Pisni izpit iz Analize 2b

30. 6. 2014

Veliko uspeha!

Ime in priimek

Vpisna številka

Σ

1. naloga (20 točk)

Za vsako od spodnjih trditev v pripadajoči kvadratek čitljivo označi, če je trditev pravilna $\boxed{\text{P}}$ oziroma napačna $\boxed{\text{N}}$.

Če ne veš, pusti kvadratek prazen, ker se nepravilni odgovor šteje negativno!

$\boxed{\text{P}}$ Če za vektorsko polje $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$ velja $\mathrm{div} F = 3$, potem velja $|\int_{S((0,0,0),1)} \vec{f} d\vec{S}| = 4\pi$.

$\boxed{\text{P}}$ Če je $F(z)$ Lapleceova transformiranka zvezne funkcije $f : [0, \infty) \to \mathbb{R}$ z eksponentno rastjo, velja $\lim_{|z|\to\infty} F(z) = 0$.

$\boxed{\text{P}}$ Vsaka omejena cela funkcija je konstantna.

$\boxed{\text{N}}$ Če so E, F in G koeficienti 1. fundamentalne forme gladke ploskve S in so koordinatne krivulje ploskve S medsebojno pravokotne, velja $G = 0$.

$\boxed{\text{N}}$ Če je funkcija f holomorfna na kolobarju $A(0; 1, 2)$ in $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ njena Laurentova vrsta, velja $\int_{|z|=1} f(z)dz = 2ic_{-1}\pi$.

$\boxed{\text{P}}$ Če je f holomorfna na $D \setminus \{a\}$, kjer je $D \subset \mathbb{C}$ odprta omejena množica s kosoma gladkim robom, in ima f v a pol stopnje n, potem je $\int_{\partial D} \frac{f'(z)}{f(z)} dz = -2ni\pi$.

$\boxed{\text{P}}$ Če je $f(x + iy) = u(x, y) + iv(x, y)$ holomorfna funkcija, je $h = u + 2v$ harmonična funkcija.

$\boxed{\text{N}}$ Integral potencialnega polja po zaključeni krivulji je enak 0.

$\boxed{\text{P}}$ Množico $H = \{z \in \mathbb{C} \mid \mathrm{Im}\, z < 0\}$ lahko konformno preslikamo na množico $B = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}\, z < 1\}$.

$\boxed{\text{N}}$ Če je $D \subset \mathbb{R}^2$ omejeno območje s kosoma gladkim robom ∂D, velja $\int_{\partial D} y^2 dx + x^2 dy = 0$.

(2) $\quad \operatorname{rot} \vec{F}$ $\overset{x}{(yze^{xyz}+Axyz}, \overset{y}{Bxze^{xyz}+x^2z}, \overset{z}{Cxye^{xyz}+x^2y})$

potencial

$$\Longrightarrow 0 = \operatorname{rot} \vec{F} = (\cancel{x^2} + Cx^2yze^{xyz} - Bx^2yze^{xyz} \cancel{-x^2},$$

$$xy^2z e^{xyz} + Axy - Cxy^2z e^{xyz} - 2xy,$$

$$B(ze^{xyz} + xyz^2e^{xyz}) + 2xz - ze^{xyz} - xyz^2e^{xyz} - Axz)$$

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$B = C$, $\boxed{A=2, \quad C = 1 = B}$

$$\vec{F} = (yze^{xyz} + 2xyz, \; xze^{xyz} + x^2z, \; xye^{xyz} + x^2y) =$$

$= \operatorname{grad} u \quad \Longrightarrow$

$$u = e^{xyz} + x^2yz + C_1(y,z)$$
$$u = e^{xyz} + x^2yz + C_2(x,z)$$
$$u = e^{xyz} + x^2yz + C_3(x,y)$$

$/\!/ \; C$

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$$\int_{(0,0,0)}^{(1,1,1)} \vec{F} \, d\vec{r} = u(1,1,1) - u(0,0,0) =$$

$$= e + 1 - 1 = \underline{\underline{e}}$$

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(3)

a) $\vec{r}(r,\varphi) = (r\cos\varphi, r\sin\varphi, r^2)$

$\vec{r}_r = (\cos\varphi, \sin\varphi, 2r)$

$\vec{r}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$$\left.\begin{array}{l} E = 1+4r^2 \\ F = 0 \\ G = r^2 \end{array}\right\} \Rightarrow \sqrt{EG-F^2} = r\sqrt{1+4r^2}$$

$$J = \int_S (x^2+y^2)\,\rho\, dS = \rho \int_0^{2\pi} d\varphi \int_0^2 r^2 \cdot r\sqrt{1+4r^2}\, dr =$$

$u^{3/2} - u^{1/2}$

$$= 2\pi\rho \int_1^{17} \frac{u-1}{4} \cdot \sqrt{u} \cdot \frac{du}{8} =$$

$u = 1+4r^2 \quad du = 8r\,dr \qquad r^2 = \frac{u-1}{4}$

$$= \frac{\pi\rho}{16}\left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right)\Big|_1^{17} =$$

$17 \cdot 5 = 85$

$$= \frac{\pi\rho}{8}\left(\frac{289\sqrt{17}}{5} - \frac{17\sqrt{17}}{3} - \frac{1}{5} + \frac{1}{3}\right) =$$

$17 \cdot 17$
17
119
$289 \cdot 3$
867
-85
782

$$= \frac{\pi\rho(782\sqrt{17}+2)}{8 \cdot 15}$$

$-3+5=2$

$$\rho = \frac{m}{S} \; ; \; S = \int_0^{2\pi} d\varphi \int_0^2 r\sqrt{1+4r^2}\, dr =$$

$$= 2\pi \int_1^{17} \sqrt{u} \cdot \frac{du}{8} = \frac{\pi}{4}\left(\frac{2u^{3/2}}{3}\right)\Big|_1^{17} =$$

$$= \frac{\pi}{6}(17\sqrt{17}-1) \Rightarrow \rho = \frac{6m}{\pi(17\sqrt{17}-1)} \Rightarrow$$

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$$\Rightarrow \boxed{J = \frac{m(782\sqrt{17}+2)}{20(17\sqrt{17}-1)}}$$

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b) $\vec{r}(x) = (x, x, 2x^2)$

$\vec{f}(x,y,z) = (x, y^2, z) = (x, x^2, 2x^2)$

$\dot{\vec{r}} = (1, 1, 4x)$

$$\Longrightarrow \int_K \vec{f}\, d\vec{r} = \int_{-\sqrt{2}}^{\sqrt{2}} (x + 8x^3 + x^2)\, dx =$$

$$= \frac{x^3}{3}\Big|_{-\sqrt{2}}^{\sqrt{2}} = \frac{4\sqrt{2}}{3}$$

$\cancel{\dot{\vec{r}}, \dot{\vec{r}}}$

$\cancel{(x, x, 2x^2)(1, 1, 4x)}$

$\cancel{|\dot{\vec{r}}|^2 \dot{\vec{r}} = 2x^2 + 8}$

$$\sqrt{2 + 16x^2} = \sqrt{2}\sqrt{1 + 8x^2}$$

(4)

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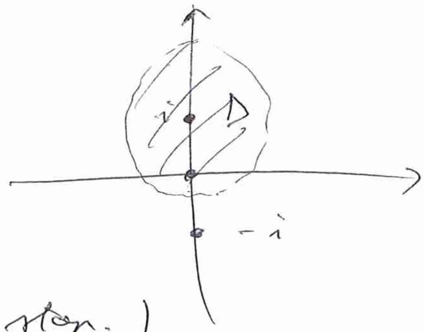
a) $$\oint\limits_{|z-i|=\frac{6}{5}} \frac{(z+1)\,dz}{z^3(z^2+1)} = I$$



Sing v D: $z=0$ (pol 3. stop.)

$z=i$ (pol 1. stop.)

$$I = 2\pi i \left(\mathrm{Res}(f,0) + \mathrm{Res}(f,i) \right)$$

$$\mathrm{Res}(f,0) = \frac{1}{2!}\left(\frac{z+1}{z^2+1}\right)''\Bigg|_{z=0} = \frac{1}{2}\left(\frac{\overbrace{z^2+1-2z^2-2z}^{-z^2-2z+1}}{(z^2+1)^2}\right)'\Bigg|_{z=0}$$

$$= \frac{1}{2}\,\frac{(-2z-2)(z^2+1)^2 - (-z^2-2z+1)\,2(z^2+1)\cdot 2z}{(z^2+1)^4}\Bigg|_{z=0} =$$

$$= \frac{1}{2}\,\frac{-2}{1} = \underline{\underline{-1}}$$

$$\mathrm{Res}(f,i) = \frac{i+1}{i^3(i+i)} = \frac{i+1}{2}$$

$$I = 2\pi i\left(\frac{i-1}{2}\right) = \underline{\underline{-\pi(i+1)}}$$

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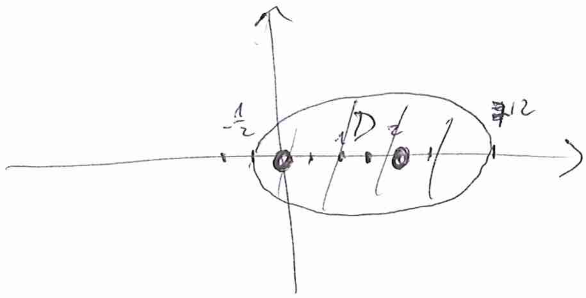
b) $I = 2\pi i \left(\mathrm{Res}(f,0) + \mathrm{Res}(f,2)\right)$



$\mathrm{Res}(f,0) = \mathrm{ch}\left(-\frac{1}{2}\right) = \mathrm{ch}\,\frac{1}{2}$

$$\mathrm{Res}(f,2):\quad \overset{=\frac{1}{2}}{\frac{1}{(z-2)+2}} \cdot \left(\underbrace{1 + \frac{1}{2!}(z-2)^{-2} + \frac{1}{4!}(z-2)^{-4} + \dots}\right)$$

$$= \frac{1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \dots\right) \cdot \left(\;\_\_\; \right) =$$

$$= \frac{1}{2} \cdot \left(\dots + \frac{1}{z-2}\left(-\frac{1}{2!} \cdot \frac{1}{2} - \frac{1}{4!}\,\frac{1}{2^3} - \frac{1}{6!}\,\frac{1}{2^5} - \dots\right) + \dots\right)$$

$$\Rightarrow c_{-1} = \mathrm{Res}(f,2) = -\left(\frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 + \frac{1}{6!}\left(\frac{1}{2}\right)^6 + \dots\right) =$$

$$= 1 - \mathrm{ch}\,\frac{1}{2}$$

$$\Rightarrow \quad I = 2\pi i$$

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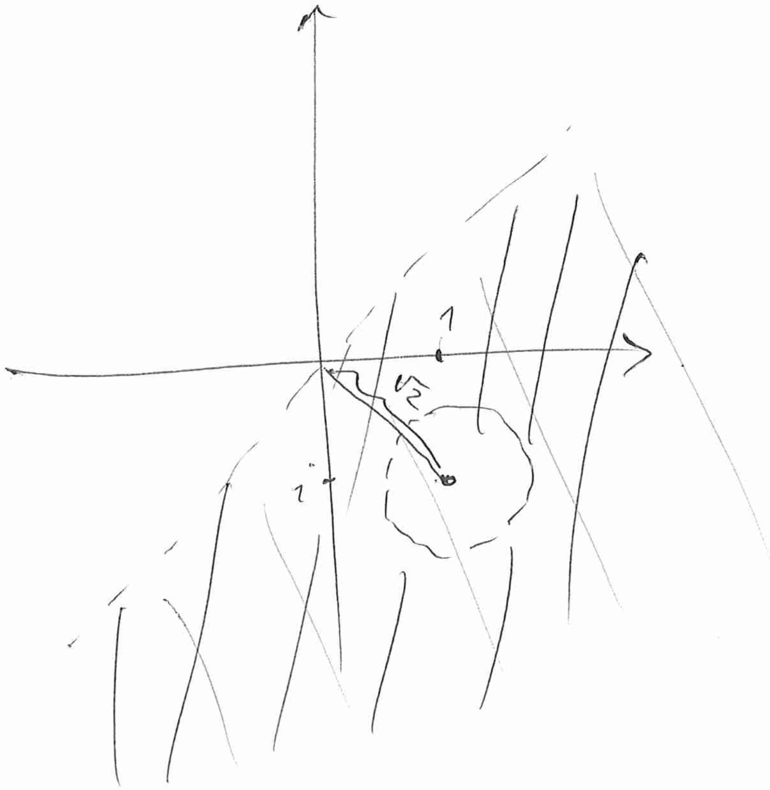
a)

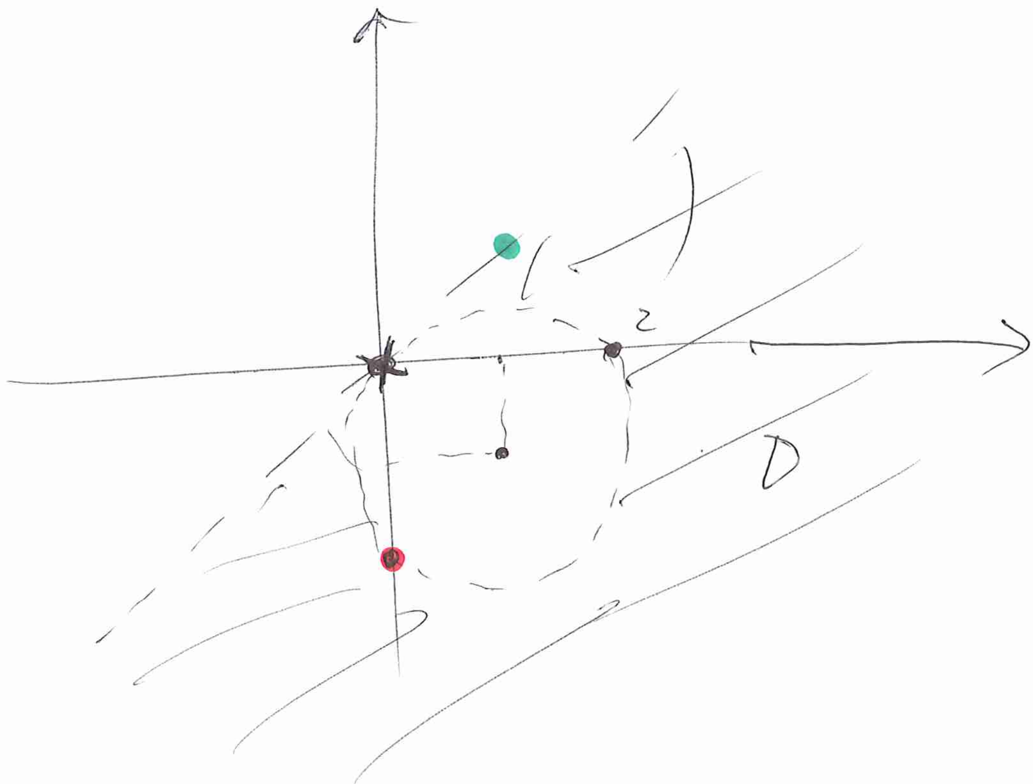


Če $a < \sqrt{2}$, D ni enost. povezano

$\Longrightarrow$ ne $\exists$ konformna presl. $f: D \to \Delta$.

Če $a \geq \sqrt{2} = a_0 \Longrightarrow D \sim \Delta$

b) $a = a_0$



$$f_1(z) = \frac{az+b}{cz+d}$$

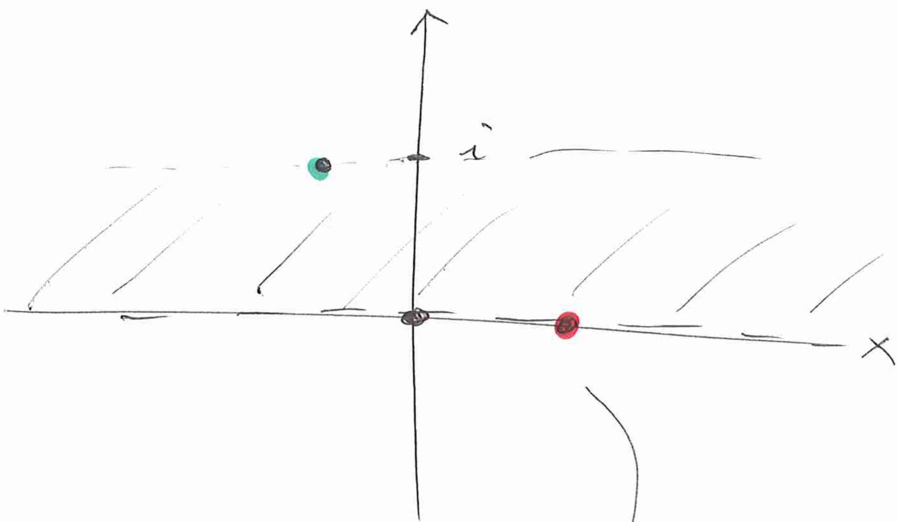
$$\left.\begin{aligned} f_1(0) &= \infty \\ f_1(2) &= 0 \end{aligned}\right\} \Rightarrow f_1(z) = \frac{z-2}{cz}$$

$$f_1(-2i) = 1$$

$$\Rightarrow \frac{-2i-2}{-2ci} = 1 \Rightarrow -2i-2 = -2ci$$

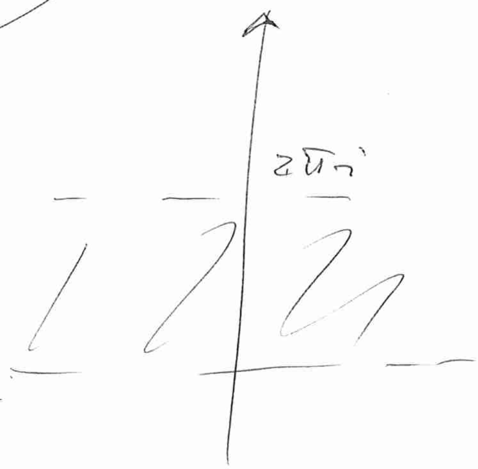
$$\Rightarrow \underline{\underline{c}} = \frac{i+1}{i} = \underline{\underline{1-i}}$$

$$D \xrightarrow{f_1}$$

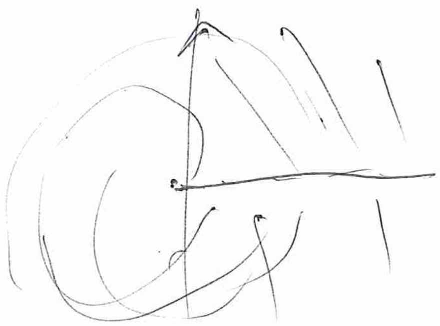


$$f_1(1+i) = \frac{1+i-2}{(1-i)(1+i)} = \frac{i-1}{2}$$

$$f_2(z) = 2\pi z \xrightarrow{f_2}$$



$$f_3(z) = e^z \longrightarrow$$



$$\xrightarrow{\sqrt{z}}$$

$$\downarrow f_0$$