ANALIZA 3 - 1. pisni izpit

31. 1. 2011

Ime in priimek:
Vpisna št.:

1. Dana je NDE
$$y' + \ln y' = y.$$
   (a) Ali ima ta NDE kako singularno rešitev ?

   (b) V parametrični obliki zapiši tisto rešitev te NDE, ki zadošča pogoju $y(3) = 1$.

2. Poišči vse krivulje z lastnostjo, da tangenta na vsako točko in zveznica te točke s koordinatnim izhodiščem oklepata kot $\frac{\pi}{4}$. Predpostaviti smeš, da je krivulja dobljena kot graf (gladke) funkcije $y = y(x)$.

3. Poišči splošno rešitev $x = x(t), y = y(t)$ linearnega sistema NDE **drugega** reda

$$\begin{aligned} \ddot{x} &= 11x + 6y, \\ \ddot{y} &= -20x - 11y. \end{aligned}$$

Pomoč: Najprej dani sistem linearnih NDE drugega reda prevedi na (večji) sistem linearnih NDE prvega reda.

4. Poišči vse ekstremale funkcionala

$$I[y] = \int_0^1 \frac{y'^2}{1+y^2}\,dx$$

na prostoru funkcij, ki ustrezajo pogojem/vezem

$$y(0) = y(1) = 0,$$

$$\int_0^1 \frac{dx}{1+y^2} = \frac{1}{2}.$$

1. a) $y' + \ln y' = y \quad / \frac{\partial}{\partial y'}$

$1 + \frac{1}{y'} = 0 \rightarrow y' = -1$ ⟶̸ $\ln y'$ definiran

$\Rightarrow$ ni sing. reš.

b) $y = u + \ln u$, $y' = u$, $x = v$ / $u = \frac{d(u + \ln u)}{dv} = \frac{du}{dv}\left(1 + \frac{1}{u}\right)$

$dv = du\left(\frac{1}{u} + \frac{1}{u^2}\right)$

$v = C + \ln u - \frac{1}{u}$

$$x = C + \ln u - \frac{1}{u}$$
$$y = u + \ln u$$

$y(x=3) = 1$

$3 = C + \ln u - \frac{1}{u}$

$1 = u + \ln u \Rightarrow u = 1$

$\Rightarrow 3 = C + \ln 1 - \frac{1}{1}$

$C = 4$

2. 

$\pm 1 = tg(\pm 45°) = tg\left(\mathrm{arctg}\frac{y}{x} - \mathrm{arctg}\, y'\right) = \frac{y/x - y'}{1 + \frac{y}{x}y'}$

obravnavajmo le $+1$, -1 gre podobno

$1 + \frac{y}{x}y' = \frac{y}{x} - y' \rightarrow y'\left(1 + \frac{y}{x}\right) = \frac{y}{x} - 1 \rightarrow y' = \frac{y/x - 1}{y/x + 1}$

$u = y/x$, $y = ux$, $y' = u'x + u \rightarrow u'x + u = \frac{u-1}{u+1} \rightarrow u'x = \frac{u - 1 - u^2 - u}{u+1}$

$\frac{du}{dx}x = -\frac{1+u^2}{u+1} \rightarrow \frac{du}{u^2+1}(u+1) = -\frac{dx}{x} \rightarrow \frac{1}{2}\ln(1+u^2) + \mathrm{arctg}\, u = -\ln x + C$

$\frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) + \frac{1}{2}\ln x^2 + \mathrm{arctg}\, y/x = C \rightarrow \ln\sqrt{x^2+y^2} + \mathrm{arctg}\, y/x = C$

polarne koord. $\rightarrow$ $\ln r + \varphi = C \rightarrow r e^{\varphi} = K \rightarrow r = K e^{-\varphi}$

logaritemska spirala

③ $\ddot{x} = 11x + 6y$ $\quad \dot{x} = u$

$\ddot{y} = -20x - 11y \rightarrow \dot{y} = v$

$\dot{u} = 11x + 6y$

$\dot{v} = -20x - 11y$

$\vec{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$, $\dot{\vec{x}} = A\vec{x}$, kjer

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 11 & 6 & 0 & 0 \\ -20 & -11 & 0 & 0 \end{pmatrix}; \quad \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & & 1 & \\ & -\lambda & & 1 \\ 11 & 6 & -\lambda & \\ -20 & -11 & & -\lambda \end{pmatrix} \begin{matrix} \cdot\lambda + \\ \cdot\lambda + \end{matrix}$$

$$= \det\begin{pmatrix} -\lambda & & 1 & \\ & -\lambda & & 1 \\ 11-\lambda^2 & 6 & 0 & 0 \\ -20 & -11-\lambda^2 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 11-\lambda^2 & 6 \\ -20 & -11-\lambda^2 \end{pmatrix} =$$

~~$= (11-\lambda^2)^2 + 120 = 121 - 22\lambda^2 + \lambda^4 + 120 = \lambda^4 - 22\lambda^2 + 241$~~

$$= -121 + \lambda^4 + 120 = \lambda^4 - 1 \rightarrow \lambda_1 = 1 \quad \lambda_3 = i$$
$$\lambda_2 = -1 \quad \lambda_4 = -i$$

$\lambda_1 = 1$: $u = x$, $v = y$, $11x + 6y = u$, $-20x - 11y = v$ $\rightarrow$ $10x + 6y = 0$, $-20x - 12y = 0$ $\rightarrow 5x + 3y = 0 \rightarrow v_1 = \begin{pmatrix} 3 \\ -5 \\ 3 \\ -5 \end{pmatrix}$

$\lambda_2 = -1$: $u = -x$, $v = -y$, $\vdots\ -u$, $-v$ $\rightarrow 5x + 3y = 0 \rightarrow v_2 = \begin{pmatrix} 3 \\ -5 \\ -3 \\ 5 \end{pmatrix}$

$\lambda_3 = i$: $u = ix$, $v = iy$, $11x + 6y = iu$, $-20x - 11y = iv$ $\rightarrow$ $12x + 6y = 0 \rightarrow v_3 = \begin{pmatrix} -1 \\ 2 \\ -i \\ 2i \end{pmatrix}$, $\lambda_4 = i$ $v_4 = \begin{pmatrix} -1 \\ 2 \\ i \\ -2i \end{pmatrix}$

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splošna rešitev: $\vec{x} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = c_1 e^t v_1 + c_2 e^{-t} v_2 + c_3 e^{it} v_3 + c_4 e^{-it} v_4$

$$x = 3c_1 e^t + 3c_2 e^{-t} - c_3 e^{it} - c_4 e^{-it}$$

$$y = -5c_1 e^t - 5c_2 e^{-t} + 2c_3 e^{it} + 2c_4 e^{-it}$$

ali v realnem (Re x ter Im x)

$$x = 3c_1 e^t + 3c_2 e^{-t} - \tilde{c}_3 \cos t - \tilde{c}_4 \sin t$$

$$y = -5c_1 e^t - 5c_2 e^{-t} + 2\tilde{c}_3 \cos t + 2\tilde{c}_4 \sin t$$

(4.) $L = \frac{y'^2 - \lambda}{1+y^2}$, ker L ni odvisen od x, je

$L - y' L_{y'} = C \Rightarrow \frac{y'^2-\lambda}{1+y^2} - y' \frac{2y'}{1+y^2} = c$

$-y'^2 - \lambda = c + cy^2$ $\quad (-\lambda - c) = A$

$\Rightarrow \quad cy^2 + y'^2 = A$ (lahko rešujemo kot DE z ločljivimi spr.; hitreje je, če odvajamo)

$2yy'C + 2y'y'' = 0 \,/ : y'$ (če bi bilo $y'=0$, bi bilo $y = \text{konst} \rightarrow y = 0 \; * \int_0^1 \frac{1}{1+y^2} = \frac{1}{2}$!) $\quad y(0)=y(1)=0$

$Cy + y'' = 0$

① $c < 0 \rightarrow y = \tilde{A} e^{\sqrt{-c}\,x} + \tilde{B} e^{-\sqrt{-c}\,x}$, $\xrightarrow{y(0)=y(1)=0} y = 0 \; * \int \dots = 1/2$

torej ni $c < 0$

② $c = 0$ podobno izločimo

③ $c > 0$; pišimo $C = \mu^2 \rightarrow y = \tilde{A} \sin \mu x + \tilde{B} \cos \mu x$

$y(0) = 0 \rightarrow \tilde{B} = 0$, $y(1) = 0 \rightarrow 0 = \tilde{A} \sin \mu$ ali $\tilde{A} = 0 \rightarrow y = 0 \; *$

$\mu = k\pi$, $k \in \mathbb{Z} - \{0\}$

Torej $y(x) = \tilde{A} \sin(k\pi x)$.

Vstavimo v integral:

tabele določenih integralov

$$\frac{1}{2} = \int_0^1 \frac{dx}{1 + \tilde{A}^2 \sin^2(k\pi x)} = \frac{1}{\sqrt{1+\tilde{A}^2}} \Rightarrow \tilde{A} = \pm\sqrt{3}$$

$$\rightarrow y(x) = \pm\sqrt{3} \sin(k\pi x)$$