DODATEK 1:

FOURIEROVE VRSTE IN TRIGONOMETRIJSKE VRSTE.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

trig. vrsta, predstavlja 2π-periodično funkcijo na $\mathbb{R}$ oz. funkcijo na krožnici S^1 potom univerzalne krovne preslikave $x \mapsto e^{ix} = \cos x + i \sin x$.

J. Fourier (1768–1830): študij prevajanja toplote.
1807: predstavitev dela francoski akademiji znanosti.

A. Zygmund: Trigonometric series (Cambridge U. Press, 1968).
Dirichlet, 1829: konvergenca Fourierovih vrst.

PROSTOR $L^2([a,b])$.

$\mathcal{C}([a,b])$ ... vektorski prostor zveznih funkcij na $[a,b]$.
$\mathcal{R}([a,b])$ ... vektorski prostor Riemannovo integrabilnih fn. (v posplošenem smislu).

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_a^b f(x) g(x)\, dx$$ skalarni produkt

$$\|f\|^2 = \|f\|_{L^2}^2 = \langle f, f \rangle = \int_a^b |f(x)|^2 dx.$$

(Kompleksne funkcije: $\langle f, g \rangle = \int_a^b f \overline{g} \cdot dx$; Hermitski produkt).

Prostor $L^2([a,b])$: merljive funkcije (oz. ekvivalenčni razredi, kjer $f = g$ če se f in g ujemata izven množice z mero 0) s skalarnim produktom

$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x)\, dx \quad \ldots \text{ Lebesguov integral.}$$

Ali tež $\rho \geq 0$:

$$\langle f, g \rangle_\rho = \int_a^b f(x)\, g(x) \cdot \rho(x) dx$$

$$\|f\|_\rho^2 = \int_a^b |f(x)|^2 \rho(x)\, dx.$$

$$\mathcal{C}([a,b]) \subset \mathcal{R}([a,b]) \subset L^2([a,b])$$

(Isti skalarni produkt.)

Vektorski prostor $L^2([a,b])$ je poln (Hilbertov) prostor; napolnitev prostora $\mathcal{C}([a,b])$ glede na metriko

$$d(f,g) = \|f - g\| = \sqrt{\int_a^b |f(x) - g(x)|^2 dx}.$$

$\mathcal{C}([a,b])$ je gost podprostor v $L^2([a,b])$.

Prav tako prostor odsekoma zveznih fn. ter $\mathcal{R}([a,b])$.

A 2bis

Cauchy–Schwarzova neenačba:

$\langle x, y \rangle$ skalarni produkt na vekt. pr. E

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

Dokaz: $0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y \rangle$

$$= \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \cdot \|y\|^2.$$

$$\frac{\partial}{\partial \lambda} = -2 \langle x, y \rangle + 2\lambda \cdot \|y\|^2 = 0$$

$$\lambda = \frac{\langle x, y \rangle}{\|y\|^2}$$

Vstavimo:

$$0 \leq \|x - \lambda y\|^2 = \|x\|^2 - 2 \cdot \frac{\langle x, y \rangle^2}{\|y\|^2} + \frac{\langle x, y \rangle^2}{(\|y\|^2)^2} \cdot \|y\|^2$$

$$= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

$$\Rightarrow |\langle x, y \rangle|^2 \leq \|x\|^2 \cdot \|y\|^2 .$$

V $L^2([a,b])$: $\left(\int_a^b f(x) g(x)\, dx \right)^2 \leq \int_a^b |f|^2 dx \cdot \int_a^b g(x)^2 \cdot dx$ ..

A3

Def.:
Funkciji f, g sta ortogonalni, če $\langle f, g\rangle = 0$.
Ortogonalni sistem funkcij: $\{\varphi_n\}_n$

$$\langle \varphi_n, \varphi_m \rangle = 0 \quad \text{za } n \neq m.$$

Primer 1. $\varphi_0(x) = 1$, $\varphi_1(x) = x$ na $x \in [-1, 1]$.
Očitno $\langle \varphi_0, \varphi_1 \rangle = \int_{-1}^{1} x\, dx = 0$.

Funkcija $f(x) = x^2$ ni ortogonalna na φ_0, je $\perp \varphi_1$.

$$\langle f, \varphi_0 \rangle = \int_{-1}^{+1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{+1} = \frac{2}{3}.$$

$$\|\varphi_0\|^2 = \int_{-1}^{+1} 1 \cdot dx = 2;$$

$$f - \frac{\langle f, \varphi_0 \rangle}{\|\varphi_0\|^2} \varphi_0 =: \varphi_2(x) = x^2 - \frac{2}{3 \cdot 2} \cdot 1 = x^2 - \frac{1}{3}.$$

$$\langle \varphi_2, \varphi_0 \rangle = \int_{-1}^{+1} \left(x^2 - \frac{1}{3}\right) dx = \frac{2}{3} - \frac{2}{3} = 0;$$

$$\langle \varphi_2, \varphi_1 \rangle = 0.$$

Ta Gramm-Schmidtov postopek lahko nadaljujemo in dobimo ortogonalen sistem polinomov

$$\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots \varphi_n(x), \ldots$$

$$\text{st}(\varphi_n) = n.$$

Primeri ortogonalnih sistemov pol. (glede na različne uteži):
Jacobijevi pol., Laguerrovi pol., Hermitovi, Čebiševi, Legendrovi, ... Uporaba: numerika, ...

Primer 2 $\{1, \sin nx, \cos nx\}_{n=1}^{\infty}$ na $[-\pi, \pi]$.

Trigonometrični sistem.

Velja:
$$\begin{cases} \sin A \cdot \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \\ \cos A \cdot \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \\ \sin A \cdot \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right). \end{cases}$$

Odtod sledi, da je ~~t~~ trig. sistem ortogonalen:

$$\langle 1, \sin nx\rangle = \int_{-\pi}^{\pi} \sin nx \cdot dx = 0$$

$$\langle 1, \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx \cdot dx = 0$$

$$\langle \sin nx, \sin mx\rangle = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(n-m)x - \cos(n+m)x\right] dx$$

$$= \begin{cases} 0, & n \neq m \\ \frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Big|_{-\pi}^{\pi} = \pi, & n = m. \end{cases}$$

$\|\sin nx\| = \sqrt{\pi}$;

$$\langle \cos nx, \cos mx\rangle = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(n-m)x + \cos(n+m)x\right) dx$$

$$= \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$\|\cos nx\| = \sqrt{\pi}$.

$$\boxed{\begin{aligned} \|\sin nx\|^2 &= \pi \\ \|\cos nx\|^2 &= \pi \end{aligned}}$$

$\langle \sin nx, \cos mx\rangle = 0, \quad \forall n, m.$

$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\}$ je ortonormiran sistem funkcij v $\mathcal{C}([-\pi,\pi]) \subset L^2([-\pi,\pi])$.

Primer 3. na intervalu $[-L, L]$:

$$\left\{ 1, \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} \right\}_{n=1}^{\infty} \quad \text{trig. sistem.}$$

$$\left\| \sin\frac{n\pi x}{L} \right\|^2 = \left\| \cos\frac{n\pi x}{L} \right\|^2 = L.$$

$$\left\{ \frac{1}{\sqrt{2L}}, \frac{\sin\frac{n\pi x}{L}}{\sqrt{L}}, \frac{\cos\frac{n\pi x}{L}}{\sqrt{L}} \right\}_n$$ ortonormiran trig. sistem na $[-L, +L]$.

Opomba: Če je $\{\varphi_n\}_n$ ortogonalen sistem, $\|\varphi_n\| > 0$, je $\left\{ \frac{\varphi_n}{\|\varphi_n\|} \right\}_n$ ortonormiran sistem.

PRIMER 4. Na $[0, L]$ sta sistema

$$\left\{ \sin\frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

$$\left\{ \cos\frac{n\pi x}{L} \right\}_{n=0}^{\infty}$$

ortogonalna: $\int_0^L \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L} \ldots = 0$, če je $m \neq n$.

Podobno za cos.

Ortogonalne projekcije

Naj bo $\{\varphi_n\} \subset L^2([a,b])$ ortogonalen sistem; $\|\varphi_n\| > 0$.

Za vsak $N \in \mathbb{N}$ je

$$S_N = \left\{ \sum_{j=1}^{N} c_j \varphi_j \in L^2([a,b]);\ c_1, \dots, c_N \in \mathbb{R} \right\}$$

N - dimenzionalen vektorski podprostor v $L^2([a,b])$.

Ortogonalna projekcija ~~(????)~~ $P_N : L^2 \longrightarrow S_N$ je določena s pogojem

$$Q_N(f) = f - P_N(f) \perp S_N, \quad \forall f \in L^2.$$

Pišimo $P_N(f) = \sum_{n=1}^{N} c_n \cdot \varphi_n$. Zgornji pogoj je ekvivalenten

$$\langle f - P_N(f), \varphi_n \rangle = 0, \quad n = 1, \dots, N$$

$$\Longleftrightarrow \quad \langle f, \varphi_n \rangle = \langle P_N(f), \varphi_n \rangle = c_n \|\varphi_n\|^2, \quad n = 1, \dots, N.$$

Torej je

$$P_N(f) = \sum_{n=1}^{N} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \varphi_n .$$

NALOGA: Dokaži: $\|f - P_N(f)\| = \min\{\|f - g\| : g \in L^2([a,b])\}$.

Definicija: Naj bo $\{\varphi_n\}$ nek ortogonalen sistem v $L^2([a,b])$ in $f \in L^2([a,b])$.

Število
$$c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} = \frac{\int_a^b f(x)\varphi_n(x)dx}{\int_a^b \varphi_n(x)^2 dx}$$

se imenuje n-ti Fourierov koeficient f glede na sistem $\{\varphi_n\}$. Vrsta (formalna!)
$$\sum_{n=1}^{\infty} c_n \cdot \varphi_n(x) = \sum_1^{\infty} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \cdot \varphi_n(x)$$
se imenuje Fourierova vrsta f po $\{\varphi_n\}$.

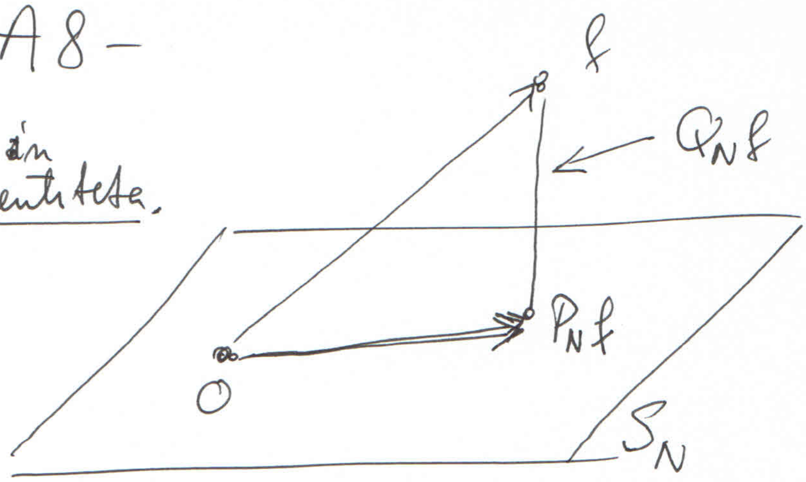
Oznaka:
$$f \sim \sum_1^{\infty} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \varphi_n \quad .$$

Opomba: Simbol $\sim$ pomeni, da (zaenkrat) ne vemo ničesar o konvergenci vrste.

Primer: $\{1, \sin nx, \cos nx\}$; f, $[-\pi,\pi]$.

$$\boxed{\begin{aligned} & f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ & a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \cdot dx \\ & \qquad\qquad\qquad\qquad\qquad\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx \end{aligned}}$$

Besselova neenačba in Parsevalova identiteta.



$$f = P_N f + Q_N f,$$

$$P_N f \perp Q_N f.$$

Sledi: $\|f\|^2 = \langle P_N f + Q_N f, P_N f + Q_N f\rangle$

$$= \langle P_N f, P_N f\rangle + 2\underbrace{\langle P_N f, Q_N f\rangle}_{=0} + \langle Q_N f, Q_N f\rangle$$

$$\|f\|^2 = \|P_N f\|^2 + \|Q_N f\|^2.$$

Pri prehodu $N \to \infty$ dobimo naslednjo trditev.

TRDITEV (a) Za vsak $f \in L^2([a,b])$ velja:

$$\sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|^2 \leq \|f\|^2; \quad c_n = \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2}$$

(b) $$\sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|^2 = \|f\|^2 \iff \lim_{N\to\infty} P_N f = f$$

$$\iff \lim_{N\to\infty} Q_N f = 0,$$

kjer sta limiti v $L^2([a,b])$-topologiji:

$$\lim_{N\to\infty} Q_N f = 0 \iff \lim_{N\to\infty} \|Q_N f\| = 0.$$

Opomba: Neenakost v (a) se imenuje Besselova neenakost, enačba v (b) pa Parsevalova ident.

Definicija: Ortogonalen sistem $\{\varphi_n\} \subset L^2$ se imenuje poln, če za $\forall f \in L^2$ velja točka (b) v prejšnji trditvi:

$$\sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|^2 = \|f\|^2 \iff \lim_{N\to\infty} P_N f = f .$$

Očitno je $\{\varphi_n\}$ poln natanko tedaj, ko velja:

$$\langle f, \varphi_n \rangle = 0, \ \forall n \iff f = 0 \ \text{v} \ L^2([a,b]).$$

TRDITEV. Če je $\{\varphi_n\}$ poln ortogonalen sistem v L^2, je preslikava

$$L^2([a,b]) \ni f \xmapsto{\ \Phi\ } \left(\frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|} \right)_{n=1}^{\infty} \in l_2$$

linearni izomorfizem (izometrija) Hilbertovega prostora $L^2([a,b])$ na Hilbertov prostor

$$l_2 = \{(a_1, a_2, a_3, \dots) : a_j \in \mathbb{R}, \ \sum_1^{\infty} a_j^2 < \infty\}.$$

s skalarnim produktom

$$\langle a, b \rangle = \sum_{j=1}^{\infty} a_j b_j .$$

<u>Dokaz</u>. Fourierovi koeficienti f so enaki

$$c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2}$$

Torej $\quad c_n \cdot \|\varphi_n\| = \dfrac{\langle f, \varphi_n \rangle}{\|\varphi_n\|}$.

Iz Besselove neenakosti sledi

$$\sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|^2 \leqq \|f\|^2 < \infty,$$

torej je zaporedje $\left(c_n \cdot \|f\|_n\right)_{n=1}^{\infty} = \left(\dfrac{\langle f, \varphi_n \rangle}{\|\varphi_n\|}\right) \in l_2$ element prostor l_2.

Ker je $\{\varphi_n\}$ poln, velja celo enakost, torej je

$$\|f\|_{L^2([a,b])} = \left\| \left(\frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|} \right)_n \right\|_{l_2} = \|\Phi(f)\|_{l_2}$$

in preslikava $\Phi : L^2([a,b]) \longrightarrow l_2$ je izometrija.

<u>OPOMBA</u>: Φ = Fourieva izometrija.

<u>POSLEDICA</u> Besselove neenakosti:

$$\lim_{n\to\infty} c_n \cdot \|\varphi_n\| = \lim_{n\to\infty} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|} = 0.$$

OPOMBA: Če je $\{\varphi_n\}$ poln ortonormiran, dobimo

$$f \sim \sum_1^\infty c_n \varphi_n \Longrightarrow \|f\|^2 = \sum_1^\infty c_n^2 \|\varphi_n\|^2 = \sum_1 c_n^2.$$

$$c_n = \langle f, \varphi_n \rangle; \quad \lim_{n\to\infty} c_n = 0$$

IZREK. Trig. sistem $\{1, \cos nx, \sin nx\}$ je poln na $[-\pi,\pi]$.

Dokaz. Glej npr. M. Stoll, Introd. to Real. Analysis Addison-Wesley 1997 (ali katerikoli drug vir iz Fourierove anal.)

POSLEDICA. $f \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\,dx; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\cdot dx$$

$$\left|\frac{a_0}{2}\right|^2 \cdot \|1\|^2 = \frac{a_0^2}{4}\cdot 2\pi = \frac{\pi}{2}a_0^2$$

$$|a_n|^2 \cdot \|\cos nx\|^2 = |a_n|^2\cdot \pi;$$

$$|b_n|^2\cdot \|\sin nx\|^2 = |b_n|^2\cdot\pi.$$

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2 dx = \pi\cdot\left(\frac{a_0^2}{2} + \sum_{n=1}^\infty (a_n^2 + b_n^2)\right).$$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2)}$$

PRIMER. $f(x) = x$

naloga: izračunaj $x \sim \sum_1^\infty \frac{2 \cdot (-1)^{n+1}}{n} \cdot \sin nx$

Parseval: $\sum_1^\infty \frac{4}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot dx = \frac{2}{3}\pi^2$

$$\frac{\pi^2}{6} = \sum_1^\infty \frac{1}{n^2} .$$

PRIMER: $[-L, +L]$; $\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\}$

$$\|1\|^2 = 2L; \quad \|\cos\frac{n\pi x}{L}\|^2 = \|\sin\frac{n\pi x}{L}\|^2 = L$$

$$f \sim \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\boxed{\frac{1}{L}\|f\|^2 = \frac{1}{L}\int_{-L}^{+L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2).}$$

PRIMER (VRSTA PO SINUSIH).

Naj bo $f \in L^2([0,\pi])$. Razširimo jo do *lihe funkcije* na $[-\pi, \pi]$; $f(-x) = -f(x)$.

Tedaj $\int_{-\pi}^{\pi} f(x) \cos nx \cdot dx = 0, \quad \forall n,$

ker je $f(x) \cdot \cos nx$. ~~sod~~ liha fn.

Torej dobimo: $a_n = 0, \ \forall n = 0, 1, 2, \dots$

$$f \sim \quad + \sum_{1}^{\infty} b_n \sin nx \qquad \text{(razvoj po sinusih)}$$

$$\|f\|^2_{L^2([0,\pi])} = \frac{1}{2} \|f\|^2_{L^2([-\pi,\pi])}$$

$$= \frac{\pi}{2} \sum_{1}^{\infty} b_n^2 .$$

VRSTA PO KOSINUSIH:

Če je f soda funkcija na $[-\pi, +\pi]$, je

$$\int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx = 0, \quad \forall n .$$

Torej $\quad f \sim \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$

$$\frac{1}{\pi} \|f\|^2 = \frac{a_0^2}{2} + \sum_{1}^{\infty} a_n^2 .$$

POSLEDICA: Družini funkcij

(a) $\left\{ \sin \frac{n\pi x}{b} \right\}_{n=1}^{\infty}$ in (b) $\left\{ 1, \cos \frac{n\pi x}{b} \right\}_{n=1}^{\infty}$

sta polna ortogonalna sistema v $L^2([0,b])$ za vsak $b > 0$.

Dokaz: Vsako $f \in L^2([0,b])$ lahko razširimo do sode ali do lihe fn. $\tilde{f} \in L^2([-b,+b])$.

Konvergenca ~~Fourierove~~ trigonometrične vrste po točkah.

Naslednja trditev očitno sledi iz Cauchyevega kriterija za absolutno in enakomerno konvergenco.

TRDITEV. Če zaporedji (a_k), (b_k) zadoščata

$$\sum_{k=1}^{\infty}\left(|a_k| + |b_k|\right) < +\infty, \qquad (*)$$

potem trigonometrična vrsta

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos kx + b_k \sin kx\right) \qquad (**)$$

konvergira enakomerno in absolutno na $\mathbb{R}$. Njena vsota je zvezna 2π-periodična funkcija.

Opomba. Pogoj $(*)$ pomeni, da je zaporedje $(a_k, b_k)_{k=1}^{\infty}$ v ℓ_1. Ker sledi $\lim\limits_{k\to\infty} a_k = 0$ in $\lim\limits_{k\to\infty} b_k = 0$ in ker je $0 \leq a_k^2 \leq |a_k|$ ~~za~~, če je $|a_k| < 1$, odtod sledi $\sum\limits_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right) < \infty$, torej vrsta $(**)$ konvergira tudi v $L^2([-\pi, +\pi])$

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Dirichletov kriterij za pogojno konvergenco trigonometričnih vrst po točkah

IZREK. Naj bo $\{b_k\}$ ~~zaporedje~~ monotono padajoče zaporedje:

$$b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0, \quad \lim_{k\to\infty} b_k = 0.$$

Potem

(a) $\sum_{k=1}^{\infty} b_k \cdot \sin kx$ konvergira za $\forall x \in \mathbb{R}$

(b) $\sum_{k=1}^{\infty} b_k \cdot \cos kx$ konvergira $\forall x \in \mathbb{R}$, razen morda za $x = 2p\pi$, $p \in \mathbb{Z}$.

Dokaz: $\sum_{k=1}^{n} \sin kx =: A_n$.

(nek p) $x = 2p\pi \Rightarrow$ člen $0 \Rightarrow$ OK.

Naj bo sedaj $x \neq 2p\pi$. $\frac{x}{2} \neq p\pi, \forall p$

$$\underset{\neq 0}{\sin\frac{x}{2}} \cdot A_n = \sum_{k=1}^{n} \sin\frac{x}{2} \cdot \sin kx$$

$$= \frac{1}{2}\sum_{1}^{n}\left[\cos\left(k-\tfrac{1}{2}\right)x - \cos\left(k+\tfrac{1}{2}\right)x\right]$$

$$= \frac{1}{2}\left(\cos\frac{x}{2} - \cos\left(n+\tfrac{1}{2}\right)x\right)$$

Omejene delne vsote: $|A_n| \leq \frac{1}{|\sin\frac{x}{2}|}$.

Torej $\sum_{k=1}^{\infty} b_k \sin kx$ konvergira po Dirichletovem testu (posplošeni Leibnitzov test).

Podobno za cos.

<u>Primer</u>: $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ konvergira.

Toda: $b_n = \frac{1}{\sqrt{n}}$; $\sum_1^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ divergira.

Iz parsevalove enačbe sledi, da to ni ~~vrsta~~ trig. vrsta nobene $f \in L^2([-\pi,\pi])$.

$$f \in L^2([-\pi,\pi]) \xrightarrow{\Phi(f)} \underbrace{(a_0, a_1, b_1, a_2, b_2, \ldots)}_{\text{Fourierovi koef.}} \in l_2$$

<u>IZREK</u> (Dirichlet).
Denimo, da ima $f \in L^2([-\pi,\pi])$ v točki $x_0 \in (-\pi,\pi)$ skok:

(a) $\exists\, f(x_0+),\ f(x_0-)$ (leva & desna limi)

(b) $$\left.\begin{aligned} |f(x_0+t) - f(x_0+)| &\leq M \cdot t \\ |f(x_0-t) - f(x_0-)| &\leq M t \end{aligned}\right\}$$ za $t > 0$ majhen (za nek M).

Tedaj je:
$$\lim_{N\to\infty} P_N(f)(x_0) = \frac{f(x_0+) + f(x_0-)}{2}$$

$$\left(P_N(f) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)\right).$$

(Brez dokaza.)

POSLEDICA. Če je f Lipschitzovo zvezna fn. na $[-\pi,\pi]$, potem $\forall x \in (-\pi,\pi)$ velja:

$$\lim_{N\to\infty} P_N(f)(x) = f(x).$$

(V krajiščih $x=\pm\pi$ to velja, če je f Lip. in 2π-per. na $\mathbb{R}$).

INTEGRIRANJE TRIG. VRST IN FOURIEROVIH VRST:

Če je $\{\varphi_n\}$ poln ortogonalen sistem v $L^2([a,b])$, je $\forall N \in \mathbb{N}$:

$$f = P_N(f) + R_N(f)$$

$$P_N(f) = \sum_{n=1}^{N} c_n \cdot \varphi_n = \sum_{1}^{N} \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2} \cdot \varphi_n$$

$$P_N(f) \perp R_N(f)$$

$$\|f\|^2 = \|P_N(f)\|^2 + \|R_N(f)\|^2; \quad \lim_{N\to\infty} \|R_N(f)\|^2 = 0.$$

Ker je $\left|\int_a^b R_N(f)(x)\,dx\right| = |\langle R_N(f), 1\rangle| \le \|R_N(f)\| \cdot \|1\|$,

sledi $\lim_{N\to\infty} \int_a^b R_N(f)\,dx = 0$, torej

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \int_a^b \sum_{n=1}^{N} c_n \varphi_n \cdot dx .$$

Fourierovo vrsto torej lahko členoma integriramo:

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b \varphi_n(x)\,dx .$$

Poseben primer: $f \in L^2([-\pi,\pi])$

$$f \sim \frac{a_0}{2} + \sum_1^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \pi a_0 .$$

ODVAJANJE TRIG. VRST:

Naj bo $f \in C^2([-\pi,\pi])$, 2x zvezno odvedljiva.

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad x \in (-\pi,\pi)$$

(Vrsta konvergira po točkah, ker je f Lipsch.)

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty}(-n a_n \sin nx + n b_n \cos nx)$$

Ker je f' odvedljiva in je odvod $(f')' = f''$ omejen, vrsta za $f'(x)$ konvergira po točkah na $x \in (-\pi,\pi)$.

Fourierove koef. funkcije $f'(x)$ dobimo z integracijo per partes... Podobno dobimo višje odvode, če je f gladka.