Analiza 2a

Pisni izpit

13. 5. 2014

Ime in priimek

Vpisna številka

Σ

Veliko uspeha!

1. naloga (20 točk)

Za vsako od spodnjih trditev v pripadajoči kvadratek čitljivo označi, če je trditev pravilna $\boxed{\text{P}}$ oziroma napačna $\boxed{\text{N}}$.

Če ne veš, pusti kvadratek prazen, ker se nepravilni odgovor šteje negativno!

N Če ima $D \subset \mathbb{R}^n$ mero 0, ima tudi volumen 0.

P Naj bo $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$, kjer sta $f, g : \mathbb{R}^2 \to \mathbb{R}$ zvezno odvedljivi funkciji, za kateri velja $f(0,0) = g(0,0) = 0$ in $f_x(0,0)g_y(0,0) \neq g_x(0,0)f_y(0,0)$. Potem obstajata taki okolici U in V za $(0,0)$, da je zožitev $F : U \to V$ difeomorfizem.

N Če je $f : \mathbb{R}^2 \to \mathbb{R}$ dvakrat zvezno odvedljiva funkcija, obstaja taka linearna preslikava $L : \mathbb{R}^2 \to \mathbb{R}$, da je $\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-L(x,y)}{x^2+y^2} = 0$.

N Taylorjeva vrsta s središčem v $(0,0)$ za funkcijo $f(x,y) = \frac{1}{1+x^2y^2}$ je povsod konvergentna.

P Funkcija $F(x) = \int_0^{x^2} \frac{\sin(x+t)}{\ln(1+x^4t^{2014})} dt$ je odvedljiva na $\mathbb{R}$.

P Denimo, da za funkcijo $f : \mathbb{R} \to \mathbb{R}$ velja $f(x) = 0$ za $x \leq 0$ in $f(x) = 2014$ za $x > 0$. Tedaj Fourierova vrsta za f na intervalu $[-\pi, \pi]$ pri $x = 0$ konvergira proti 1007.

N Velja $B(\frac{1}{2}, \frac{1}{2}) = \pi^2$.

P Funkcija $f(x,y) = \frac{1}{x^2+y^2}$ je integrabilna na kvadru $[0,1] \times [1,2]$.

P Če je $f : [1,4] \times [2,6] \to \mathbb{R}$ zvezna funkcija, velja $\int_2^3 dx \int_3^5 f(x,y)dy = \int_3^5 dy \int_2^3 f(x,y)dx$.

P Enačba $(1 + x^2 + y^2)^2 - z^2 = 0$ določa gladko podmnogoterost v $\mathbb{R}^3$.

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Čas pisanja je 100 minut. Možno je doseči 100 točk.

Veliko uspeha!

Naloga 2

Naj bosta in r in φ polarni koordinati ter $x = r\cos\varphi$ in $y = r\sin\varphi$ običajni kartezični koordinati. Za dvakrat zvezno odvedljivo funkcijo u prevedi izraz

$$I = ru_{r\varphi} - u_\varphi$$

v kartezične koordinate.

Naloga 3

Vrtičkar želi iz tanke pločevine zgraditi lopo za orodje, ki ima obliko kvadra. Vse štiri stene želi zaradi trdnosti narediti iz dvojne plasti pločevine, strop lope iz enojne plasti pločevine, tal pa v lopi ne namerava narediti. Kako naj zgradi lopo, da bo imela ta največjo možno prostornino, če ima na voljo $192m^2$ pločevine?

Naloga 4

Naj bosta $a, \alpha > 0$. Neenačba $(x^2 + y^2 + z^2)^3 \leq a^2(x^2 + y^2)^2$, podaja nehomogeno telo $D \subset \mathbb{R}^3$, ki ima v točki (x, y, z) gostoto $\rho(x, y, z) = (x^2 + y^2)^\alpha$.

a) Izračunaj maso telesa D.

b) Izračunaj vztrajnostni moment telesa D okrog osi z za $\alpha = 2$.

Naloga 5

Izračunaj integral

$$\int_0^\infty \frac{1 - \cos x}{e^x x^2} dx.$$

POMOČ: $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

(2) $x = r\cos\varphi \qquad y = r\sin\varphi \implies$

$$\implies \frac{\partial x}{\partial r} = \cos\varphi \quad \frac{\partial y}{\partial r} = \sin\varphi$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\varphi = -y \qquad \frac{\partial y}{\partial \varphi} = r\cos\varphi = x$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\cdot\frac{\partial}{\partial y} = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\cdot\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$r\cdot\frac{\partial^2}{\partial r\partial\varphi} = r\cdot\left(\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) =$$

$$= \overbrace{r\cos\varphi}^{x}\left(-y\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y}\right) +$$

$$+ \underbrace{r\sin\varphi}_{y}\left(-\frac{\partial}{\partial x} - y\frac{\partial^2}{\partial x\partial y} + x\frac{\partial^2}{\partial y^2}\right) =$$

$$= -xy\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} + x^2\frac{\partial^2}{\partial x\partial y} - y\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial x^2\partial y^2}$$

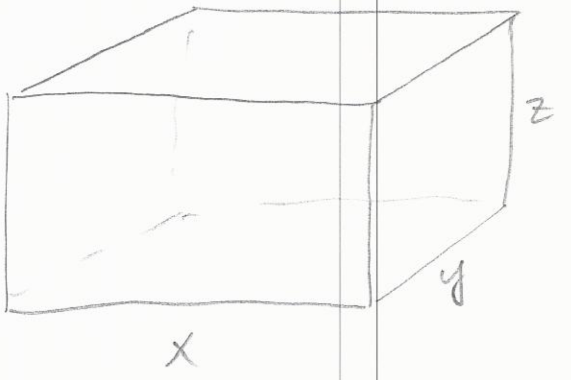
$$+ xy\frac{\partial^2}{\partial y^2}$$

$$r u_{r\varphi} - u_\varphi = -xy\,u_{xx} + \cancel{x u_y} + (x^2 - y^2)\,u_{xy} \cancel{- y u_x} + xy\,u_{yy}$$

$$- (\cancel{-y u_x} + \cancel{x u_y}) =$$

$$= xy(u_{yy} - u_{xx}) + (x^2 - y^2)\,u_{xy}$$

3



$V = xyz$

$S = xy + \underbrace{4xz + 4yz}_{\text{dvojne stene!}} =$

Vezani ekstrem: $L = xyz + \lambda(xy + 4xz + 4yz)$

$L_x = yz + \lambda(y + 4z) = 0$

$L_y = xz + \lambda(x + 4z) = 0$

$L_z = xy + \lambda(4x + 4y) = 0$

Ker so $x, y, z > 0 \implies \lambda = \dfrac{-yz}{y+4z} \overset{(1)}{=} \dfrac{-xz}{x+4z} \overset{(2)}{=} \dfrac{-xy}{4x+4y}$

$(1) \iff -\cancel{xyz} - 4yz^2 = -\cancel{xyz} - 4xz^2 \xRightarrow{z \neq 0} \boxed{x = y}$

$(2) \xRightarrow{x=y} -8\cancel{x^2}z = -\cancel{x^2}(x + 4z) \xRightarrow{x \neq 0} \boxed{x = 4z}$

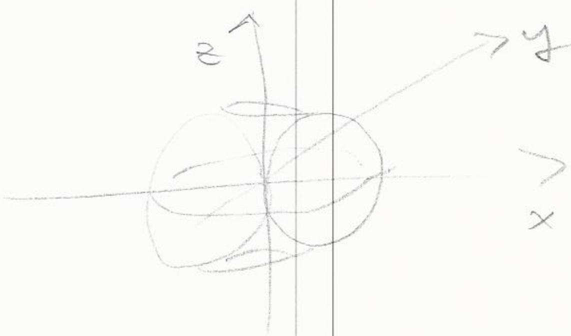
$\implies S = 16z^2 + 16z^2 + 16z^2 = 48z^2 = 192$

$\implies z^2 = 4 \implies \boxed{z = 2 \,,\; x = y = 8}$

4

$(x^2+y^2+z^2)^3 \le a^2(x^2+y^2)^2$ v sferič. koor.

$r^6 \le a^2 r^4 \cos^4\vartheta \iff r \le a\cos^2\vartheta$



$$m = \int_D \rho(x,y,z)\,dV = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\vartheta \int_0^{a\cos^2\vartheta} r^{2\alpha}\cos^{2\alpha}\vartheta \cdot r^2\cos\vartheta\, dr =$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \left[\frac{r^{3+2\alpha}}{3+2\alpha}\right]\Bigg|_0^{a\cos^2\vartheta} \cos^{2\alpha+1}\vartheta\, d\vartheta =$$

$7+6\alpha = 2p-1$ $\quad$ $2q-1=0$

$$= \frac{4\pi a^{3+2\alpha}}{3+2\alpha}\int_0^{\frac{\pi}{2}} \cos^{6+4\alpha+2\alpha+1}\vartheta\, d\vartheta =$$

$$= \frac{4\pi a^{3+2\alpha}}{3+2\alpha}\,\frac{1}{2}\,B(p,q) = \frac{2\pi a^{3+\alpha}}{3+2\alpha}\,B\left(4+3\alpha, \frac{1}{2}\right) \quad 3$$

$p = 4+\frac{3\alpha}{2}$

$q = \frac{1}{2}$

$$J = \int_D \overbrace{(x^2+y^2)\rho(x,y,z)}^{\text{"kot } \rho(x,y,z) \text{ pri } \alpha+1\text{"}}\,dV = \ldots\ldots \int_0^{a\cos^2\vartheta} r^{2\alpha+2}\cos^{2\alpha+2}\vartheta \ldots =$$

$$= \frac{2\pi a^9}{9}\,B\left(\overbrace{4+9}^{13}, \frac{1}{2}\right) = \frac{2\pi a^9}{9}\,\frac{12!\,\sqrt{\pi}}{\Gamma\left(\frac{27}{2}\right)} =$$

$\alpha+1=3$ v vlogi α

$$= \frac{2\pi a^9\, 12!\, \cancel{\sqrt{\pi}}}{\frac{25\cdot 23\cdot \ldots \cdot 3\cdot 1}{2^{13}}\cancel{\sqrt{\pi}}} = \frac{2^{14}\pi a^9\, 12!}{25!!}$$

(5) $F(a) = \int_0^\infty \frac{\sin^2(ax)}{e^x x^2}\, dx$

$$F'(a) = \int_0^\infty \frac{2\sin(ax)\cos(ax)\cdot x}{e^x x^2}\, dx =$$

$$= \int_0^\infty \frac{\sin(2ax)}{e^x\, x}\, dx$$

$$F''(a) = \int_0^\infty \frac{\cos(2ax)\cdot 2\cancel{x}}{e^x \cdot \cancel{x}}\, dx =$$

$$= 2\cdot\left(\frac{e^{-x}}{1+4a^2}\left(-\cos 2ax + 2a\sin 2ax\right)\right)\Big|_0^\infty =$$

$$= \frac{2}{1+4a^2}$$

$$F'(a) = \operatorname{arctg} 2a + C \xRightarrow{a=0} C = 0$$

$$F(a) = \int \operatorname{arctg} 2a\, da = a\operatorname{arctg} 2a - \int \frac{2a\, da}{\underbrace{1+4a^2}_{u}} =$$

$du = 8a\,da$

$$= a\operatorname{arctg} 2a - \frac{1}{4}\int \frac{du}{u} = a\operatorname{arctg} 2a - \frac{1}{4}\ln(1+4a^2) + D$$

$$\xRightarrow{a=0} D = 0 \;\;\&\;\; F(a) = a\operatorname{arctg} 2a - \frac{1}{4}\ln(1+4a^2)$$

$$\boxed{I = F\left(\tfrac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4}\ln 2}$$