Topics in Combinatorics 2011

Homework 10 (due December 16)

1. In this exercise, we prove that if $\ell(\lambda) \le n$, then

$$s_\lambda(1,q,\dots,q^{n-1}) = q^{n(\lambda)} \prod_{x\in[\lambda]} \frac{1-q^{n+c(x)}}{1-q^{h(x)}}, \qquad (*)$$

where $c(x)$ and $h(x)$ were defined in Homework 1, Problem 3.

(a) Show that if $x_i = q^{i-1}$ for $1 \le i \le n$, $x_i = 0$ for $i > n$, and $\ell(\lambda) \le n$,

$$a_{\lambda+\delta} = \prod_{i<j}(q^{\lambda_j+n-j} - q^{\lambda_i+n-i}) = q^{n(\lambda)+\binom{n}{3}} \prod_{i<j}(1-q^{\lambda_i-\lambda_j-i+j})$$

and

$$a_\delta = \prod_{i<j}(q^{n-j}-q^{n-i}) = q^{\binom{n}{3}} \prod_{i<j}(1-q^{j-i}).$$

(b) (Bonus problem.) Show that

$$\prod_{x\in[\lambda]}(1-q^{h(x)}) = \frac{\prod_{i=1}^n (1-q)(1-q^2)\cdots(1-q^{\lambda_i+n-i})}{\prod_{i<j}(1-q^{\lambda_i-\lambda_j-i+j})}.$$

Hint: Add $n-i$ squares to the i-th row of $[\lambda]$, and write the number $\lambda_i+n-i-j+1$ in cell (i,j). What happens if you remove all columns j for j of the form $\lambda_k+n-k+1$ for some k?

(c) Show that

$$\prod_{x\in[\lambda]}(1-q^{n+c(x)}) = \prod_{i=1}^n \frac{(1-q)(1-q^2)\cdots(1-q^{\lambda_i+n-i})}{(1-q)(1-q^2)\cdots(1-q^{n-i})}.$$

(d) Use (a), (b) and (c) to prove formula $(*)$. Use $(*)$ to find $s_\lambda(1,q,q^2,\dots)$ and $s_\lambda(1,\dots,1)$ (n ones). Compare with Homework 5, exercise 1.

2. Let $p(n)$ denote the number of partitions of n. Prove that

$$\det(p(i-j+1))_{1\le i,j\le n}$$

equals ±1 or 0, depending on whether n is a pentagonal number or not.

3. Show that under the specialization $h_n \mapsto X$ for $n \ge 1$, we have $s_\lambda \mapsto 0$ if $\lambda_2 > 1$, and if $\lambda = (a,1^b)$, then $s_\lambda \mapsto X(X-1)^b$. Use this to prove that

$$\sum_\lambda m_\lambda(x) X^{\ell(\lambda)} = \sum_{a\ge1,b\ge0} s_{(a,1^b)}(x) X(X-1)^b.$$

Use this identity to reprove the expansion of $p_n^{(k)}$ in terms of Schur functions from Homework 9, Exercise 2b.