Topics in Combinatorics 2011

Homework 2 (due October 21)

1. Recall that we defined a metric d on $K[[x]]$ by
$$d(a,b) = 2^{-\min\{i\colon a_i \neq b_i\}}$$
for $a \neq b$. Prove the following statements.
   (a) The metric space $(K[[x]], d)$ is complete.
   (b) We have $a^{(k)} \to a$ if and only if $v(a^{(k)} - a) \to \infty$.
   (c) Recall that if $x = (0,1,0,0,\ldots)$, $a = (a_n)_{n\in\mathbb{N}}$ and $F(x) = \sum_{n\in\mathbb{N}} a_n x^n$ are equal in $K[[x]]$. Prove that the sum $\sum_{k=1}^{\infty} F_k(x)$ converges if and only if $v(F_k(x)) \to \infty$, and that the product $\prod_{k=1}^{\infty}(1+F_k(x))$, where $F_k(0) = 1$ for all k, converges if and only if $v(F_k(x)) \to \infty$.

2. Prove that composition of formal power series is associative. More precisely, prove that if $G(0) = H(0) = 0$, then $F\circ(G\circ H) = (F\circ G)\circ H$. Then prove that $(\{F \in K[[x]]\colon v(F) = 1\}, \circ)$ is a group.

3. A permutation $\pi \in \mathfrak{S}_n$ is called an *involution* if $\pi^2 = \mathrm{id}$. Let a_n denote the number of involutions in $\mathfrak{S}_n$.
   (a) Prove that $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (n-1)a_{n-2}$ for $n \geq 2$.
   (b) Use (a) to find the exponential generating function for a_n.

4. A permutation w is called *alternating* if $w_1 > w_2 < w_3 > w_4 < \ldots$, and *reverse alternating* if $w_1 < w_2 > w_3 < w_4 > \ldots$
   (a) Prove that the number of alternating permutations of $\mathfrak{S}_n$ is equal to the number of reverse alternating permutations of $\mathfrak{S}_n$.
   (b) Denote the number of (reverse) alternating permutations of $\mathfrak{S}_n$ by E_n. Prove that
$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}.$$
   (c) Denote the generating function
$$\sum_{n\geq 0} E_n \frac{x^n}{n!}$$
by y. Prove that
$$2y' = y^2 + 1, \qquad y(0) = 1.$$
   (d) Solve the differential equation from (c).
   (e) Find Taylor expansion for $\tan x$ and $1/\cos x$.