ANALIZA 3 - 2. kolokvij

17. 1. 2011

Ime in priimek:
Vpisna št.:

1. Poišči rešitvi $x = x(t), y = y(t)$ naslednjega sistema NDE

$$\begin{aligned}\dot{x} &= 3x + 2y + e^{5t}\\ \dot{y} &= 4x + y + e^{5t},\end{aligned}$$

ki zadoščata pogojem $x(0) = y(0) = 0$.

Homogeni del!

$$\begin{pmatrix}x\\y\end{pmatrix}' = \underset{\overset{\shortparallel}{A}}{\begin{pmatrix}3&2\\4&1\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix},\quad \det(A-\lambda) = \lambda^2 - 4\lambda - 5$$

$$= (\lambda-5)(\lambda+1)$$

$$\lambda_1 = 5 \quad \lambda_2 = -1$$

$$Av_1 = \lambda_1 v_1 \rightarrow v_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$Av_2 = \lambda_2 v_2 \rightarrow v_2 = \begin{pmatrix}1\\-2\end{pmatrix}$$

nesing. mat. reš.: $X = \begin{pmatrix}e^{5t} & e^{-t}\\ e^{5t} & -2e^{-t}\end{pmatrix}$

Var. konst.: $\begin{pmatrix}x\\y\end{pmatrix} = X\begin{pmatrix}c_1\\c_2\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}(t=0) = 0 \Rightarrow c_1(0) = c_2(0) = 0$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}' = X^{-1}\begin{pmatrix}e^{5t}\\e^{5t}\end{pmatrix} = \frac{1}{-3}\begin{pmatrix}-2e^{-5t} & -e^{-5t}\\ -e^{t} & e^{t}\end{pmatrix}\begin{pmatrix}e^{5t}\\e^{5t}\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\left.\begin{aligned}c_1 &= t + A\\ c_2 &= B\end{aligned}\right\} \longrightarrow \begin{aligned}c_1 &= t\\ c_2 &= 0\end{aligned}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = X\begin{pmatrix}t\\0\end{pmatrix} = \begin{pmatrix}te^{5t}\\te^{5t}\end{pmatrix},\quad \boxed{x = y = te^{5t}}$$

2. Če veš, da $y_1(x) = x$ ter $y_2(x) = x^2$ rešita naslednjo linearno NDE

$$0 = \frac{-6}{x^3+1}y + \frac{6x}{x^3+1}y' - \frac{3x^2}{x^3+1}y'' + y''',$$

poišči njeno splošno rešitev.

$$W = e^{-\int \frac{-3x^2}{x^3+1}dx} = K(x^3+1)$$

$$\overset{\parallel}{\begin{vmatrix} y & x & x^2 \\ y' & x' & (x^2)' \\ y'' & x'' & (x^2)'' \end{vmatrix}} = \begin{vmatrix} y & x & x^2 \\ y' & 1 & 2x \\ y'' & 0 & 2 \end{vmatrix}$$

$$\rightarrow 2y - 2y'x + y''x^2 = \underbrace{\overset{A}{K(x^3}+1)}_{B}$$

Eulerjeva DE! hom. del: $y_H = Ax + Bx^2$

part. rej.: A: $\overset{\text{nastavek}}{y_{PA} = x^3 \cdot C} \rightarrow y_{PA} = \frac{K}{2}x^3$

B: $y_{PB} = D \rightarrow y_{PB} = \frac{K}{2}$

$$\rightarrow \boxed{y_{spl.} = Ax + Bx^2 + \overset{K/2}{\overset{\parallel}{C}}(x^3+1)}$$

3. Na ploskvi

$$\Sigma = \{(x,y,z) \in (-\frac{\pi}{2},\frac{\pi}{2}) \times \mathbb{R}^2 \,|\, y = \ln\cos x\} \subset \mathbb{R}^3$$

poišči geodetko, ki povezuje točki $A(0,0,0)$ ter $B(\frac{\pi}{4}, -\frac{1}{2}\ln 2, 1)$.

$z = z(x)$, minimiziramo

pov. geod:

$x \mapsto (x, \ln\cos x, z(x)) \Rightarrow \quad I[z] = \int_0^{\pi/4} dx \sqrt{1+\mathrm{tg}^2 x + z'^2}$

$$z(0)=0, \quad z(\tfrac{\pi}{4}) = 1$$

$$L_{z'} = K \Rightarrow \frac{z'}{\sqrt{1+\mathrm{tg}^2x+z'^2}} = K$$

$$z'^2\overbrace{\left(\frac{1}{K^2}-1\right)}^{=\frac{1}{\beta^2}} = \frac{1}{\cos^2 x}$$

$$z'/\beta = \frac{1}{\cos x}$$

$$z' = \frac{\beta}{\cos x}, \quad z(x) = \overset{0}{z(0)} + \beta\int_0^x \frac{dx}{\cos x} = \beta\frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}$$

$$1 = z(\pi/4) = \beta\frac{1}{2}\ln\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} \rightarrow \beta = \frac{2}{\ln\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}}$$

$$\rightarrow \boxed{z = \frac{\ln\frac{1+\sin x}{1-\sin x}}{\ln\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}}}$$

4. Na prostoru funkcij iz $C^1[0,1]$, ki zadoščajo pogoju $y(0) = 0$, poišči vse ekstremale funkcionala

& $y(1)=1$

$$I[y] = \left(\int_0^1 y(x)\,dx\right)^2 + \int_0^1 y'^2(x)\,dx$$

tako, da določiš krepki odvod funkcionala DI_y, odtod določiš pogoje, ki jim mora ustrezati ekstremala in vse ekstremale poiščeš.

$$DI_y(\eta) = 2\left(\int_0^1 y\right)\cdot\int_0^1 \eta + \int_0^1 2y'\eta'$$

$$= \int_0^1 dx\,\eta(x)\left(2\left[\int_0^1 y\right] - 2y''(x)\right) + 2y'(1)\cancel{\eta(1)} - 2y'(0)\cancel{\eta(0)}$$

za dopustne η

$\{\eta \in C^1[0,1] \mid \eta(0)=\eta(1)=0\}$

Ker $C_c^\infty(0,1) \subseteq$ dopustne η, smemo uporabiti OLVR dobimo $2\left(\int_0^1 y\right) - 2y'' = 0$ } (A)

Rešimo: (A) $y'' = \int_0^1 y =: \alpha \in \mathbb{R}$

(B1) $y(0)=0$ (B2) $y(1)=1$

(A) $\rightarrow$ $y = \frac{\alpha}{2}x^2 + \beta x + \gamma$

(B1) $\rightarrow$ $\gamma = 0$ (B2) $\rightarrow$ $\frac{\alpha}{2} + \beta = 1$

Torej $y = \frac{\alpha}{2}x^2 + \left(1 - \frac{\alpha}{2}\right)x$

Konsistenčni pogoj: $\alpha = \int_0^1 y = \frac{\alpha}{2}\frac{1}{3} + \left(1 - \frac{\alpha}{2}\right)\frac{1}{2}$

$\alpha = \frac{1}{2} + \frac{\alpha}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{2} - \frac{\alpha}{12} \rightarrow \frac{13}{12}\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{6}{13}$

$$\boxed{y = \frac{3}{13}x^2 + \frac{10}{13}x}$$