Rešitve kolokvija iz verjetnosti in statistike z dne 10. 1. 2011

Matematika – bolonjski univerzitetni študij in pedagoška matematika

1. Označimo z A zamudo Ahačičevih, z B pa zamudo Berkopčevih. Tedaj je $T = \min\{A, B\}$. Sledi:

$$\begin{aligned} F_T(t) &= P(T \le t) = P\big(\{A \le t\} \cup \{B \le t\}\big) = \\ &= P(A \le t) + P(B \le t) - P(A \le t)\, P(B \le t)\,. \end{aligned}$$

Za $t \le 0$ je gotovo $F_T(t) = 0$ (ker še nihče ne pride), za $t \ge 10$ pa velja $F_T(t) = 1$ (ker Ahačičevi zagotovo pridejo). Za $0 < t < 10$ pa velja $P(A \le t) = t/10$ in $P(B \le t) = t/20$, od koder sledi:

$$F_T(t) = \frac{3t}{20} - \frac{t^2}{200}\,, \quad f_T(t) = \frac{3}{20} - \frac{t}{100}\,.$$

Torej je:

$$f_T(t) = \begin{cases} \frac{3}{20} - \frac{t}{100} & ;\ 0 < t < 10 \\ 0 & ;\ \text{sicer} \end{cases}\,.$$

2. Velja:

$$\begin{aligned} P(Y = 0) &= P(X = 1) + P(X = 2) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}\,, \\ P(Y = k) &= P(X = 3k) + P(X = 3k+1) + P(X = 3k+2) = \\ &= \frac{1}{3}\left[\left(\frac{2}{3}\right)^{3k-1} + \left(\frac{2}{3}\right)^{3k} + \left(\frac{2}{3}\right)^{3k+1}\right] = 19 \cdot \frac{2^{3k-1}}{3^{3k+2}}\,; \quad k \in \mathbb{N}\,. \end{aligned}$$

3. a) *Prvi način.* Velja:

$$\begin{aligned} f_Z(z) &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, f\left(\frac{z+x}{2}\right) \mathrm{d}x = \\ &= \frac{\lambda^2}{2} \int_{\substack{x \ge 0 \\ (z+x)/2 \ge 0}} e^{-\lambda x}\, e^{-\lambda(z+x)/2}\, \mathrm{d}x = \\ &= \frac{\lambda^2}{2}\, e^{-\lambda z/2} \int_{\max\{-z,0\}}^{\infty} e^{-3\lambda x/2}\, \mathrm{d}x\,. \end{aligned}$$

Za $z \ge 0$ je torej:

$$f_Z(z) = \frac{\lambda^2}{2}\, e^{-\lambda z/2} \int_0^{\infty} e^{-3\lambda x/2}\, \mathrm{d}x = \frac{\lambda}{3}\, e^{-\lambda z/2}\,,$$

za $z \leq 0$ pa velja:

$$f_Z(z) = \frac{\lambda^2}{2}\, e^{-\lambda z/2} \int_{-z}^{\infty} e^{-3\lambda x/2}\, \mathrm{d}x = \frac{\lambda}{3}\, e^{\lambda z}\,.$$

Drugi način. Velja:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f(2y-z)\, f(y)\, \mathrm{d}y = \\ &= \lambda^2 \int_{\substack{2y-z\geq 0\\ y\geq 0}} e^{-\lambda(2y-z)}\, e^{-\lambda y}\, \mathrm{d}y = \\ &= \lambda^2\, e^{\lambda z} \int_{\max\{z/2,0\}}^{\infty} e^{-3\lambda y}\, \mathrm{d}y\,. \end{aligned}$$

Za $z \geq 0$ je torej:

$$f_Z(z) = \lambda^2\, e^{\lambda z} \int_{z/2}^{\infty} e^{-3\lambda y}\, \mathrm{d}y = \frac{\lambda}{3}\, e^{-\lambda z/2}\,,$$

za $z \leq 0$ pa velja:

$$f_Z(z) = \lambda^2\, e^{\lambda z} \int_{0}^{\infty} e^{-3\lambda y}\, \mathrm{d}y = \frac{\lambda}{3}\, e^{\lambda z}\,.$$

b) Velja $E(Z) = 2\,E(X) - E(Y) = \dfrac{2}{\lambda} - \dfrac{1}{\lambda} = \dfrac{1}{\lambda}$ (ni potrebno integrirati $z\, f_Z(z)$).

4. Velja:

$$\begin{aligned} E\big(|Z|\big) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|\, e^{-z^2/2}\, \mathrm{d}z = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z\, e^{-z^2/2}\, \mathrm{d}z = \frac{2}{\sqrt{2\pi}} \doteq 0{\cdot}798\,, \\ D(|Z|) &= E\big(|Z|^2\big) - \Big(E\big(|Z|\big)\Big)^2 = E\big(Z^2\big) - \Big(E\big(|Z|\big)\Big)^2 = \\ &= D(Z) - \Big(E\big(|Z|\big)\Big)^2 = 1 - \frac{2}{\pi} \doteq 0{\cdot}363\,. \end{aligned}$$

V splošnem za vsako slučajno spremenljivko velja:

$$\begin{aligned} D\big(|X|\big) &= E\big(|X|^2\big) - \Big(E\big(|X|\big)\Big)^2 = E(X^2) - \Big(E\big(|X|\big)\Big)^2 \leq E(X^2) - \big(E(X)\big)^2 = \\ &= D(X)\,. \end{aligned}$$

Zadnja neenakost velja zato, ker je $\big|E(X)\big| \leq E\big(|X|\big)$ in zato $\big(E(X)\big)^2 \leq \Big(E\big(|X|\big)\Big)^2$.