①

valovna $\Rightarrow u(x,t) = F(x-t) + G(x+t)$

~~r.p.~~ $\Rightarrow$ z.p. $\Rightarrow$ $F(\alpha) = \frac{1}{2} e^{-\alpha} = G(\alpha)$ za $\alpha > 0$

potrebujemo še $F(\alpha)$ za $\alpha < 0$

$$\left[u_x = F'(x-t) + G'(x+t) \right]$$

r.p. $\Rightarrow$ $F(-t) + G(t) + F'(-t) + G'(t) = 0, \ \forall t > 0$

torej $F(-t) + F'(-t) = 0 \quad t > 0$

ali $F(\alpha) + F'(\alpha) = 0 \quad \alpha < 0$

$$\Downarrow$$

$$F(\alpha) = K e^{-\alpha} \quad \alpha < 0$$

Da bo $u \in C^2$, mora biti $F \in C^2$,

torej iz $F(\alpha) = \begin{cases} \frac{1}{2} e^{-\alpha} & \alpha > 0 \\ K e^{-\alpha} & \alpha < 0 \end{cases}$

sledi $K = \frac{1}{2}$.

Torej $F(\alpha) = \frac{1}{2} e^{-\alpha}$ povsod.

$$\underline{u(x,t) = e^{-x} \operatorname{ch} t}$$

(2) Robni problem: $y'' = \lambda y$

$y(0) = y'(1) = 0$

$\boxed{\nabla \atop 0}$ $k=0$ tudi?

Rešitev: $y_k = \cos(k\pi x)$, $k \in \mathbb{N}_0$

$\lambda_k = -k^2\pi^2$

Torej $u(x,t) = \sum_{k=0}^{\infty} c_k(t) \cos(k\pi x)$

NDE za $c_k(t)$: $\dot{c}_k = \lambda_k c_k + d_k e^{-t}$

$c_k(t=0) = 0$,

kjer $d_k = \dfrac{\int_0^1 x\cos(k\pi x)\,dx}{\int_0^1 \cos^2(k\pi x)\,dx} = \begin{cases} \frac{1}{2}, & k=0 \\ \frac{-2}{(k\pi)^2}(1-(-1)^k), & k>0 \end{cases}$

Rešitev:

$\cancel{c_k(t) = \frac{d_k}{1+k^2\pi^2}(e^{-k^2\pi^2 t} - e^{-t})}$ $\quad c_k(t) = \dfrac{d_k}{k^2\pi^2 - 1}\left(-e^{-k^2\pi^2 t} + e^{-t}\right)$

Torej $u(x,t) = \sum_{k=0}^{\infty} \dfrac{d_k}{k^2\pi^2-1}\left(e^{-t} - e^{-k^2\pi^2 t}\right)\cos k\pi x =$

$= \underbrace{-\frac{1}{2}(e^{-t}-1)}_{\downarrow t\to\infty \atop \frac{1}{2}} + \underbrace{\sum_{k=1}^{\infty} \frac{2}{(k\pi)^2}\frac{(1-(-1)^k)}{1-k^2\pi^2}\left(e^{-t} - e^{-k^2\pi^2 t}\right)\cos k\pi x}_{\downarrow t\to\infty \atop 0}$

$\lim_{t\to\infty} u = \underline{\underline{\frac{1}{2}}}$

③ ① $G(z,z_0) - \frac{1}{2\pi}\operatorname{Re}\ln(z-z_0)$ harm. na Ω (kot f-ja z, ker $z_0\in\Omega$)

$$= \frac{1}{2\pi}\operatorname{Re}\ln\frac{(R^2-z\bar z_0)}{(z-\bar z_0)(R^2-zz_0)}$$

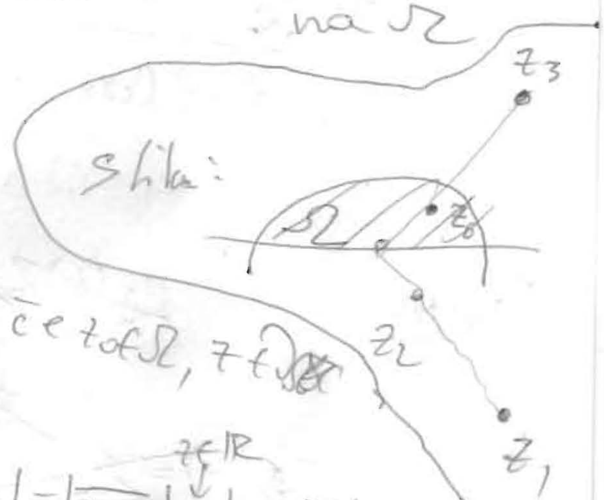
singularnosti: • $z_1 = R^2/z_0$; ker $|z_1| = \frac{R^2}{|z_0|} \overset{|z_0|<R}{>} R \Rightarrow z_1\notin\Omega$

• $z_2=\bar z_0$; ker $\operatorname{Im} z_0>0 \Rightarrow \operatorname{Im} z_2<0 \Rightarrow z_2\notin\Omega$

• $z_3 = R^2/\bar z_0$; ker $|z_3| = \frac{R^2}{|z_0|}>R \Rightarrow z_3\notin\Omega$

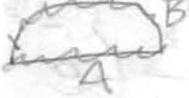
torej sing. v Ω ni; torej je izraz harm. na Ω

slika:



② $G(z,z_0)=0$, če $z_0\in\Omega$, $z\in\partial\Omega$

dovolj videti, da je $\left|\frac{z-z_0}{z-\bar z_0}\cdot\frac{R^2-z\bar z_0}{R^2-zz_0}\right|=1$, če $z_0\in\Omega$, $z\in\partial\Omega$

$\partial\Omega$: 

primer A: $z\in\mathbb{R} \Rightarrow$ • $\left|\frac{z-z_0}{z-\bar z_0}\right|=1$, saj $|z-z_0| = |\overline{z-z_0}| \overset{z\in\mathbb{R}}{=} |z-\bar z_0|$

• $\left|\frac{R^2-z\bar z_0}{R^2-zz_0}\right|=1$, saj $|R^2-\bar z_0 z| = |\overline{R^2-\bar z_0 z}| \overset{z\in\mathbb{R}}{=} |R^2-z z_0|$

primer B: $|z|=R$ • $\left|\frac{z-z_0}{R^2-z\bar z_0}\right|^2 = \frac{(z-z_0)(\overline{z-z_0})}{(R^2-z\bar z_0)(R^2-\bar z z_0)} \overset{z\bar z=R^2}{=} \frac{|z|^2 - z\bar z_0 - \bar z z_0 + |z_0|^2}{R^4 - z\bar z_0 R^2 - R^2\bar z z_0 + R^2|z_0|^2} = \frac{1}{R^2}$

• $\left|\frac{R^2-zz_0}{z-\bar z_0}\right|^2 = \dots = \frac{(R^2-zz_0)(R^2-\bar z\bar z_0)}{(z-\bar z_0)(\bar z-z_0)} = \frac{R^4-R^2 z z_0 - R^2\bar z\bar z_0 + R^2|z_0|^2}{R^2 - \bar z\bar z_0 - zz_0+|z_0|^2} = R^2$

(4) $U(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} u(x,t)\,dx$

$$\begin{cases} U_{tt} + 2\alpha U_t + \alpha^2 U = -y^2 U \\ U(y,0) = F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iyx} dx \\ U_t(y,0) = 0 \end{cases} \quad (\&)$$

$' = \frac{d}{dt}$ $\quad U'' + 2\alpha U' + (\alpha^2 + y^2) U = 0$

$\Rightarrow U(y,t) = A(y) e^{(-\alpha+iy)t} + B(y) e^{(-\alpha-iy)t}$

$$(\&) \Rightarrow U(y,t) = e^{-\alpha t} F(y) \left[\underbrace{\tfrac{1}{2} e^{iyt}}_{(1)} + \underbrace{\tfrac{1}{2} e^{-iyt}}_{(2)} + \underbrace{\tfrac{\alpha}{2} \frac{e^{iyt} - e^{-iyt}}{iy}}_{(3)} \right]$$

Inverzni Fourier:

$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(y,t) e^{ixy} dy$

(1): $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t} F(y) \frac{1}{2} e^{iy(x+t)} dy = \frac{e^{-\alpha t}}{2} \tilde{\mathcal{F}}^{-1}(F)\Big|_{x+t} = \frac{e^{-\alpha t}}{2} f(x+t)$

(2): $- \; - \; - \; -$ $\quad \frac{e^{-\alpha t}}{2} f(x-t)$

(3): $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t} e^{ixy} F(y) \frac{\alpha}{2} \frac{e^{iyt} - e^{-iyt}}{iy} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha t} F(y) \frac{\alpha}{2} \left[\int_{-t}^{t} ds\, e^{iys} \right] dy\, e^{ixy}$

$= e^{-\alpha t} \frac{\alpha}{2} \int_{-t}^{t} ds \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, F(y) e^{iy(s+x)} = e^{-\alpha t} \frac{\alpha}{2} \int_{-t}^{t} ds\, f(x+s)$

$$\boxed{u(x,t) = \frac{e^{-\alpha t}}{2} \left[f(x+t) + f(x-t) + \alpha \int_{-t}^{t} ds\, f(x+s) \right]}$$