This expression is the same as the Schrödinger equation for one-dimensional motion in the region $0 \leqslant r < \infty$ with an effective potential

$$U_{eff}(r) = U(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}.$$

Since $\chi_{nl} = rR_{nl}$ vanishes for $r = 0$, then we can take $U = +\infty$ for $r < 0$ in this one-dimensional problem.

2. In the equation for $\chi = Rr$

$$\chi'' + \left\{\frac{2\mu}{\hbar^2}[E - U(r)] - \frac{l(l+1)}{r^2}\right\}\chi = 0$$

we make the substitution

$$\chi = A \exp\left(i \frac{S}{\hbar}\right),$$

where A and S are real functions.

Equating to zero the real and imaginary parts separately, we obtain

$$2A'S' + S''A = 0, \tag{1}$$

$$S'^2 - \frac{\hbar^2 A''}{A} = 2\mu[E - U(r)] - \frac{\hbar^2 l(l+1)}{r^2}. \tag{2}$$

From the first equation we have

$$A = \frac{\text{const}}{\sqrt{S'}}.$$

We shall find an approximate solution of the second equation by considering $\hbar^2$ to be a small quantity. Here it is necessary, however, to remember that in passing over to classical mechanics ($\hbar \to 0$) $\hbar l$ should be regarded as a finite quantity, since $\hbar l$ represents the angular momentum in classical mechanics. Hence only the term $\hbar^2 A''/A$ in Eq. (2) can be considered to be a small quantity. For small r, when the dominating term on the right-hand side of (2) becomes $\hbar^2 l(l+1)/r^2$, we have $S' \approx i\hbar\sqrt{l(l+1)}\, r^{-1}$, $A \sim \sqrt{r}$, from which we obtain the approximate expression $\hbar^2 A'' A^{-1} \approx -\frac{\hbar^2}{4} r^{-2}$. Therefore, we obtain a better approximation of S if we take this term into account and insert in (2) this approximate relation (for large r the correction is not important). Hence we obtain

$$S = \int \sqrt{2\mu[E - U(r)] - \frac{\hbar^2\left(l + \frac{1}{2}\right)^2}{r^2}}\, dr,$$

$$A = \text{const}\left[2\mu[E - U(r)] - \frac{\hbar^2\left(l + \frac{1}{2}\right)^2}{r^2}\right]^{-1/4}.$$

3. We represent the Hamiltonian in the form

$$\hat{H} = \hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}, \quad \text{where } \hat{H}_0 = -\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + U(r).$$

Then the minimum values of energy and the eigenfunctions corresponding to them are related in the following way:

$$E_l^{\min} = \int \psi_l^* \left\{\hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right\}\psi_l \, d\tau,$$

$$E_{l+1}^{\min} = \int \psi_{l+1}^* \left\{\hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{(l+1)(l+2)}{r^2}\right\}\psi_{l+1} \, d\tau.$$

The last expression can be rearranged to give

$$E_{l+1}^{\min} = \int \psi_{l+1}^* \left\{\hat{H}_0 + \frac{\hbar^2 l(l+1)}{2\mu r^2}\right\}\psi_{l+1} \, d\tau + \int \frac{\hbar^2}{\mu} \cdot \frac{l+1}{r^2}\psi_{l+1}^* \psi_{l+1} \, d\tau.$$

Let us compare the first term of this expression with $E_l^{\min}$. Since ψ_l corresponds to the minimum eigenvalue of the operator $\hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}$, then

$$\int \psi_{l+1}^* \left\{\hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right\}\psi_{l+1} \, d\tau > \int \psi_l^* \left\{\hat{H}_0 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right\}\psi_l \, d\tau.$$

The integral $\int \frac{\hbar^2}{\mu}\frac{(l+1)}{r^2}\psi_{l+1}^*\psi_{l+1} \, d\tau$, is always greater than zero. Hence we have $E_l^{\min} < E_{l+1}^{\min}$, which was to be proved.

4. $\hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 \equiv \hat{\mathbf{P}} = -i\hbar\nabla_R$; $\hat{\mathbf{l}}_1 + \hat{\mathbf{l}}_2 \equiv \hat{\mathbf{L}} = [\mathbf{R}\hat{\mathbf{P}}] + [\mathbf{r}\hat{\mathbf{p}}]$, where $\hat{\mathbf{p}} = -i\hbar\nabla_r$.

5. The potential energy is $U(r) = \frac{1}{2}\mu\omega^2 r^2$.

The radial part R of the wave function satisfies the equation

$$R'' + \frac{2}{r}R' + \left\{\frac{2\mu E}{\hbar^2} - \frac{\mu^2\omega^2 r^2}{\hbar^2} - \frac{l(l+1)}{r^2}\right\}R = 0.$$

Substituting $\chi = Rr$ and introducing the notation

$$k = \frac{1}{\hbar}\sqrt{2\mu E}, \quad \frac{\mu\omega}{\hbar} = \lambda,$$

we have

$$\chi'' + \left\{k^2 - \lambda^2 r^2 - \frac{l(l+1)}{r^2}\right\}\chi = 0. \tag{1}$$

Problems in Quantum Mechanics Gol'dman & Krivchenkov

Taking into account the asymptotic behaviour of χ for $r \to 0$ and for $r \to \infty$, we look for a solution χ in the form

$$\chi = r^{l+1} \exp\left(-\frac{\lambda}{2} r^2\right) u(r). \tag{2}$$

Substituting (2) into (1), we get the following equation for the function $u(r)$:

$$u'' + 2\left\{\frac{l+1}{r} - \lambda r\right\} u' - \left\{2\lambda\left(l+\frac{3}{2}\right) - k^2\right\} u = 0. \tag{3}$$

Introducing a new independent variable $\xi = \lambda r^2$, we obtain from (3) the following differential equation:

$$\xi \frac{d^2 u}{d\xi^2} + \left\{\left(l+\frac{3}{2}\right) - \xi\right\} \frac{du}{d\xi} + \left\{\frac{1}{2}\left(l+\frac{3}{2}\right) - \frac{1}{2} s\right\} u = 0,$$

where

$$s = \frac{k^2}{2\lambda} = \frac{E}{\hbar\omega}.$$

The solution of this equation is a confluent hypergeometric function

$$u = F\left\{\frac{1}{2}\left(l+\frac{3}{2}-s\right),\ l+\frac{3}{2};\ \xi\right\}.$$

The condition that R decrease for $r \to \infty$ gives us

$$\frac{1}{2}\left(l+\frac{3}{2}-s\right) = -n_r \quad (n_r = 0, 1, 2, \ldots),$$

and consequently the energy levels are given by $E_{n_r l} = \hbar\omega\left(l + 2n_r + \frac{3}{2}\right)$, and the wave functions are

$$\psi_{n_r lm} = r^l \exp\left(-\frac{\lambda}{2} r^2\right) F\left\{-n_r,\ l+\frac{3}{2},\ \lambda r^2\right\} Y_{lm}(\vartheta, \varphi).$$

6. The wave functions are

$$\Phi_{n_1 n_2 n_3}(x, y, z) = \varphi_{n_1}(x)\varphi_{n_2}(y)\varphi_{n_3}(z),$$

where

$$\varphi_n(x) = (2^n \lambda^{n-\frac{1}{2}} n!)^{-1/2} \pi^{-1/4} \left(\lambda x - \frac{\partial}{\partial x}\right)^n \exp\left(-\frac{1}{2}\lambda x^2\right).$$

The corresponding energy levels are

$$E_{n_1 n_2 n_3} = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) \quad \text{(see Prob. 6, § 1).}$$

The relation between $\psi_{n_r lm}$ and $\Phi_{n_1 n_2 n_3}$ for $n_r = 0$, $l = 1$ has the form

$$\psi_{011} = \frac{1}{\sqrt{2}}(\Phi_{100} + i\,\Phi_{010}),$$

$$\psi_{010} = \Phi_{001},$$

$$\psi_{01,-1} = \frac{1}{\sqrt{2}}(\Phi_{100} - i\,\Phi_{010}).$$

7.

$$Z_n = (n+1)(n+2),$$

where

$$n = 2n_r + l.$$

8. For ${}_2He^4$

$$\varrho(\mathbf{r}) = \frac{4}{(r_0\sqrt{2\pi})^3} \exp\left(-\frac{1}{2}\frac{r^2}{r_0^2}\right),$$

where

$$r_0 = \sqrt{\frac{\hbar}{2\mu\omega}}; \qquad R = r_0.$$

For ${}_8O^{16}$

$$\varrho(\mathbf{r}) = \frac{4}{(r_0\sqrt{2\pi})^3}\left(1 + \frac{r^2}{r_0^2}\right)\exp\left[-\frac{1}{2}\left(\frac{r}{r_0}\right)^2\right],$$

$$R = 3{\cdot}73\, r_0.$$

9. The equation for the radial wave function has the form

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} R + \left\{U(r) - E\right\} R = 0,$$

where μ is the reduced mass; $\mu = \frac{M_p M_n}{M_p + M_n} \approx \frac{M}{2}$, since $M_p \approx M_n = M$.

Setting $l = 0$ and $R = \chi(r)/r$, we get

$$\frac{d^2\chi}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + A\exp\left(-\frac{r}{a}\right)\right]\chi = 0.$$

Introducing the change of variables

$$\xi = \exp\left(-\frac{r}{2a}\right),$$