① $F = p^3 + q^3 + u^3 = 0$, $u(s, -s) = 1$

$\dot{x} = 3p^2 \quad \dot{y} = 3q^2 \quad \dot{u} = 3(p^3 + q^3) = -3u^3$

$\dot{p} = -3pu^2 \quad \dot{q} = -3qu^2$

$\Rightarrow \quad \frac{du}{u^3} = -3dt \quad -\frac{1}{2}\frac{1}{u^2} = -3(t-t_0)$

$$\underline{u^2 = \frac{1}{6(t-t_0)}}$$

$\frac{\dot{p}}{p} = -3u^2 \rightarrow \frac{dp}{p} = \frac{-dt}{2(t-t_0)} \quad \ln p = -\frac{1}{2}\ln(t-t_0) + \tilde{c}$

$$\underline{p = \frac{K}{\sqrt{t-t_0}}}$$

Podobno $\underline{q = \frac{L}{\sqrt{t-t_0}}}$

$\dot{x} = \frac{3K^2}{(t-t_0)} \rightarrow \underline{x = 3K^2 \ln|t-t_0| + A}$

$y = 3L^2 \ln|t-t_0| + B$

z.p. $^{(t=0)}$: $(p, q, -1)(1, -1, 0) = 0 \Rightarrow p = q \overset{F=0}{\Rightarrow} p(t=0) = q(t=0) = -\frac{1}{\sqrt[3]{2}}$

x: $3K^2 \ln|t_0| + A = s$

y: $3L^2 \ln|t_0| + B = -s$ $\Big\} \Rightarrow A + B = -\cancel{B}\left(\ln\frac{1}{6}\right)\frac{1}{6}\frac{1}{\sqrt[3]{4}}$

p: $\frac{K}{\sqrt{-t_0}} = -\frac{1}{\sqrt[3]{2}} \rightarrow K = -\frac{1}{\sqrt[3]{2}}\frac{1}{\sqrt{6}}$

q: $\frac{L}{\sqrt{-t_0}} = -\frac{1}{\sqrt[3]{2}} \rightarrow L = -\frac{1}{\sqrt[3]{2}}\frac{1}{\sqrt{6}}$

u: $-\frac{1}{6t_0} = 1 \rightarrow t_0 = -\frac{1}{6}$

$x = 3\frac{1}{\sqrt[3]{4}}\frac{1}{6}\ln\left(t+\frac{1}{6}\right) + A$

$y = 3\frac{1}{\sqrt[3]{4}}\frac{1}{6}\ln\left(t+\frac{1}{6}\right) + B$

$u = \frac{1}{\sqrt{6(t-t_0)}}$

$\rightarrow u(x,y) = e^{-\frac{x+y}{\sqrt[3]{2}}}$

(2)

$$(2y(x+y)-u)u_x + (u-2x(x+y))u_y = 2(y-x)u$$

$$\dot{x} = 2y(x+y)-u$$

$$\dot{y} = u-2x(x+y)$$

$$\dot{u} = 2(y-x)u$$

en prvi integral

$$\dot{x}+\dot{y} = 2(x+y)(y-x)$$

$$= (x+y)\frac{\dot{u}}{u}$$

$$\frac{(x+y)'}{x+y} = \frac{\dot{u}}{u}$$

$$\Rightarrow \frac{x+y}{u} = \psi$$

drug prvi integral

$$\dot{x}x+\dot{y}y = -ux+uy = u(y-x) = \frac{\dot{u}}{2}$$

$$\Rightarrow \phi = (x^2+y^2-u)$$

$$\Phi\left(\frac{x+y}{u}, x^2+y^2-u\right) = 0$$

z.p. $\rightarrow \phi(\underset{\alpha}{2s}, \underset{\beta}{2s^2-1}) = 0$

$$s = \frac{\alpha}{2}$$

$$\beta = -1+2\frac{\alpha^2}{4}$$

$$x^2+y^2-u = -1+\frac{1}{2}\left(\frac{x+y}{u}\right)^2$$

(3) $Q(y,t) = \sum_{n=1}^{\infty} x_n(t) y^n$

$\sum_{n=1}^{\infty} \dot{x}_n y^n = t \sum_{n=1}^{\infty} n\, x_{n+1}\, y^n$

z.p. $x_n(0) = \frac{1}{(n-1)!}$

$$Q_t = t y \frac{\partial}{\partial y}\left(\frac{Q}{y}\right), \quad Q(t=0,y) = y e^y$$

$$Q_t = t\left(Q_y - \frac{Q}{y}\right)$$

$$Q_t - t Q_y = -\frac{t}{y} Q$$

$\dot{t} = 1$ $\qquad t = \tau + A$

$\dot{y} = -t$ $\qquad y = -\frac{\tau^2}{2} - \tau A + B$

$\dot{Q} = -\frac{t}{y} Q$ $\qquad \frac{dQ}{Q} = -\frac{\tau + A}{-\frac{\tau^2}{2} - \tau A + B}$

$$\ln Q = \ln\left|B - \tau A - \frac{\tau^2}{2}\right| + \tilde{C}$$

$$Q = K\left(B - \tau A - \frac{\tau^2}{2}\right)$$

z.p. $Q(y=s, t=0) = s e^s$

$A = 0 \qquad s = \cancel{\ldots} B \Rightarrow B = s\cancel{\ldots}$

$$s e^s = K(s) \Rightarrow K = e^s$$

$t = \tau$

$y = \cancel{\ldots} s - \tau^2/2$

$Q = \cancel{\ldots}\, e^s(s - \tau^2/2)$

$\Rightarrow Q(y,t) = e^{y + \frac{t^2}{2}} y \Rightarrow \boxed{x_n(t) = \frac{1}{(n-1)!} e^{t^2/2}}$

(4)

(a) kar. sistem

$$\left.\begin{aligned}\dot{\tilde{x}} &= a(\tilde{x},\tilde{y})\\ \dot{\tilde{y}} &= b(\tilde{x},\tilde{y})\\ \dot{\tilde{u}} &= -\tilde{u}^2\end{aligned}\right\} \quad (\ast)$$

Dokaz s protislovjem: denimo, da $\exists (x_0,y_0) \ni$:

$u(x_0,y_0) = u_0 < 0$.

Potem si oglejmo rešitev kar. sistema $(\ast)$, za katero je $\tilde{x}(t=0) = x_0$, $\tilde{y}(t=0) = y_0$, $\tilde{u}(t=0) = u_0$

Velja $\tilde{u}(t) = \dfrac{1}{t+\frac{1}{u_0}}$ ker $\tilde{x}\dot{\tilde{x}} + \tilde{y}\dot{\tilde{y}} \leq 0$

$$\tilde{x}\dot{\tilde{x}} + \tilde{y}\dot{\tilde{y}} = \frac{1}{2}(\tilde{x}^2+\tilde{y}^2)^{\cdot} \leq 0$$

iz zadnjega sledi, da je $\tilde{x}^2+\tilde{y}^2$ padajoča f-ja.

Torej velja $u(\tilde{x}(t),\tilde{y}(t)) = \underbrace{\dfrac{1}{t+\frac{1}{u_0}}}_{(\Delta)}$ za $\forall t \geq 0$

in je $\tilde{x}(t)^2 + \tilde{y}(t)^2 \leq x_0^2+y_0^2$, $\forall t \geq 0$.

Ker ~~iz~~ (Δ) pri $t \nearrow -\frac{1}{u_0}$ (>0) postane neomejen, odtod sledi, da je u na $\overline{K}((0,0),\sqrt{x_0^2+y_0^2}) =: \overline{\Omega}$ neomejena, to pa je protislovje z $u \in C(\overline{\Omega})$ in $\overline{\Omega}$ kompaktna.

b) $u = x^2+y^2$, $a = \frac{1}{2}x(x^2+y^2)$, $b = -\frac{1}{2}y(x^2+y^2)$