be useful for some purposes. It is called the stretch parameter. The value $\lambda = 1.22474$ which corresponds to $p_m = \frac{1}{3}$ is reported to be a good choice for an at the money Call (or Put) option.

Trinomial algorithm of Kamrad Ritchen

To compute $e^{-rT}\mathbb{E}_q\Big[f(S_N)\Big]$, one has to solve :

$$\begin{cases} u(N\Delta T, x) = f(x), \\ u(n\Delta T, x) = e^{-r\Delta T}\Big[p_u u\Big((n+1)\Delta T, xu\Big) + p_m u\Big((n+1)\Delta T, x\Big) + p_d u\Big((n+1)\Delta T, xd\Big)\Big]. \end{cases}$$

In particular if, $f(S_N) = (K - S_N)_+$ then :

Trinomial model of Kamrad Ritchken

```
/*Up-Down factors*/
h=T/N;
lambda=1.22474;
u=exp(lambda*sigma*sqrt(h));
d=1./u;
/*Probabilities*/
z=r-SQR(sigma)/2.;
pu=(1./(2.*SQR(lambda))+z*sqrt(h)/(2.*lambda*sigma));
pm=(1.-1./SQR(lambda));
pd=1./(2.*SQR(lambda))-z*sqrt(h)/(2.*lambda*sigma));

/* Condition at maturiy */
for (j=0;j<=2*N;j++)
    P[j]=MAX(0.,K-x*pow(u,N-j)  induction */
  for (i=1;i<=N;i++)
    for (j=0;j<=2*N-2*i;j++)
 P[j]=exp(-r*k)*(pu*P[j]+pm*P[j+1]+pd*P[j+2]);

/* E(f(S_N)) is given in P[0] */
```

American option

The value at time $t = 0$ of an American Put option on the risky underlying, with maturity T and payoff function $\psi(x) = (K - x)_+$, is, in the connection with Optimal Stopping Theory, given by:

$$v(0, s_0) = \sup_{\tau \in \mathcal{T}_{0,T}} E_Q \left(e^{-r\tau} \psi(S_\tau) \right)$$

where $\mathcal{T}_{0,T}$ is the set of all stopping times with values in $[0, T]$.

CRR algorithm

$$u = e^{\sigma\sqrt{\Delta T}}$$

$$d = e^{-\sigma\sqrt{\Delta T}}$$

$$q = \frac{e^{r\Delta T} - d}{u - d}$$

The price of an American out v_0 is obtained solving:

$$\begin{cases} v(N, x) = (K - x)_+, \\ v(n, x) = MAX\left(e^{-r\Delta T} qv(n+1, xu) + (1-q)v(n+1, xd), (K-x)_+\right). \end{cases}$$

Binomial algorithm
American Put option

```
/*Up-Down factors*/
h=T/N;
u=exp(sigma*sqrt(h));
d=1./u;
/*Risk neutral probability*/
pu=(exp(r*h)-d)/(u-d);
pd=1-pu;

/*Intrinsic values*/
for (j=0;j<=2*N;j++)
    InV[j]=max(0.,K-xpow(u,N-j));
/*Terminal condition*/
for (j=0;j<=N;j++)
    P[j]=InV[2*j];

/*Dynamic programming*/
  for (i=1;i<=N;i++)
    for (j=0;j<=N-i;j++)
 P[j]=MAX(exp(-r*k)*(pu*P[j]+pd*P[j+1]),InV[i+2*j]);

  /* Price in P[0] */
```

Finite difference approximation

Consider a function $u(x) : \mathbb{R} \to \mathbb{R}$, $u \in C^4(\mathbb{R})$.
By Taylor expansion

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2 u''(x+\nu h), \quad 0 \le \nu \le 1$$

So we have

$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h)$$

Finite difference approximation

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}\Delta x^2 u''(x) + \frac{1}{6}h^3 u^{(3)}(x) + \frac{1}{24}\Delta x^4 u^{(4)}(x+\nu^+ h)$$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2 u''(x) - \frac{1}{6}h^3 u^{(3)}(x) + \frac{1}{24}h^4 u^{(4)}(x+\nu^- h)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

Greeks

Delta

$$\Delta = \frac{\partial C}{\partial x} = \frac{v(\Delta T, xu) - v(\Delta T, xd)}{xu - xd}.$$

Gamma

In order to have second order accuracy in space, for $\frac{\partial^2 v}{\partial x^2}$ computation, we have to modify the finite difference stencils with

$$\frac{2}{h_i + h_{i+1}}\left(\frac{\tilde{v}(x_{i+1}) - \tilde{v}(x_i)}{h_{i+1}} - \frac{\tilde{v}(x_i) - \tilde{v}(x_{i-1})}{h_i}\right).$$

Let $h = \frac{1}{2}(xu^2 - xd^2)$. Then

$$\Gamma = \frac{\partial^2 C}{\partial x^2} = \frac{\frac{\left(v(2\Delta T, xu^2) - v(2\Delta T, x)\right)}{\left(xu^2 - x\right)} - \frac{\left(v(2\Delta T, x) - v(2\Delta T, xd^2)\right)}{\left(x - xd^2\right)}}{h}.$$

Cox-Ross-Rubinstein Tree

The price at time 0 $v(0, s_0)$ of the European (resp. American) Put option can be computed by the following backward dynamic programming equations

$$\begin{cases} v_N(N, x) = (K - x)^+ \\ v_N(n, x) = \max\left(\psi(x), e^{-r\Delta T}\left[p_u v_N(n+1, xu) + (1 - p_u) v_N(n+1, xd)\right]\right), \end{cases}$$

where $\psi \equiv 0$ (resp. $\psi(x) = (K - x)^+$).

The Cox-Ross-Rubinstein tree (CRR) corresponds to the choice $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}}$. This leads to $p_u = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}$.

The Cox-Ross-Rubinstein tree satisfy local consistency conditions, that is the matching at the first order of the first and second moments of the logarithmic increments of the approximating chain with those of the continuous-time limit.

Convergence results

The price may converge slowly or even oscillate significantly, especially for Barrier type option (Boyle and Lau Journal of Derivatives 94).

Figlewsky and Gao (Journal of Financial Economics 99) identify two types of error:

- the distribution error. The distribution error arises from approximating the continuous distribution with a discrete distribution;
- the nonlinearity error. The nonlinearity error occurs at certain price or time level.
  The vanilla option has a critical point at maturity with the stock price equal to the stock price.
  For the continuous barrier options the critical price occurs along the barrier price.

Asymptotics results for CRR

Diener-Diener(MF 2002)

$$P_N^{\mathrm{CRR}} = P_{\mathrm{BS}} - \frac{Ke^{-rT}}{N} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} \left[\kappa_N(\kappa_N - 1)\sigma\sqrt{T} + D\right] + \mathcal{O}\left(\frac{1}{N^{3/2}}\right),$$

where

- D is a constant
- κ_N denotes the fractional part of $\frac{\log(\frac{K}{s_0})}{2\sigma}\sqrt{\frac{N}{T}} - \frac{N}{2}$.

In the at the money case i.e. that $K = s_0$, then one has $\kappa_N = 0$, and then for $N = 2m$ even, the Strike K coincides with the $(m+1)$-th final node of the (CRR) tree, and one has

$$P_{2m}^{\mathrm{CRR}} = P_{\mathrm{BS}} - \frac{DKe^{-rT}}{2m} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right).$$

Richardson extrapolation

$$P_{2m}^{\mathrm{CRR}} = P_{\mathrm{BS}} - \frac{DKe^{-rT}}{2m} e^{-\frac{d_2^2}{2}} \sqrt{\frac{2}{\pi}} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right).$$

In the approximation $2P_{4m}^{\mathrm{CRR}} - P_{2m}^{\mathrm{CRR}}$ of P_{BS} obtained using Richardson extrapolation, the term with order $1/N$ vanish.
As a consequence the rate of convergence of

$$2P_{4m}^{\mathrm{CRR}} - P_{2m}^{\mathrm{CRR}}$$

to

$$P_{\mathrm{BS}}$$

is

$$\mathcal{O}\left(\frac{1}{m^{3/2}}\right)$$

which explains the good numerical behaviour of this approximation in at the money case.

- The BIR method of Gaudenzi-Pressacco (DEF 2003), is Binomial Interpolated with Richardson extrapolation. The logic of the BI approach then is to create a set of computational options, each one with a computational Strike lying exactly on a final node of the tree. The value of the option with the contractual Strike is then obtained by interpolation of the values of the computational options. Furthermore, it is possible to exploit the recovered regularity a two-points Richardson extrapolation : this leads to the BIR method.
- The BBSR method introduced by Broadie and Detemple (RFS 96) replaces at any node of the last but one time before maturity, the binomial continuation value with the Black-Scholes European one.
- The Adaptive Mesh Model AMM introduced by Figlewski and Gao (JFE 99) resorts to refining the grid around the strike and at maturity.

MSM

Matching Moments and Strike

When N is even, if the Strike K is equal to one of the final nodes $(s_0 e^{(2k-N)\sigma\sqrt{\Delta T}})_{0 \leq k \leq N}$ of the tree, then one has $\kappa_N = 0$ (so that the term with order $1/N$ vanish in Richardson extrapolation).

This justifies our interest in trees such that the Strike coincides with one of the final nodes.
MSM is based on two matching conditions :

- **Strike Condition** : the Strike K is equal to one of the final nodes of the tree.
- **Local Consistency Condition** : the tree is consistent with the Black-Scholes model in the limit of an infinite step number.

Strike Matching

The Strike K is equal to one of the final nodes of the tree .

Instead of requiring $u = \frac{1}{d}$ as in the Cox-Rubinstein model, we propose in the MSM method to ensure that the Strike K is the $(k+1)$-th (with $k \in \{1, \dots, N-1\}$) final node of the tree :

$$K = s_0 u^k d^{N-k}$$

which also writes

$$\frac{1}{N} \log(\frac{K}{s_0}) = q \log u + (1-q) \log d$$

where $q = \frac{k}{N}$.

Remark Degree of fredom of k choiche.

Moments Matching : Local consistency condition

The two first moments matching conditions read

$$\begin{cases} p_u \log u + (1 - p_u) \log d = (r - \frac{1}{2}\sigma^2)\Delta T \\ p_u (\log u)^2 + (1 - p_u)\,(\log d)^2 = \sigma^2 \Delta T. \end{cases}$$

Kushner's theorem says that the local consistency conditions, that is the matching at the first order of the first and second moments of the logarithmic increments of the approximating chain with those of the continuous-time limit grants the convergence of the expectations of smooth functionals.

Linear System

We want to find $(\log u, \log d, p_u)$ with $\log u > \log d$ and $p_u \in]0,1[$ solving the following system of equations with unknowns (x, y, p)

$$\begin{cases} qx + (1-q)y = \alpha \\ px + (1-p)y = \beta \\ px^2 + (1-p)y^2 = \gamma \end{cases}$$

where $\alpha = \frac{1}{N}\log(\frac{K}{s_0})$, $\beta = (r - \frac{\sigma^2}{2})\Delta T$ and $\gamma = \sigma^2 \Delta T$, $q = \frac{k}{N}$.

For $q = k/N$ with $k \in \{1, \dots, N-1\}$, the solutions $(p_i, x_i, y_i)_{i \in \{1,2\}}$ of the system provide two trees :

- **the first one** with $p_u = p_1$, $\log u = x_1$ and $\log d = y_1$ is such that the $(k+1)$-th final node $s_0 u^k d^{N-k}$ of the tree is equal to the Strike K,
- **the second one** with $p_u = 1 - p_2$, $\log u = y_2$ and $\log d = x_2$ is such that the $(N-k)$-th final node $s_0 u^{N-k} d^k$ of the tree is equal to the Strike K.
  When N is even and $k = N/2$, both trees are equal.

Remark The tree is **recombining** since u and d remain constant within the tree but not symmetric .

Tree MSM parameters

For any $k \in \{1, \ldots, N-1\}$, there is a unique MSM tree with N steps and parameters $(p_u, \log u, \log d)$ (with $p_u \in]0,1[$ and $\log u > \log d$) satisfying the two first moment matching conditions and such that the strike K is equal to the $(k+1)$-th final node of the tree : $K = s_0 u^k d^{N-k}$.

$$\begin{cases} p_u = \frac{(\alpha-\beta)^2 + 2q(\gamma-\beta^2) - (\alpha-\beta)\sqrt{(\alpha-\beta)^2 + 4q(1-q)(\gamma-\beta^2)}}{2\left((\alpha-\beta)^2 + (\gamma-\beta^2)\right)} \\ \log u = \alpha + (1-q)\frac{\beta-\alpha}{p_u - q} \\ \log d = \alpha - q\frac{\beta-\alpha}{p_u - q} \end{cases}.$$

where

$$\begin{cases} q = \frac{k}{N} \\ \alpha = \frac{1}{N}\log\left(\frac{K}{s_0}\right) \\ \beta = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{N} \\ \gamma = \sigma^2 \frac{T}{N} \end{cases}.$$

Asymptotics results for MSM

Following Diener-Diener(MF 2002) results, we prove in the European case, when N is even and $k = N/2$, as m tends to infinity,

$$P_{2m} = P_{\mathrm{BS}} + \frac{C_P}{m} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right)$$
$$\delta_{2m} = \delta_{\mathrm{BS}} + \frac{C_\delta}{m} + \mathcal{O}\left(\frac{1}{m^{3/2}}\right),$$

$$\text{with } C_P = Ke^{-rT}\left(\eta\mathcal{N}(-d_2) + \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_1^3 + d_2 - d_2^3 - d_1}{8} + \nu - \mu\right)\right),$$

and

$$\text{with } C_\delta = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left(\frac{d_1^3 - d_1}{8} + \nu\right).$$

MSMR method

We have not been able to obtain asymptotic expansions for the price and delta of the American Put Option in the MSM tree with $2m$ steps and $k = m$. Nevertheless, because of the expansions obtained for the European Put option, we propose to use Richardson extrapolation even when computing the price and the delta of the American Put. This leads to MSMR method:

Price

$$2P_{4m}^{\mathrm{A}} - P_{2m}^{\mathrm{A}}$$

Delta

$$2\frac{v_{4m}^{\mathrm{A}}(1, s_0 u_{4m}) - v_{4m}^{\mathrm{A}}(1, s_0 d_{4m})}{s_0 (u_{4m} - d_{4m})} - \frac{v_{2m}^{\mathrm{A}}(1, s_0 u_{2m}) - v_{2m}^{\mathrm{A}}(1, s_0 d_{2m})}{s_0 (u_{2m} - d_{2m})}.$$

Numerical results

American Put Options in Black-Scholes Model

- We compare our algorithm MSMR with the procedures we have mentionned (CRR,BIR,BBSR,AMM) for pricing and hedging American Put options in the Black-Scholes model.
- A sample of 5.000 options was extracted randomly from a population whose parameters are the ones used in Gaudenzi-Pressacco-Zanette-Ziani[04].
- Several options of the sample have been discarded for various reasons. 4.443 options survived.
- For each option of the sample a neutral reliable price benchmark was computed as the CRR at 96.000 steps.
- the errors for the whole sample are summarized by the Mean Relative Error (MRE) and by the Squared Root of the Mean Quadratic Relative Error (RMSRE).

Price

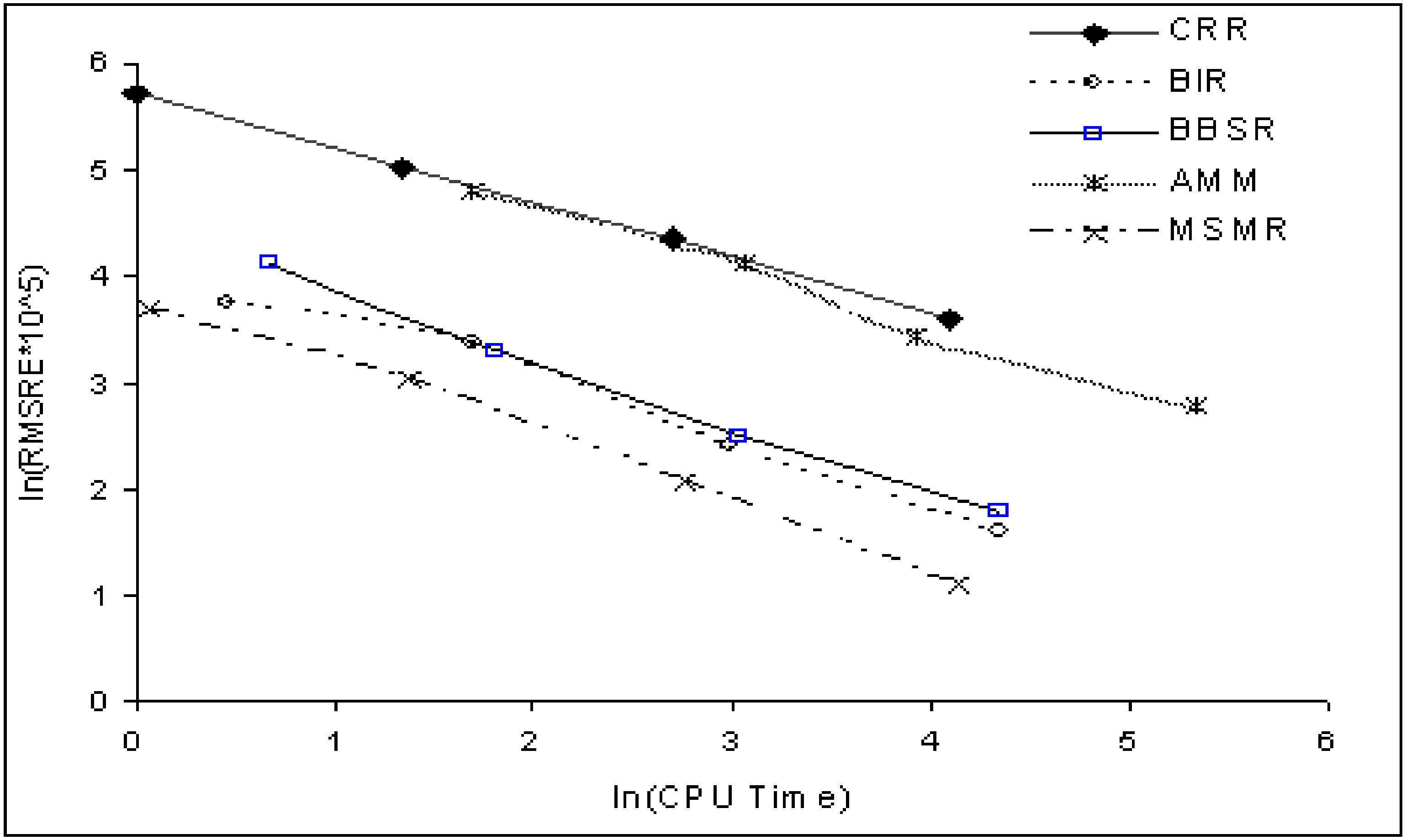

Figure 1: Price: speed-precision efficiency for tfhe 4.443 samples.

Delta

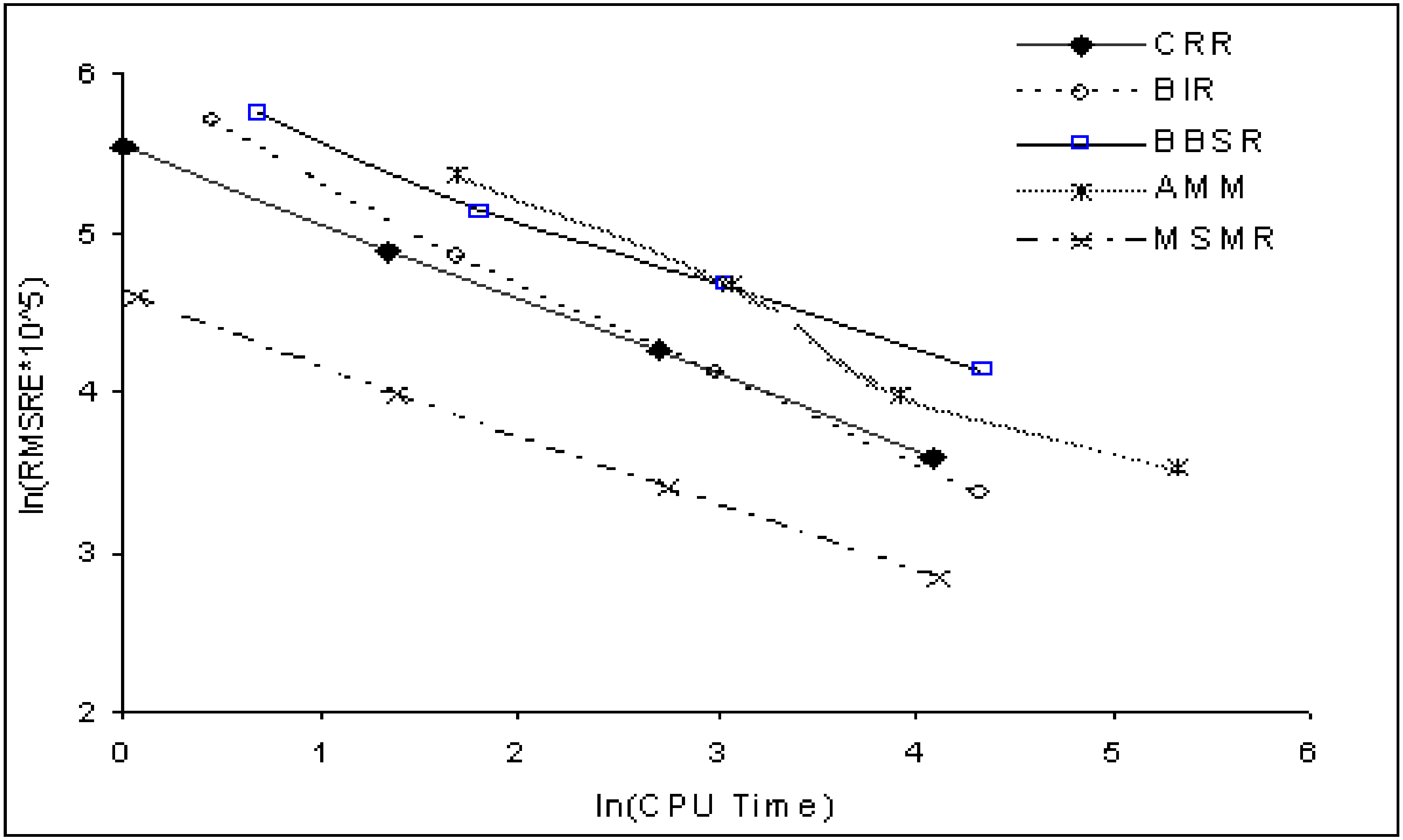

Figure 2: Delta: speed-precision efficiency for the 4.443 samples.

Recent results

- J.B.Walsh The rate of convergence of the binomial tree scheme. Finance Stochastics 2003.
- L.B.Chang K.Palmer Smooth convergence in the binomial model. Finance Stochastics 2007.
- M.Joshi Achieving smooth asymptotics for the prices of European Options in binomial trees. Quantitative Finance 2009.
- M.Joshi The convergence of binomial trees for pricing the American put. Journal of Risk 2009.

Chang Palmer method

- They use flexible binomial tree of Tian (99) *A flexible binomial option pricing model. The Journal of Future Markets 1999.* $u = e^{\sigma\sqrt{\Delta}T+\lambda\sigma^2\Delta T}$, $d = e^{-\sigma\sqrt{\Delta}T+\lambda\sigma^2\Delta T}$ choosing λ so that $K = S_0 u^{j_0} d^{n-j_0}$.

- The price of an Europen call option with the flexible binomial model satisfied

$$C_n = C_{\mathrm{BS}} + \frac{C_1}{n} + o\left(\frac{1}{n}\right).$$

- The price of an Europen digital option with the flexible binomial model satisfied

$$D_n = e^{-rT}N(d2) + \frac{C_2}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right).$$

- They obtain a new binomial model the center binomial model which permits to obtain convergence $\frac{1}{n}$ for the digital options.
- They choose λ so that K is the geometric average of $S_0 u^{j_0} d^{n-j_0}$ and $S_0 u^{j_0-1} d^{n-j_0+1}$.
- The intuitive reason for the better convergence for the digital options is that the payoff function for a digital option is that the payoff function has a jump at the strike price if it coincides with a terminal stock price but has no jump if the strike is situated between stock prices.

- The price of an Europen call option with the center binomial model satisfied

$$C_n = C_{\mathrm{BS}} + \frac{C_1}{n} + o\left(\frac{1}{n}\right).$$

- The price of an Europen digital option with the center binomial model satisfied

$$D_n = e^{-rT} N(d2) + \frac{C_3}{n} + o\left(\frac{1}{n}\right).$$

Joshi 1

The propose adjusted tree is obtained using

- $u = e^{\mu \Delta T + \sigma \sqrt{T}}$
- $d = e^{\mu \Delta T - \sigma \sqrt{T}}$
- $p = \frac{e^{r\Delta T - d}}{u-d}$
- μ is choosen so that the tree is centered on the strike in log space.

$$\mu = \frac{1}{T}(\log K - \log S_0)$$

- $C_n = C_{\mathrm{BS}} + \frac{C_1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$

Tian third moment matching tree

Tian model. J.R. Tian (93) *A modified lattice approach to option pricing. Journal of Future Markets*

The equations that u, d, q should satisfy are

$$\begin{aligned} qu + (1-q)\,d &= e^{r\Delta T}, \\ qu^2 + (1-q)\,d^2 - e^{2r\Delta T} &= e^{2r\Delta T}\left(e^{\sigma^2\Delta T} - 1\right). \end{aligned}$$

Since one degree of freedom remains, a natural idea is to match also the third moment, which gives the equation

$$qu^3 + (1-q)\,d^3 = e^{3r\Delta T}e^{3\sigma^2\Delta T}.$$

The solution of this system is

$$
\begin{aligned}
u &= \frac{e^{r\Delta T}Q}{2}\left[1+Q+\sqrt{Q^2+2Q-3}\right], \\
d &= \frac{e^{r\Delta T}Q}{2}\left[1+Q-\sqrt{Q^2+2Q-3}\right], \\
q &= \frac{e^{r\Delta T}-d}{u-d},
\end{aligned}
$$

with $Q = e^{\sigma^2 \Delta T}$. Notice that $ud = e^{2r\Delta T}Q^2 > 1$: this tree is not symmetric.

Joshi 2

Joshi study 20 different implementation methodologies for pricing American Put options.

The best method is obtained with the Tian three moment matching method combined with

- Smoothing BBS.
- Richardson extrapolation.
- Truncation.