NUMERIČNA APROKSIMACIJA IN INTERPOLACIJA

Rešitve 1. pisnega izpita

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1. Naj bo $B_n f$ Bernsteinova aproksimacija za funkcijo f na intervalu $[0,1]$,

$$B_n f(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) b_{n,i}(x), \quad b_{n,i}(x) = \binom{n}{i} x^i \,(1-x)^{n-i}.$$

Zapišite polinom $B_n f$ v bazi $(b_{n+1,i})_{i=0}^{n+1}$: določite koeficiente $(\alpha_i)_{i=0}^{n+1}$, da bo veljalo

$$B_n f(x) = \sum_{i=0}^{n+1} \alpha_i \, b_{n+1,i}(x).$$

Namig: Uporabite $1 = x + (1-x)$ na levi strani enačbe in primerjajte koeficiente pri enakih polinomskih členih na obeh straneh enačbe.

Rešitev:
Z upoštevanjem namiga dobimo

$$\begin{aligned} B_n f(x) &= (x + (1-x)) B_n f(x) = \\ &= \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^{i+1} (1-x)^{n-i} + \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \binom{n}{i} x^{i} (1-x)^{n+1-i}. \end{aligned}$$

V prvi vsoti prestavimo indeks $i \to i-1$ in upoštevamo, da je $\binom{n}{-1} = 0$, $\binom{n}{n+1} = 0$:

$$\begin{aligned} B_n f(x) &= \sum_{i=1}^{n+1} f\left(\frac{i-1}{n}\right) \binom{n}{i-1} x^i (1-x)^{n-i+1} + \sum_{i=0}^{n+1} f\left(\frac{i}{n}\right) \binom{n}{i} x^i (1-x)^{n+1-i} = \\ &= \sum_{i=\not{1}\,0}^{n+1} f\left(\frac{i-1}{n}\right) \frac{\binom{n}{i-1}}{\binom{n+1}{i}} b_{n+1,i}(x) + \sum_{i=0}^{n+1} f\left(\frac{i}{n}\right) \frac{\binom{n}{i}}{\binom{n+1}{i}} b_{n+1,i}(x) = \\ &= \sum_{i=0}^{n+1} f\left(\frac{i-1}{n}\right) \frac{i}{n+1} b_{n+1,i}(x) + \sum_{i=0}^{n+1} f\left(\frac{i}{n}\right) \frac{n+1-i}{n+1} b_{n+1,i}(x) = \\ &= \sum_{i=0}^{n+1} \left(f\left(\frac{i-1}{n}\right) \frac{i}{n+1} + f\left(\frac{i}{n}\right) \frac{n+1-i}{n+1} \right) b_{n+1,i}(x). \end{aligned}$$

Ker tvorijo $(b_{n+1,i}(x))_{i=0}^{n+1}$ bazo prostora polinomov stopnje $\leq n+1$ sledi iz

$$B_n f(x) = \sum_{i=0}^{n+1} \alpha_i \, b_{n+1,i}(x),$$

da so koeficienti α_i enaki

$$\alpha_i = f\left(\frac{i-1}{n}\right) \frac{i}{n+1} + f\left(\frac{i}{n}\right) \frac{n+1-i}{n+1}, \quad i = 0, 1, \ldots, n+1.$$

2. Skalarni produkt je definiran kot

$$\langle f, g\rangle := \int_0^1 f(x)\, g(x)\, \frac{1}{\sqrt{x}}\, dx.$$

Izračunajte prve tri ortogonalne polinome. Uporabite tričlensko rekurzivno formulo.

Rešitev:
Z uporabo algoritma dobimo

$$\begin{aligned}
&p_0(x) = 1, \quad \|p_0\|^2 = \int_0^1 \frac{1}{\sqrt{x}} dx = 2, \\
&\alpha_1 = -\frac{1}{2}\int_0^1 x\frac{1}{\sqrt{x}} dx = -\frac{1}{3}, \\
&p_1(x) = x - \frac{1}{3}, \quad \|p_1\|^2 = \int_0^1 \left(x - \frac{1}{3}\right)^2 \frac{1}{\sqrt{x}} dx = \frac{8}{45}, \\
&\alpha_2 = -\frac{45}{8}\int_0^1 x\left(x - \frac{1}{3}\right)^2 \frac{1}{\sqrt{x}} dx = -\frac{11}{21}, \quad \beta_2 = -\frac{\|p_1\|^2}{\|p_0\|^2} = -\frac{4}{45}, \\
&p_2(x) = \left(x - \frac{11}{21}\right)\left(x - \frac{1}{3}\right) - \frac{4}{45} = x^2 - \frac{6}{7}x + \frac{3}{35}.
\end{aligned}$$

Prvi trije ortogonalni polinomi so torej enaki

$$p_0(x) = 1, \quad p_1(x) = x - \frac{1}{3}, \quad p_2(x) = x^2 - \frac{6}{7}x + \frac{3}{35}.$$

3. Naj bodo $x_0, x_1, \ldots, x_n$ paroma različne točke in naj bo

$$V = V(x_0, x_1, ..., x_n) = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}$$

Vandermondova matrika. Izrazite njen inverz V^{-1} s pomočjo Lagrangeevih baznih polinomov. Posebej izpišite V^{-1} za primer $x_0 = -1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$.

Rešitev:
Iščemo matriko $V^{-1} = [X_1, X_2, \ldots, X_n]$, tako da bo $VV^{-1} = I$. Za izračun i-tega stolpca inveza moramo rešiti linearni sistem

$$VX_i = e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^T.$$

Naj bo $X_i = (a_{0,i}, a_{1,i}, \ldots, a_{n,i})^T$. Na reševanje tega linearnega sistema lahko pogledamo tudi takole. Iščemo polinom $p_i(x) = \sum_{k=0}^{n} a_{k,i}x^k$ stopnje n, za katerega velja $p_i(x_j) = \delta_{i,j}$. Polinom, ki temu zadošča, pa je natanko Lagrangeev bazni polinom $\ell_{i,n}(x) = \prod_{j=0,j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$. Če ta polinom zapišemo v standardni bazi, potem so koeficienti polinoma v tej bazi ravno elementi i-tega stolpca inverza. Ker velja

$$\ell_{i,n}(x) = \sum_{k=0}^{n} \frac{1}{k!}\ell_{i,n}^{(k)}(0)x^k,$$

je

$$X_i = \begin{pmatrix} \ell_{i,n}(0) \\ \ell'_{i,n}(0) \\ \frac{1}{2!}\ell''_{i,n}(0) \\ \vdots \\ \frac{1}{n!}\ell_{i,n}^{(n)}(0) \end{pmatrix},$$

oziroma

$$V^{-1} = \left(\frac{1}{k!}\ell_{i,n}^{(k)}(0)\right)_{k,i=0}^{n}.$$

Če izberemo $x_0 = -1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ dobimo

$$\begin{aligned}
\ell_{0,3}(x) &= \frac{(x-1)(x-2)(x-4)}{-30} = -\frac{1}{30}x^3 + \frac{7}{30}x^2 - \frac{7}{15}x + \frac{4}{15}, \\
\ell_{1,3}(x) &= \frac{(x+1)(x-2)(x-4)}{6} = \frac{1}{6}x^3 - \frac{5}{6}x^2 + \frac{1}{3}x + \frac{4}{3}, \\
\ell_{2,3}(x) &= \frac{(x+1)(x-1)(x-4)}{-6} = -\frac{1}{6}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{2}{3}, \\
\ell_{3,3}(x) &= \frac{(x+1)(x-1)(x-2)}{30} = \frac{1}{30}x^3 - \frac{1}{15}x^2 - \frac{1}{30}x + \frac{1}{15},
\end{aligned}$$

od koder sledi

$$V^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{4}{3} & -\frac{2}{3} & \frac{1}{15} \\ -\frac{7}{15} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{30} \\ \frac{7}{30} & -\frac{5}{6} & \frac{2}{3} & -\frac{1}{15} \\ -\frac{1}{30} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{30} \end{pmatrix}.$$

4. Naj bo $f(x) = \sin(\pi x)$, $\boldsymbol{x} = (x_i)_{i=0}^n$ ekvidistantno zaporedje stičnih točk na intervalu $[-1, 1]$ in $S : [-1, 1] \to \mathbb{R}$ parabolični zlepek, za katerega velja

$$S\Big|_{[x_i, x_{i+1}]} = P_i \in \mathbb{P}_2, \quad i = 0, 1, \ldots, n-1,$$
$$P_i(x_i) = f(x_i), \quad P_i\left(\frac{x_i + x_{i+1}}{2}\right) = f\left(\frac{x_i + x_{i+1}}{2}\right), \quad P_i(x_{i+1}) = f(x_{i+1}).$$

Na koliko delov moramo razdeliti interval $[-1, 1]$, da bo napaka $\|f - S\|_{\infty,[-1,1]}$ manjša od 10^{-3}.

Rešitev:
Naj bo $x \in [x_i, x_{i+1}]$ in naj bodo točke ekvidistantne s korakom h. Po formuli za napako interpolacijskega polinoma dobimo

$$f(x) - P_i(x) = \underbrace{(x - x_i)\left(x - x_i - \frac{h}{2}\right)(x - x_i - h)}_{\omega(x)}\left[x_i, x_i + \frac{h}{2}, x_i + h, x\right] f =$$
$$= \omega(x)\frac{1}{6} f^{(3)}(\xi), \quad \xi \in (x_i, x_{i+1}),$$

oziroma

$$|f(x) - P_i(x)| \leq \|\omega\|_{\infty,[x_i,x_{i+1}]} \frac{1}{6} \left\|f^{(3)}\right\|_{\infty,[x_i,x_{i+1}]}.$$

Iz

$$\omega'(x) = 3(x - x_i)^2 - 3h(x - x_i) + \frac{h^2}{2} = 0$$

izračunamo, da je ekstrem ω dosežen v točkah $x = x_i + \frac{h}{6}\left(3 \pm \sqrt{3}\right)$ in je enak

$$\|\omega\|_{\infty,[x_i,x_{i+1}]} = \frac{\sqrt{3}}{36} h^3.$$

Dalje je

$$\left\|f^{(3)}\right\|_{\infty,[x_i,x_{i+1}]} = \pi^3,$$

od koder sledi

$$\|f - P_i\|_{\infty,[x_i,x_{i+1}]} \leq \frac{\sqrt{3}}{36}\frac{\pi^3}{6} h^3.$$

Ker je ta ocena neodvisna od i, velja

$$\|f - S\|_{\infty,[-1,1]} \leq \frac{\sqrt{3}}{6^3}\pi^3 h^3$$

in napaka bo manjša od 10^{-3}, če bo veljalo

$$h < \sqrt[3]{\frac{6^3 \cdot 10^{-3}}{\sqrt{3}\pi^3}} = 0.159031$$

oziroma, če bo $n > 2/0.159031 = 12.5762$. Interval moramo torej razdeliti na vsaj 13 delov.