(1) integrabilnost $\begin{bmatrix} y \\ -x \\ w(x,y) \end{bmatrix} \cdot \operatorname{rot} \begin{bmatrix} y \\ -x \\ w(x,y) \end{bmatrix} = 0$

$$x w_x + y w_y - 2w = 0, \quad w(s,1) = -e^{-s}$$

$$\dot{x} = x, \quad \dot{y} = y, \quad \dot{w} = 2w$$

spl. reš. $\phi\big(\underbrace{x/y}_{\alpha}, \underbrace{w/y^2}_{\beta}\big) = 0$

zač. p.: $\alpha = s, \ \beta = -e^{-s} \rightarrow \beta = -e^{-\alpha}$

$$w/y^2 = -e^{-x/y}$$

$$\underline{\underline{w(x,y) = -y^2 e^{-x/y}}}$$

Pfaff: $y\,dx - x\,dy - y^2 e^{-x/y} dz = 0$

$z_x = \lambda y$

$z_y = -\lambda x$

$-1 = \lambda(-y^2 e^{-x/y}) \rightarrow \lambda = \frac{1}{y^2} e^{x/y}$

$$\rightarrow \left.\begin{aligned} z_x &= \frac{1}{y} e^{x/y} \\ z_y &= -\frac{x}{y^2} e^{x/y} \end{aligned}\right\} \Rightarrow \underline{\underline{z(x,y) = e^{x/y} + C}}$$

② $-u_y + u_x^2 y\left(1+\frac{1}{x^2}\right) = 0, \quad u(s,1) = \sqrt{s^2+1}$

$$F = q - p^2 y\left(1+\frac{1}{x^2}\right) = 0$$

$$\dot{x} = -2py\left(1+\frac{1}{x^2}\right)$$

$$\dot{y} = 1 \xrightarrow{F=0} \underline{y = t - t_0}$$

$$\dot{u} = -2p^2 y\left(1+\frac{1}{x^2}\right) + q \overset{F=0}{=} -q \longrightarrow \dot{u} = -A(t-t_0)$$

$$\underline{u = -A\frac{(t-t_0)^2}{2} + B}$$

$$\dot{p} = -2p^2 y / x^3$$

$$\dot{q} = p^2\left(1+\frac{1}{x^2}\right) \overset{F=0}{=} q/y \longrightarrow \dot{q} = \frac{q}{t-t_0} \rightarrow \underline{q = A(t-t_0)}$$

$$\cancel{\dot{x} = -2py\left(1+\frac{1}{x^2}\right) \overset{F=0}{=} -\frac{2q}{p} = -\frac{2A(t-t_0)}{p}}$$

$$\frac{dx}{dp} = \frac{-2py\left(1+\frac{1}{x^2}\right)}{-2p^2 y/x^3} = \frac{x(x^2+1)}{p} \rightarrow \frac{dx}{x(x^2+1)} = \frac{dp}{p} \rightarrow$$

$$\rightarrow \underline{p = C\frac{x}{\sqrt{x^2+1}}}; \quad (*)$$

$$0 = F = A(t-t_0) - \frac{C^2x^2}{x^2+1}(t-t_0)\left(1+\frac{1}{x^2}\right) \rightarrow \underline{A = C^2}$$

$$\dot{x} = -\frac{2Cx}{\sqrt{x^2+1}}(t-t_0)\left(1+\frac{1}{x^2}\right) \rightarrow \frac{dx\, x}{\sqrt{x^2+1}} = -2C(t-t_0)\,dt$$

$$\underline{\sqrt{x^2+1} = -C(t-t_0)^2 + D}$$

$t=0$:

$$\left.\begin{aligned} x(t=0) &= s \\ y(t=0) &= 1 \\ u(t=0) &= \sqrt{s^2+1} \\ p(t=0) &= \frac{s}{\sqrt{s^2+1}} \\ q(t=0) &= 1 \end{aligned}\right\} \rightarrow$$

$$\sqrt{s^2+1} = -Ct_0^2 + D = -C + D \rightarrow D = \pm 1 + \sqrt{s^2+1}$$

$$\underline{t_0 = -1}$$

$$\sqrt{s^2+1} = -\frac{At_0^2}{2} + B = -\frac{A}{2} + B$$

$$A = C^2$$

$$-At_0 = 1 \Rightarrow \underline{A = 1} \Rightarrow \underline{B = \sqrt{s^2+1} + \frac{1}{2}}, \quad \underline{C = \pm 1}$$

Torej: (rešitev parametrično) $\sqrt{x^2+1} = \mp(t+1)^2 \pm 1 + \sqrt{s^2+1}$, $y = t+1$, $u = -\frac{1}{2}(t+1)^2 + \frac{1}{2} + \sqrt{s^2+1}$

Enostavno videti iz upr. (*), da $C=1 \Rightarrow u(x,y) = \sqrt{x^2+1} + \frac{1}{2}(y^2-1)$

3. (a) Rešujemo Pfaffovo enačbo $\vec{n}\cdot(dx,dy,dz)=0$, kjer je vektor $\vec{n}$ na premici iz dane družine dan s smernim vektorjem te premice. Torej

$$\vec{n}\big(\underbrace{\alpha+\gamma f(\alpha), \beta+\gamma g(\beta), 0+\gamma\cdot 1}\big) = (f(\alpha), g(\beta), 1)$$

parametrično zapisana premica iz družine, ki ustreza parametroma α, β. Parameter premice smo označili γ.

Torej $\quad n_1(\alpha+\gamma f(\alpha), \beta+\gamma g(\beta), \gamma) = f(\alpha)$

$\frac{\partial}{\partial\alpha}$: $\quad n_{1x}(1+\gamma f'(\alpha)) = f'(\alpha) \Rightarrow \underline{n_{1x} = f'(\alpha)/(1+\gamma f'(\alpha))}$

$\frac{\partial}{\partial\beta}$: $\quad n_{1y}(1+\gamma g'(\beta)) = 0 \Rightarrow \underline{n_{1y} = 0}$

$\frac{\partial}{\partial\gamma}$: $\quad n_{1x}f(\alpha) + \underset{\overset{\parallel}{0}}{n_{1y}}g(\beta) + n_{1z} = 0 \Rightarrow \underline{n_{1z} = -\frac{f'(\alpha)f(\alpha)}{1+\gamma f'(\alpha)}}$

Podobno: $\quad n_{2x} = 0, \; n_{2y} = g'(\beta)/(1+\gamma g'(\beta)), \; \underline{n_{2z} = -\frac{g'(\beta)g(\beta)}{1+\gamma g'(\beta)}}$

$$n_{3x} = n_{3y} = n_{3z} = 0$$

Torej $\quad \operatorname{rot}\vec{n} = \begin{bmatrix} n_{3y}-n_{2z} \\ n_{1z}-n_{3x} \\ n_{2x}-n_{1y} \end{bmatrix} = \begin{bmatrix} g'(\beta)g(\beta)/(1+\gamma g'(\beta)) \\ -f'(\alpha)f(\alpha)/(1+\gamma f'(\alpha)) \\ 0 \end{bmatrix}$

Pogoj za integrabilnost:

$$0 = \vec{n}\cdot\operatorname{rot}\vec{n} = \begin{bmatrix} f(\alpha) \\ g(\beta) \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \, ? \, \\ \\ \end{bmatrix} = f(\alpha)g(\beta)\left(\frac{g'(\beta)}{1+\gamma g'(\beta)} - \frac{f'(\alpha)}{1+\gamma f'(\alpha)}\right) = 0 \quad \forall \alpha,\beta,\gamma$$

$$\Longrightarrow \quad f(\alpha)g(\beta)(g'(\beta)-f'(\alpha)) = 0$$

3.(b) $\frac{x-\alpha}{f(\alpha)} = \frac{y-\beta}{g(\beta)} = z \quad \forall \alpha,\beta \in \mathbb{R}$

iz (a)
$f'(\alpha) = g'(\beta) \ \forall \alpha,\beta \Rightarrow f'(\alpha) = g'(\beta) = k \Rightarrow f(\alpha) = k(\alpha - \alpha_0)$
$g(\beta) = k(\beta - \beta_0), \ k \neq 0.$

rešujemo Pfaffovo enačbo $n_1 dx + n_2 dy + n_3 dz = 0$,

kjer $\vec{n}(\underbrace{\alpha + \gamma k(\alpha - \alpha_0)}_{x}, \underbrace{\beta + \gamma k(\beta - \beta_0)}_{y}, \underbrace{\gamma}_{z}) = \cancel{(f(\alpha), g(\beta), 1)} \ \forall \alpha,\beta$
$(k(\alpha-\alpha_0), k(\beta - \beta_0), 1)$

$\Rightarrow z = \gamma, \quad \alpha(1+\gamma k) - \gamma k \alpha_0 = x \Rightarrow \alpha = \frac{x + \gamma k \alpha_0}{1 + \gamma k}$

podobno $\beta = \frac{x + \gamma k \alpha_0}{1 + \gamma k}$

odtod $\vec{n}(x,y,z) = \left(k \frac{x - \alpha_0}{1 + zk}, k \frac{y - \beta_0}{1 + zk}, 1\right)$

tako $z_x = k \frac{x - \alpha_0}{1 + zk} \lambda = -k \frac{x - \alpha_0}{1 + zk} \Rightarrow$

$z_y = k \frac{y - \beta_0}{1 + zk} \lambda = -k \frac{y - \beta_0}{1 + zk} \Rightarrow$

$-1 = \lambda \Rightarrow \lambda = -1$

$z + \frac{z^2}{2} k = -\frac{k(x-\alpha_0)^2}{2} - k\frac{(y-\beta_0)^2}{2} + c$

Torej $\frac{k}{2}(z + 1/k)^2 = -\frac{k}{2}(x-\alpha_0)^2 - \frac{k}{2}(y - \beta_0)^2 + C$

ali $(z + 1/k)^2 + (x - \alpha_0)^2 + (y - \beta_0)^2 = \text{konst.}$

sfere

(4) Izberimo poljubno $(x_0, y_0) \in \mathbb{R}^2$ in označimo ~~[illegible]~~ $u(x_0, y_0) = A$

Naj bo $(\tilde{x}, \tilde{y}, \tilde{u})$ karakteristika, za katero je

$$\tilde{x}(t=0) = x_0,\ \tilde{y}(t=0) = y_0,\ \tilde{u}(t=0) = A$$

ker γ $\quad \dot{\tilde{x}} = a(\tilde{x}, \tilde{y}),\ \dot{\tilde{y}} = b(\tilde{x}, \tilde{y}),\ \dot{\tilde{u}} = u$

$\Rightarrow$ 1) $\tilde{u}(t) = A e^t$

2) $(\tilde{x}^2 + \tilde{y}^2)' = 2(\tilde{x}\dot{\tilde{x}} + \tilde{y}\dot{\tilde{y}}) = 2(\tilde{x}\, a(\tilde{x}, \tilde{y}) + \tilde{y}\, b(\tilde{x}, \tilde{y})) \le 0 \quad \forall t$

torej je $\tilde{x}^2 + \tilde{y}^2$ padajoča funkcija. Torej $\tilde{x}^2(t) + \tilde{y}^2(t) \le x_0^2 + y_0^2$ za $\forall t \ge 0$.

~~Ker karakteristični siste~~ Odtod sledi dvoje:

A) rešitev karakterističnega sist. obstaja za $\forall t \ge 0$
(ker $\tilde{x}^2 + \tilde{y}^2$ ne more iti proti ∞, kar je edini možen problem z eksistenco rešitve kar. sistema)

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B) $|\tilde{u}(\tilde{x}(t), \tilde{y}(t))| \le M := \sup_{x^2+y^2 \le x_0^2+y_0^2} |u(x,y)| < \infty$ za $\forall t \ge 0$.

zaradi kompaktnosti množice $\{(x,y) \mid x^2+y^2 \le x_0^2+y_0^2\} \subseteq \mathbb{R}^2$

Torej je $|u(\tilde{x}(t), \tilde{y}(t))| = |\tilde{u}(t)| = |A| e^t \le M$ za $\forall t \ge 0$

To je možno le, če $A = 0$.