Topics in Combinatorics 2011

Homework 12 (due January 20)

1. Prove that every Knuth equivalence class contains exactly one involution. Find the involution that is Knuth equivalent to $w = 914362758$.

2. Suppose that w maps to (P, Q) with the RSK algorithm. We know that $w_0 w w_0$ maps to $(\mathrm{evac}(P), \mathrm{evac}(Q))$. What do ww_0 and $w_0 w$ map to? *Hint:* Use Greene's theorem to prove that the insertion tableau of ww_0 is P^T (the transpose of P).

3. Fix a tree t with n vertices and root r. Let $S(v)$ denotes the shortest path from v to r. Let $h(v)$ be the number of vertices v' such that $S(v) \subseteq S(v')$. *E.g.*, $h(r) = n$, and $h(v) = 1$ for every leaf $v \neq r$. An *increasing tree of shape* t is a bijection $\gamma\colon t \to [n]$ such that $\gamma(v) < \gamma(v')$ for all vertices $v, v' \in t$ with $S(v) \subseteq S(v')$; note that this implies that $\gamma(r) = 1$. Prove that the number of increasing trees of shape t equals
$$\frac{n!}{\prod_{v \in t} h(v)}.$$

4. In this exercise, we prove that the hook walk ends in corner (r, s) with probability
$$\frac{1}{n} \prod_{i=1}^{r-1} \left(1 + \frac{1}{h_{is} - 1}\right) \prod_{j=1}^{s-1} \left(1 + \frac{1}{h_{rj} - 1}\right), \qquad (*)$$
which completes the proof of the hook-length formula.

   (a) For a hook walk $(i_1, j_1) \to (i_2, j_2) \to \ldots \to (r, s)$ on the diagram, call $I = \{i_1, i_2, \ldots, r\}$ and $J = \{j_1, j_2, \ldots, s\}$ the *horizontal* and *vertical* projection. For nonempty sets $I \subseteq [r]$, $J \subseteq [s]$ with $\max I = r$ and $\max J = s$, denote by $P(I, J)$ the probability that a hook walk that starts in $(\min I, \min J)$ has horizontal projection I and vertical projection J (in particular, it ends in (r, s)). Prove that
$$P(I,J) = \frac{1}{h_{\min I, \min J} - 1}\left(P(I \setminus \{\min I\}, J) + P(I, J \setminus \{\min J\})\right)$$
$$P(I,\{s\}) = \frac{1}{h_{\min I, s} - 1} P(I \setminus \{\min I\}, \{s\})$$
$$P(\{r\}, J) = \frac{1}{h_{r, \min J} - 1} P(\{r\}, J \setminus \{\min J\})$$
if $|I| > 1$ and $|J| > 1$.

   (b) Use induction to prove that for nonempty sets $I \subseteq [r]$, $J \subseteq [s]$ with $\max I = r$ and $\max J = s$,
$$P(I,J) = \prod_{i \in I \setminus \{r\}} \frac{1}{h_{is} - 1} \prod_{j \in J \setminus \{s\}} \frac{1}{h_{rj} - 1}.$$

   (c) Prove $(*)$.