1 Barvanje zemljevidov

Leta 1852 je Francis Guthrie opazil, da se regionalni zemljevid Anglije da obarvati s štirimi barvami tako, da sta katerikoli dve sosednji regiji različno obarvani. (Regiji sta sosednji le, če imata skupni rob.)

Ugotovil je, da so v splošnem potrebne vsaj štiri barve ter je postavil domnevo, da to število barv tudi zadostuje.

Problem štirih barv (Francis Guthrie, 1852). *Regije poljubnega zemljevida se lahko obarvajo s štirimi barvami tako, da sta katerikoli dve sosednji regiji različno obarvani.*

Izrek o petih barvah (Heawood, 1890). *Vsak ravninski graf je 5-obarvljiv.*

Izrek o štirih barvah (Appel in Haken, 1977). *Vsak ravninski graf je 4-obarvljiv.*

Izrek 9 (Grötzsch) *Vsak ravninski graf brez trikotnikov je 3-obarvljiv.*

List-colorings

- $L(v)$ – list of admissible colors of v.
- <u>L-coloring</u> of G is a function $c: V(G) \to \mathbb{C}$
  1. $\forall v: \quad c(v) \in L(v)$
  2. $c(u) \neq c(v)$ for every pair of adjacent vertices u, v.

- G is <u>k-choosable</u> if G is L-colorable for every list a. L with $|L(v)| \geq k, \forall v$.

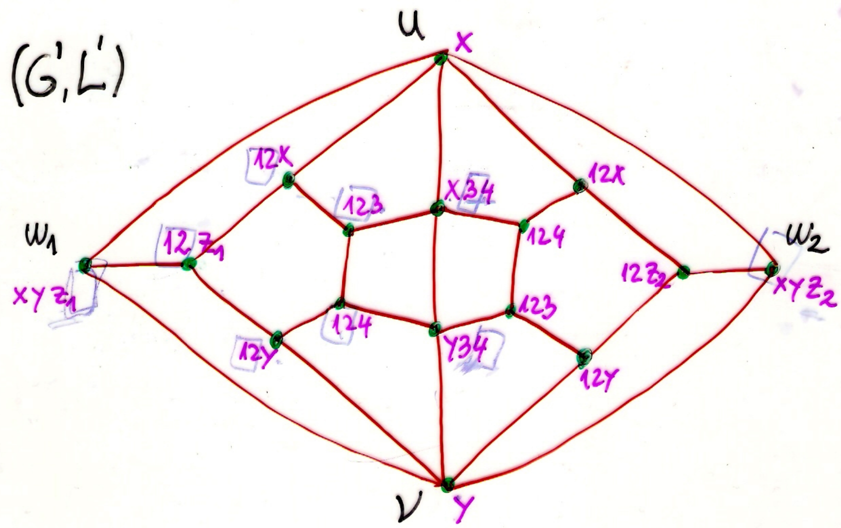
  (Erdös, Rubin, Taylor 1980)
  Vizing 1976

List coloring Planar graphs

- (Thomassen) (1994) Every planar graph is 5-choosable.
- (Voigt) (1993)
  1. $\exists$ planar graph which is not 4-choosable on 238 vertices
  2. $\exists$ Δ-free planar graph which is not 3-choosabl on 166 vertices.
- (Mirzakhani) (1996) $\exists$ planar graph which is not 4-choosable on 63 vertices.
- (Gutner) (1996) $\exists$ Δ-free planar graph on 164 vertices which is not 3-choosable

Thm (Š) 2001 $\exists$ Δ-free planar graph which is not 3-choosable on 119 vertices!

(G', L')

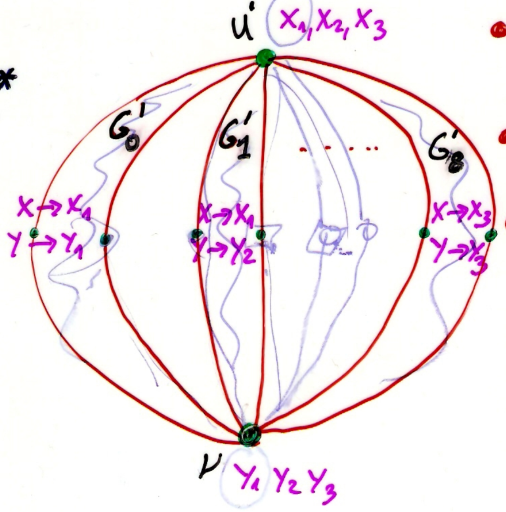


<u>CLAIM</u> G' is not L'-colorable, has 16 vertices.

FAST CONSTRUCTION:

- identify all $u_i' \to u'$ and all $v_i' \to v'$
- $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \emptyset$
- G^* is NOT L^*-colorable
- G^* has $(16-2)\cdot 9 + 2 = 128$ vertices

G^*, L^*



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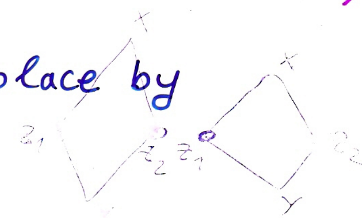
- Identify w_2 of G_i with w_1 of G_{i+1} (index modulo 9)
- Colors $t = (z_1, x, y, z_2)$ of G' replace by t_i in G_i'



$t_0 = (7, 5, 8, 9)$ $t_1 = (8, 5, 9, 10)$ $t_2 = (9, 5, 10, 6)$

$t_3 = (5, 6, 10, 7)$ $t_4 = (6, 7, 10, 9)$ $t_5 = (10, 7, 9, 6)$

$t_6 = (7, 6, 9, 8)$ $t_7 = (9, 6, 8, 7)$ $t_8 = (6, 7, 8, 5)$

L-edge-coloring

$G=(V,E)$ $\quad L: E \to 2^{N}$

$L(e)$ – list of admissible colors for e

- An **L-edge-coloring** of G is a function $\lambda: E \to \mathbb{N}$ such that $\lambda(e) \in L(e)$, $e \in E$ and $\lambda(e) \neq \lambda(f)$ for every pair of adjacent edges e, f.

- G is **k-edge-choosable** if for every list assignment L such that $\forall e: |L(e)| \geq k$, G admits an L-edge color.

- $\chi'_{\ell}(G)$ – **list chromatic index** is the smallest k for which G is k-edge-choosable.

The study of list coloring was introduced independently by:

Vizing (1976) and

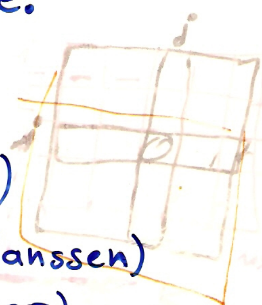
Erdős, Rubin, Taylor (1979)

The list coloring conjecture

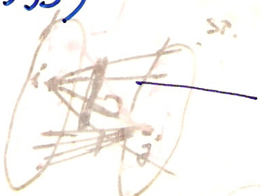
Vizing (1976)

Conjecture: For every graph G: $\chi'(G) = \chi_\ell'(G)$.

Results concerning this conjecture:



- trees; $\Delta(G)=2$ -(trivially)
- $\Delta(G)=3$ and $\chi'(G)=4$ -(Harris, 1985)
- K_{2n+1} (Häggvist, Janssen)
- biportite multigraphs (Galvin, 1995)



Vizing Theorem

(a) Every simple graph G satisfies: $\Delta \leq \chi' \leq \Delta+1$

(b) Every multigraph G satisfies: $\Delta \leq \chi' \leq \Delta+\mu$

Lower and Upper bound of χ_ℓ'.

(a) If G is a simple graph then $\Delta \leq \chi_\ell' \leq \Delta+1$

(b) If G is a multigraph then $\Delta \leq \chi_\ell' \leq \Delta+\mu$.

Total List Coloring

L- total coloring
k- total choosable
χ''_l- list total-chromatic number

<u>Conjecture.</u> $\chi''_l(G) = \chi''(G)$.

<u>Conjecture.</u> $\chi''_l(G) \leq \Delta(G)+2$.

Results conserning second conjecture:

- trees trivially
- cycles:

<u>Theorem 1.</u> Let L be a list assignment such that: $|L(s)| \geq \begin{cases} 3 & s \text{ is vertex} \\ 4 & s \text{ is edge.} \end{cases}$
Then C_n admits an L-total coloring.

<u>Theorem 2.</u> Every multigraph G with $\Delta(G) \leq 3$ is 5-total-choosable.

Total coloring of graphs

The **total coloring** is a coloring of $V(G) \cup E(G)$ such that two adjacent or incident elements of $V(G) \cup E(G)$ are differently colored.

$\chi''(G)$ – **total chromatic number** is the smallest number of colors needed for total coloring of G.

The study of total coloring was introduced by Vizing (1964) and Behzad (1965).

Conjecture: For every simple graph G: $\chi''(G) \leq \Delta(G)+2$
(multigraph version: $\chi''(G) \leq \Delta(G)+\mu(G)+1$.

Some results:

- $\Delta(G) \leq 2$ trivially
- $\Delta(G) = 3$ Rosenfeld, Vijayaditya (1971)
- $\Delta(G) = 4,5$ Kostochka (1977, 1978)

- $\forall v \in V(G)$: $L(v) \neq \emptyset$

Problem: Let G be a planar graph and let L - list assignment such that:

(a) $|L(u) \cap L(v)| \geq 4$ for every adjacent u, v

Is G L-colorable?

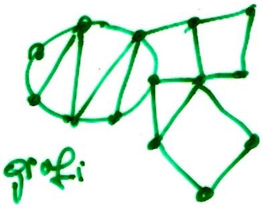
Problem: Let G be a Δ-free planar graph and let L - list assignment s.t.

(b) $|L(u) \cap L(v)| \geq 3$ for every adjacent u, v

Is G L-colorable?

$\boxed{\text{1. TRDITEV.}}$ Vsak ravninski graf brez trikotnikov je 4-izbirljiv. (Ima točko stopnje največ 3)

2. GRAF je zunanjeravninski če vse točke ležijo na zunanjem licu.



$\boxed{\text{TRDITEV}}$. Zunanje-ravninski grafi so 3-izbirljivi (3-obarvljivi)

TRDITEV Vsak zun-rav graf ima točko stopnje ≤ 2.