Topics in Combinatorics 2011

Homework 1 (due October 14)

1. A partition λ is *odd* (respectively *distinct*) if all its parts are odd (resp. distinct). Denote by $o(n)$ (resp. $d(n)$) the number of odd (resp. distinct) partitions of n. Find the generating functions

$$\sum_{n\geq 0} o(n)x^n \quad \text{and} \quad \sum_{n\geq 0} d(n)x^n.$$

Your computation should show that the number of odd partitions of size n is equal to the number of distinct partitions of size n. Find a bijective proof of this statement.

2. Take the Ferrers diagram of a partition with distinct parts. Denote by k the smallest part, and by s the rightmost 45 degree line of the diagram. In other words, $s = \max\{i \colon \lambda_j = \lambda_1 + 1 - j \text{ for } j \leq i\}$. If $k > s$, take the rightmost 45 degree line and move it to form a new row. Otherwise, move the bottom row to form a new rightmost 45 degree line. This gives a new partition into distinct parts *except in certain special cases.* Find those special cases and show how this implies the *pentagonal number theorem*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}.$$

Use this theorem to find a recursive formula for $p(n)$.

3. For λ a partition, define the *hook length* of a cell $x = (i,j) \in [\lambda]$ by $h(x) = h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ and the *content* of x by $c(x) = c(i,j) = j - i$.

   (a) Draw two diagrams of 5332 and fill one with hook lengths and the other one with contents. Why do we call $h(x)$ the hook length?

   (b) Prove that
   $$\sum_{x\in[\lambda]} h(x) = n(\lambda) + n(\lambda') + |\lambda|.$$

   (c) Prove that
   $$\sum_{x\in[\lambda]} c(x) = n(\lambda') - n(\lambda).$$

   (d) Prove that
   $$\sum_{x\in[\lambda]} (h^2(x) - c^2(x)) = |\lambda|^2.$$

   Hint. Use induction on $|\lambda|$.