① $\mathcal{L}(y''+3y'+2y)=\mathcal{L}(2t^2+1)$, $Y=\mathcal{L}(y)$

$$s^2Y-4s+3+3sY-12+2Y=\frac{4}{s^3}+\frac{1}{s}$$

$$\rightarrow \quad Y=\frac{4+s^2+4s^4+9s^3}{s^3(s+2)(s+1)} \quad \text{poli} \quad \begin{array}{ll} s=-1 & 1.\ st. \\ s=-2 & 1.\ st. \\ s=0 & 3.\ st. \end{array}$$

$$\left.\begin{array}{l} \mathrm{Res}(Ye^{st},-1)=0 \\ \mathrm{Res}(Ye^{st},-2)=0 \end{array}\right\} \Rightarrow Y \text{ nima pola } v\ -1,-2$$

$\Rightarrow$ števec deljiv ~~\[illegible\]~~ z $(s+2)(s+1)$

$$(4s^4+9s^3+s^2+4):(s^2+3s+2)=4s^2-3s+2$$

$$-(4s^4+12s^3+8s^2)$$

$$-3s^3-7s^2+4$$

$$-(-3s^3-9s^2-6s)$$

$$2s^2+6s+4$$

$$\rightarrow \quad Y=\frac{1}{s^3}(4s^2-3s+2)=\frac{4}{s}-\frac{3}{s^2}+\frac{2}{s^3} \Rightarrow \mathrm{Res}(Ye^{st},0)=$$

$$=\frac{1}{2}\left((4s^2-3s+2)e^{st}\right)''\Big|_{s=0}=\frac{1}{2}(8-6t+2t^2)=\underline{\underline{4-3t+t^2}}$$

(2) $L = y'^2(1-y)^2$ ker L ni odvisen od x

$\Rightarrow \quad L - y' L_{y'} = c$

$-y'^2(1-y)^2 = C \quad / \sqrt{\ }$

$\Rightarrow \quad y'(1-y) = D \quad (D = \sqrt{-c})$

NDE z ločljivimi spremenljivkami

$dy(1-y) = D\,dx \quad / \int$

$y - y^2/2 = Dx + c$

$y(0)=0 \Rightarrow \quad c = 0$

$y(1)=1 \Rightarrow \quad D = 1/2$

$\rightarrow \quad y - y^2/2 = \frac{1}{2}x \quad \rightarrow \quad y^2 - 2y + 1 = -x + 1$

$(y-1)^2 = 1 - x$

ni možno,

$y - 1 = \pm\sqrt{1-x}$

ker mora biti $y(0)=0$

$y = 1 \pm \sqrt{1-x}$

torej $\underline{\underline{y = 1 - \sqrt{1-x}}}$

(3) $y = \sum a_k z^{k+r}$.

$$zy'' = \sum a_k (k+r)(k+r-1) z^{k+r-1}$$

$$4y' = \sum a_k 4(k+r) z^{k+r-1}$$

$$-zy = -\sum a_k z^{k+r+1} \overset{k \to k-2}{=} -\sum a_{k-2} z^{k+r-1}$$

$$\text{NDE} \rightarrow a_k \left((k+r)(k+r-1) + 4(k+r)\right) - a_{k-2} = 0$$

$$a_k \underbrace{(k+r)(k+r+3)} - a_{k-2} = 0$$

kar. enačba: $k=0$ $\quad r(r+3)=0$ $\quad r_1 = 0,$ $\quad r_2 = -3$

Izberemo $r_1 = 0$, ker $\operatorname{Re} r_1 > \operatorname{Re} r_2$.

$\Rightarrow$ $\underline{a_k k(k+3) - a_{k-2} = 0}$, ker $\underline{a_{-1} = 0, a_{-2} = 0}$

$k=0$: a_0 poljuben

$k=1$: $a_1 \cdot 4 = 0 \Rightarrow a_1 = 0$

splošen k $\quad a_k = \dfrac{a_{k-2}}{k(k+3)}$, ker ta zveza poveže indekse, ki se razl. za 2, se splača pisati

$k = 2n+p$ $\quad$ kjer $p = 0$ ali 1

③ nadaljevanje

$$a_{2n+p} = \frac{a_{2n+p-2}}{(2n+p)(2n+p+3)} = \frac{a_{2n+p-4}}{(2n+p)(2n+p+3)(2n+p-2)(2n+p+1)} =$$

$$= \frac{a_{2n+p-6}}{\underbrace{(2n+p)}(2n+p+3)\underbrace{(2n+p-2)}(2n+p+1)\underbrace{(2n+p-4)}(2n+p-1)} =$$

$$= \dots = \frac{a_p}{(2n+p+3)!!\,\underbrace{(2n+p)!!}}$$

ker $a_1 = 0 \Rightarrow a_{2n+1} = 0$

sodi člen

$$a_{2n} = \frac{a_0}{(2n+3)!!\,(2n)!!}$$

Torej $$\boxed{y = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n+3)!!\,(2n)!!} z^{2n}}$$

$$\cancel{(2n)!! = (2n)(2n-2)(2n-4)\dots 2 = 2^n n!}$$

$$\cancel{y = a_0 \sum_{n=0}^{\infty} \frac{z^{2n}}{2^n n!}}\ (2n+3)!!\,(2n)!! =$$

$$= \left[(2n+3)(2n+1)(2n-1)(2n-3)\dots\right]\left[(2n)(2n-2)(2n-4)\dots\right]$$

$2n+3=1$

$$= \frac{2n+2}{(2n+3)!} \rightarrow y = \sum \frac{z^{2n}(2n+2)}{(2n+3)!} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+2}}{(2n+2)!} - \frac{1}{z^3}\frac{z^{2n+3}}{(2n+3)!} =$$

$$= \frac{1}{z^2}(\operatorname{ch} z - 1) - \frac{1}{z^3}(\operatorname{sh} z - z) = \frac{z \operatorname{ch} z - \operatorname{sh} z}{z^3}$$

(4) S-L l. problem

$$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = \lambda y \qquad / \cdot x^2$$

$$y(1) = 0, \quad y(0) \text{ omej.}$$

$$x^2 y'' + x y' - 4y - \lambda x^2 y = 0$$

z raj: $y = A J_2(\sqrt{-\lambda}\, x) + B N_2(\sqrt{-\lambda}\, x)$

$y(0)$ omej $\Rightarrow B = 0$

$y(1) = 0 \qquad J_0(\sqrt{-\lambda}) = 0 \Rightarrow \sqrt{-\lambda}$ je ničla J_2

označimo ξ_k = k-ta ničla $J_2 \rightarrow \sqrt{-\lambda} = \xi_k$

$\Rightarrow$ l.f-je: $\quad y_k = J_2(\xi_k x), \quad \lambda_k = -\xi_k^2$

razvoj x^2 po y_k:

števec $\qquad \xi_k x = y$

$$x^2 = \sum_{k=1}^{\infty} a_k y_k, \quad a_k = \frac{\int_0^1 x J_2(\xi_k x) x^2 \, dx}{\int_0^1 x J_2^2(\xi_k x)\, dx} = \frac{\frac{1}{\xi_k^4} \int_0^{\xi_k} y^3 J_2(y)\, dy}{\frac{1}{2} J_2'^2(\xi_k)}$$

$$= \frac{2 J_3(\xi_k)}{\xi_k J_2'^2(\xi_k)}$$

(4 mark) $u = \sum_{k=1}^{\infty} y_k(x) c_k(t)$

PDE $\rightarrow$ $\sum -\xi_k^2 y_k(x) c_k(t) = \sum y_k \ddot{c}_k + \sum a_k y_k$

$\rightarrow -\xi_k^2 c_k = \ddot{c}_k + a_k \quad (*)$

I.C. $\rightarrow$ $c_k(0) = 0, \quad \dot{c}_k(0) = 0$

hom: $c_k^{(hom)} = A\cos(\xi_k t) + B\sin(\xi_k t)$

part: $c_k^{(part.)} = C \xrightarrow{(*)} -\xi_k^2 C = a_k$

$\Rightarrow C = -\frac{a_k}{\xi_k^2}$

$\rightarrow$ gen. res.: $c_k(t) = A\cos(\xi_k t) + B\sin(\xi_k t) - \frac{a_k}{\xi_k^2}$

$t = 0$: $c_k(0) = 0 \rightarrow A = a_k/\xi_k^2$

: $\dot{c}_k(0) = 0 \Rightarrow B = 0$

$\rightarrow$ $c_k(t) = \frac{a_k}{\xi_k^2}\left(\cos(\xi_k t) - 1\right)$

$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} J_2(\xi_k x)\left(\cos(\xi_k t) - 1\right) \frac{2 J_3(\xi_k)}{\xi_k^3 J_2'^2(\xi_k)}$