1) $\det\begin{pmatrix}-14-\lambda & -9\\ 16 & 10-\lambda\end{pmatrix} = \lambda^2+4\lambda+4=(\lambda+2)^2$

$\dim\ker\begin{pmatrix}\overset{-12}{-14+2} & -9\\ 16 & \underset{12}{10+2}\end{pmatrix}=1 \Rightarrow A=\begin{pmatrix}-14 & -9\\ 16 & 10\end{pmatrix}=P\begin{pmatrix}-2 & 1\\ & -2\end{pmatrix}P^{-1}$

$P=\left((A+2I)v, v\right)$, kjer $(A+2I)v\neq 0$. Vzamemo $v=\begin{pmatrix}0\\1\end{pmatrix}$

Torej $A=\underbrace{\begin{pmatrix}-9 & 0\\ 12 & 1\end{pmatrix}}_{P}\begin{pmatrix}-2 & 1\\ & -2\end{pmatrix}\begin{pmatrix}-9 & 0\\ 12 & 1\end{pmatrix}^{-1}$.

Nesingul. matrična rešitev:

$$X=\begin{pmatrix}-9 & 0\\ 12 & 1\end{pmatrix}\begin{pmatrix}e^{-2t} & te^{-2t}\\ & e^{-2t}\end{pmatrix}=e^{-2t}\begin{pmatrix}-9 & 0\\ 12 & 1\end{pmatrix}\begin{pmatrix}1 & t\\ & 1\end{pmatrix}$$

$$=e^{-2t}\begin{pmatrix}-9 & -9t\\ 12 & 12t+1\end{pmatrix}$$

spl. rešitev

$$\begin{pmatrix}x\\y\end{pmatrix}=C_1e^{-2t}\begin{pmatrix}-9\\12\end{pmatrix}+C_2e^{-2t}\begin{pmatrix}-9t\\12t+1\end{pmatrix}$$

(2) nesing. mat. rešitev $\tilde{X} = \begin{pmatrix} 1-4t^2 & 2t^2 & x \\ t(1-2t^2) & t^3 & y \\ t^2(1-2t^2) & t^4 & z \end{pmatrix}$

(kjer je x, y, z poljubna fj-splošna rešitev).

Liouville: $(\det\tilde{X})^{\cdot} = (\operatorname{tr} A)(\det\tilde{X})$

$$\frac{d(\det\tilde{X})}{\det\tilde{X}} = \frac{4}{t}\,dt$$

$$\ln|\det\tilde{X}| = 4\ln|t| + \tilde{C}$$

$$\det\tilde{X} = K t^4, \text{ torej } \det\tilde{X} = -zt^3 + yt^4 = Kt^4$$

Torej je $K = y - z/t$ prvi integral in lahko zamenjamo y: $y = K + z/t$

$$\dot{x} = \frac{4}{t}x + \frac{11}{2t^2}\underbrace{(K + z/t)}_{y} - \frac{19}{2t^3}z$$

$$\dot{z} = 2tx + \frac{5}{2}\underbrace{(K + z/t)}_{y} - \frac{5}{2t}z$$

Torej

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 4/t & -4/t^3 \\ 2t & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + K\begin{pmatrix} 11/2t^2 \\ 5/2 \end{pmatrix}$$

Homogeni del: $X = \begin{pmatrix} 1-4t^2 & 2t^2 \\ t^2(1-2t^2) & t^4 \end{pmatrix}$ (to vemo ker $\vec{X}_1$, $\vec{X}_2$ pri $K=0$ rešitev originalni sistem)

Variacija konstante: $\begin{pmatrix} x \\ z \end{pmatrix} = X\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = KX^{-1}\begin{pmatrix} 11/2t^2 \\ 5/2 \end{pmatrix} = \frac{-K}{t^4}\begin{pmatrix} t^4 & -2t^2 \\ -t^2(1-2t^2) & 1-4t^2 \end{pmatrix}\begin{pmatrix} 11/2t^2 \\ 5/2 \end{pmatrix}$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \frac{-K}{t^4}\begin{pmatrix} t^2/2 \\ -3+t^2 \end{pmatrix} = -K\begin{pmatrix} 1/2t^2 \\ -3/t^4 + 1/t^2 \end{pmatrix} \quad \begin{aligned} \alpha &= C + \frac{K}{2t} \\ \beta &= D - Kt^{-3} + \frac{K}{t} \end{aligned}$$

(2) nedoločevanje: $\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1-4t^2 & 2t^2 \\ t^2(1-2t^2) & t^4 \end{pmatrix} \begin{pmatrix} C + {}^{K}\!/_{2t} \\ D - {}^{K}\!/_{t^3} + {}^{K}\!/_{4t} \end{pmatrix} = C \begin{pmatrix} 1-4t^2 \\ t^2(1-2t^2) \end{pmatrix} +$

$$+ D \begin{pmatrix} 2t^2 \\ t^4 \end{pmatrix} + K \begin{pmatrix} -\frac{3}{2}\frac{1}{t} \\ -t/2 \end{pmatrix}$$

Torej je splošna rešitev (upoštevamo $y = K + z/t$)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C \begin{pmatrix} 1-4t^2 \\ t(1-2t^2) \\ t^2(1-2t^2) \end{pmatrix} + D \begin{pmatrix} 2t^2 \\ t^3 \\ t^4 \end{pmatrix} + K \begin{pmatrix} -3/2 \, \frac{1}{t} \\ 1/2 \\ -t/2 \end{pmatrix}$$

(3) vez: $\int_0^1 2yy' = 1 \Rightarrow L = y^2 + 4y'^2 - 2\lambda y y'$

E.L. $\rightarrow$ $8y'' = 2y \rightarrow y = Ae^{x/2} + Be^{-x/2}$

$(y' = \frac{1}{2}Ae^{x/2} - \frac{1}{2}Be^{-x/2})$

$L_{y'} = 8y' - 2\lambda y$ pogoji: vez + $L_{y'}|_0 = 0$ + $L_{y'}|_{x=1} = 0$

Torej: $(Ae^{1/2} + Be^{-1/2})^2 - (A+B)^2 = 1$ (*)

$$\left.\begin{aligned} &4\left(\tfrac{1}{2}A - \tfrac{1}{2}B\right) - \lambda(A+B) = 0 \\ &4\left(\tfrac{1}{2}Ae^{1/2} - \tfrac{1}{2}Be^{-1/2}\right) - \lambda(Ae^{1/2} + Be^{-1/2}) = 0 \end{aligned}\right\} \begin{pmatrix} 2-\lambda & -2-\lambda \\ (2-\lambda)e^{1/2} & (-2-\lambda)e^{-1/2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

ker A in B ne smeta biti hkrati 0 (zaradi (*)) $\rightarrow \det(\;) = 0 \Rightarrow \lambda = \pm 2$

če $\lambda = 2 \rightarrow B = 0$	če $\lambda = -2 \rightarrow A = 0$
$A^2 e - A^2 = 1 \Rightarrow A = \pm\sqrt{\frac{1}{e-1}}$	$B^2 e^{-1} - B^2 = 1 \Rightarrow$ ni rešitve
$y = \pm\sqrt{\frac{1}{e-1}}\, e^{x/2}$	

(4)

$$(DI)_{y_0}(h) = \Big[\overbrace{\int_{-1}^{1} e^{y'}}^{A}\Big] \int_{-1}^{1} 2y_0' h' + \Big[\overbrace{\int_{-1}^{1} (1+y_0^2)}^{B}\Big] \int_{-1}^{1} (-e^{-y_0'}) h'$$

$$= \int_{-1}^{1} dx \, [\underbrace{A \cdot 2y_0' - Be^{-y_0'}}_{f(x)}] h' = f(1)h(1) - f(0)h(0) - \int_{-1}^{1} f'(x) h(x) dx$$

Ker so $\mathcal{C}_c^\infty$ dopustne variacije (t.j. $h \in \mathcal{C}_c^\infty \Rightarrow h(-1) + h(1) = 0$)

lahko uporabimo OLVR; dobimo $f'(x) = 0 \Rightarrow$

$$\Rightarrow C = f(x) = A \cdot 2y_0' - Be^{-y_0'}$$

$\Rightarrow$ (ker je desna stran odvisna samo od y_0' in nič od x/y_0) $y_0' = \text{Konst.} = K$

$$\Rightarrow y_0 = Kx + C \xrightarrow{y_0(-1)+y_0(1)=0} C = 0 \Rightarrow y_0 = Kx.$$

Iz $(DI)_{y_0}(h) = 0$ za $\forall h \in \underbrace{\{h(-1) + h(1) = 0\}}_{\text{dopustne variacije}}$ sledi (npr. $h(x) = x$)

$$\int_{-1}^{1} dx \, (A2K - Be^{-K}) h' = (2AK - Be^{-K})(h(1) - h(-1))$$

$$(2AK - Be^{-K})(1 - (-1)) = 0 \Rightarrow$$

$$2AK - Be^{-K} = 0, \text{ ker je } B = \int_{-1}^{1} (1+K^2) = 2(1+K^2)$$

$$A = \int_{-1}^{1} e^{-K} = 2e^{-K},$$

sledi $2K \cdot 2e^{-K} - 2(1+K^2) e^{-K} = 0 \Rightarrow 4K - 2 - 2K^2 = 0$

$$\Rightarrow K = 1 \Rightarrow K = +1 \Rightarrow \boxed{y_0 = +x}$$